1. Find a linear differential equation (homogeneous, with constant coefficients) of smallest order which has \( f(x) = x \sin(x) \) as a solution.

**Solution:** take the first few derivatives and see how they linearly interact:

\[
\begin{align*}
  f'(x) &= \sin(x) + x \cos(x) \\
  f''(x) &= 2 \cos(x) - x \sin(x) \\
\end{align*}
\]

We can move \( f(x) \) to the LHS

\[
f''(x) + f(x) = 2 \cos(x)
\]

and differentiate twice more:

\[
f^{(4)}(x) + f''(x) = -2 \cos(x)
\]

Sum \( f^{(4)}(x) + f''(x) \) with \( f''(x) + f(x) \):

\[
f^{(4)}(x) + 2f''(x) + f(x) = 0
\]

is the linear ODE we are looking for.

We cannot find an ODE of lower order, as the first 3 derivatives of \( f \), namely \( x \sin(x), \sin(x) + x \cos(x), 2 \cos(x) - x \sin(x) \), are linearly independent (enough to evaluate their values at 0, \( \pi/2, \pi \) to see that).
2. Let

\[ A_t := \exp \left[ t \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \right] \]

Find the smallest \( t > 0 \) such that \( A_t = E \) (the identity matrix).

**Solution:** this matrix is merely a rotation about the axis

\[ \hat{p} = (1 \ 1 \ 1). \]

Finding the characteristic polynomial of the matrix inside the exponential \((z^3 + 3z)\) implies that the eigenvalues of \( A_t \) are

\[ \exp(\pm i \sqrt{3}t), 1 \]

which are all equal to 1 first when \( t_* = 2\pi/\sqrt{3} \).
3. Consider the vector space $U$ spanned by the functions

\[ e_0 = 1 \quad e_1 = e^{-x} \quad e_3 = e^{-2x} \quad e_3 = e^{-3x} \]

Find the Gram matrix, its determinant, and signature of the following bilinear forms:

\[
Q_1(f, g) = f(1)g(-1) + f(0)g(0) + f(-1)g(1)
\]

\[
Q_2(f, g) = \int_0^\infty f(x)g(x)e^{-x}dx
\]

**Solution:** we begin with $Q_1$.

Let \( e_k(x) := e^{-kx} \), and the \((k, m)\) entries of the Gram matrix are

\[
G_{k,m} := Q_1(e_k, e_m) = e^{k-m} + 1 + e^{m-k} \quad 0 \leq m, k \leq 3
\]

For the signature, observe that

\[
Q_1(f, f) = \frac{1}{2}h_1^2 - \frac{1}{2}h_2^2 + h_3^2,
\]

where

\[
h_1 = f(1) + f(-1), \quad h_2 = f(1) - f(-1), \quad \text{and} \quad h_3 = f(0).
\]

Hence, the signature of the quadratic form

\[
q_1(f) = Q_1(f, f)
\]

has \( n_- = n_0 = 1 \), and \( n_+ = 2 \). From that we also learn that \( \det(G) = 0 \).

For $Q_2$ we have

\[
G_{k,m} := Q_2(e_k, e_m) = \int_0^\infty e^{-(m+k+1)x}dx = \frac{1}{m+k+1} \quad 0 \leq m, k \leq 3
\]

whose determinant is the Cauchy determinant equal to

\[
\det(G) = \frac{\prod_{0 \leq k < m \leq 3}(m-k)^2}{\prod_{0 \leq k, m \leq 3}(1+k+m)}
\]

and since

\[
q_2(f) = \int_0^\infty f^2(x)e^{-x}dx \geq 0
\]

and zero only when \( f(x) = 0 \) a.e, the signature is \( n_+ = 4 \), and \( n_0 = n_- = 0 \).
4. Consider the operator defined by the matrix

\[ A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \]

Let \( B = A^n \). Compute \( B_{11} \), \( \text{tr}(B) \) and \( \det(B) \).

**Solution:** The characteristic polynomial is

\[
P_A(\lambda) = \det \begin{pmatrix} t & -1 & -2 \\ -1 & t-1 & -1 \\ -2 & -1 & t \end{pmatrix} = t \cdot \det \begin{pmatrix} t-1 & -1 \\ -1 & t \end{pmatrix} - (-1) \det \begin{pmatrix} -1 & -2 \\ -1 & t \end{pmatrix} - 2 \det \begin{pmatrix} -1 & -2 \\ t-1 & -1 \end{pmatrix}
\]

\[
= t(t(t-1)-1) - (t+2) - 2(1+2(t-1))
= t^3 - t^2 - 6t = t(t^2 - t - 6)
\]

So the eigenvalues are \( t = 0 \) and \( t = \frac{1}{2}(1 \pm \sqrt{25}) = 3, -2 \).

Clearly, the \( A \) is singular and so is \( A^n \), and the trace is the sum of eigenvalues of \( A^n \), each of which is the \( n \)-th power of of an eigenvalue of \( A \):

\[
\det(A^n) = \det(A)^n = 0
\]
\[
\text{tr}(A^n) = 3^n + (-2)^n.
\]

For \( A^n \) we need to gather eigenvectors for the full diagonal form. It is easy to note that \( e_1^* = (1, 1, 1)/\sqrt{3} \) is a norm 1 eigenvector of \( A \) with \( t = 3 \). Next, find the eigenvector corresponding to \( t = -2 \): \( e_2^* = (1, 0, -1)/\sqrt{2} \). Whichever the third eigenvector \( e_3 \), the matrix \( A \) can be represented as

\[ A = 3e_1e_1^* + (-2)e_2e_2^* + 0e_3e_3^*, \]

whence

\[ B = 3^n e_1e_1^* + (-2)^n e_2e_2^*, \]

and

\[ B_{11} = \frac{1}{3} \times 3^n + \frac{1}{2} \times (-2)^n. \]