1

First order differential equations

1.1 General remarks about differential equations

1.1.1 Terminology

A differential equation is an equation involving a function and its derivatives. An example which we will study in detail in this book is the pendulum equation

$$\frac{d^2x}{dt^2} = -\sin(x), \tag{1.1}$$

which is a differential equation for a real-valued function x of one real variable t. The equation expresses the equality of two *functions*. To make this clear we could write (1.1) more explicitly as

$$\frac{d^2x}{dt^2}(t) = -\sin(x(t)) \qquad \text{for all } t \in \mathbb{R},$$

but this would lead to very messy expressions in more complicated equations. We therefore often suppress the independent variable.

When formulating a mathematical problem involving an equation, we need to specify *where* we are supposed to look for solutions. For example, when looking for a solution of the equation $x^2 + 1 = 0$ we might require that the unknown x is a real number, in which case there is no solution, or we might allow x to be complex, in which case we have the two solutions $x = \pm i$. In trying to solve the differential equation (1.1) we are looking for a twice-differentiable function $x : \mathbb{R} \to \mathbb{R}$. The set of all such functions is very big (bigger than the set of all real numbers, in a sense that can be made precise by using the notion of cardinality), and this is one basic reason why, generally speaking, finding solutions of differential equations is not easy.

Differential equations come in various forms, which can be classified as follows. If only derivatives of the unknown function with respect to one variable appear, we call the equation an *ordinary differential equation*, or ODE for short. If the function depends on several variables, and if partial derivatives with respect to at least two variables appear in the equation, we call it a *partial differential equation*, or PDE for short. In both cases, the order of the differential equation is the order of the highest derivative occurring in it. The independent variable(s) may be real or complex, and the unknown function may take values in the real or complex numbers, or in $\mathbb{R}^n$ of $\mathbb{C}^n$ for some positive integer n. In the latter case we can also think of the differential equation as a set of differential equations for each of the n components. Such a set is called a *system of differential equations of dimension n*. Finally we note that there may be parameters in a differential equation, which play a different role from the variables. The difference between parameters and variables is usually clear from the context, but you can tell that something is a parameter if no derivatives with respect to it appear in the equation. Nonetheless, solutions still depend on these parameters. Examples illustrating the terms we have just introduced are shown in Fig. 1.1.

In this book we are concerned with ordinary differential equations for functions of one real variable and, possibly, one or more parameters. We mostly consider real-valued functions but complex-valued functions also play an important role.

1.1.2 Approaches to problems involving differential equations

Many mathematical models in the natural and social sciences involve differential equations. Differential equations also play an important role in many branches of pure mathematics. The following system of ODEs for real-valued functions a, b and c of one

$$\frac{dy}{dx} - xy = 0$$

first order ordinary differential equation

$$\frac{du}{dt} = -v, \ \frac{dv}{dt} = u$$

first order system of ordinary differential equations, dimension two

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^3$$

second order ordinary differential equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

second order partial differential equation with parameter c

Fig. 1.1. Terminology for differential equations

real variable r arises in differential geometry:

$$\begin{aligned} \frac{da}{dr} &= \frac{1}{2rc}(a^2 - (b-c)^2) \\ \frac{db}{dr} &= \frac{b}{2acr}(b^2 - (a-c)^2) \\ \frac{dc}{dr} &= \frac{1}{2ra}(c^2 - (a-b)^2), \end{aligned}$$

One is looking for solutions on the interval $[\pi, \infty)$ subject to the initial conditions

$$a(\pi) = 0, \quad b(\pi) = \pi, \quad c(\pi) = -\pi.$$

At the end of this book you will be invited to study this problem as a part of an extended project. At this point, imagine that a colleague or friend had asked for your help in solving the above equations. What would you tell him or her? What are the questions that need to be addressed, and what methods do you know for coming up with answers? Try to write down some ideas before looking at the following list of issues and approaches.

(a) **Is the problem well-posed?** In mathematics, a problem is called well-posed if it has a solution, if that solution is unique and if the solution depends continuously on the data given in the problem, in a suitable sense. When a differential equation has one solution, it typically has infinitely many. In order to obtain a well-posed problem we therefore need to complement the differential equation with additional requirements on the solution. These could be initial conditions (imposed on the solution and its derivatives at one point) or boundary conditions (imposed at several points).

(b) **Are solutions stable?** We would often like to know how a given solution changes if we change the initial data by a small amount.

(c) **Are there explicit solutions?** Finding explicit solutions in terms of standard functions is only possible in rare lucky circumstances. However, when an explicit formula for a general solution can be found, it usually provides the most effective way of answering questions related to the differential equation. It is therefore useful to know the types of differential equation which can be solved exactly.

(d) **Can we find approximate solutions?** You may be able to solve a simpler version of the model exactly, or you may be able to give an approximate solution of the differential equation. In all approximation methods it is important to have some control over the accuracy of the approximation.

(e) **Can we use geometry to gain qualitative insights?** It is often possible to derive general, qualitative features of solutions without solving the differential equation. These could include asymptotic behaviour (what happens to the solution for large r?) and stability discussed under (b).

(f) **Can we obtain numerical solutions?** Many numerical routines for solving differential equations can be downloaded from open-source libraries like SciPy. Before using them, check if the problem you are trying to solve is well-posed. Having some insight into approximate or qualitative features of the solution usually helps with the numerical work.

In this text we will address all of these issues. We begin by looking at first order differential equations.

1.2 Exactly solvable first order ODEs

1.2.1 Terminology

The most general first order ODE for a real-valued function x of one real variable t is of the form

$$F\left(t, x, \frac{dx}{dt}\right) = 0, \tag{1.2}$$

for some real-valued function F of three variables. The function x is a solution if it is defined at least on some interval $I \subset \mathbb{R}$ and if

$$F\left(t, x(t), \frac{dx}{dt}(t)\right) = 0 \text{ for all } t \in I.$$

We call a first order ODE *explicit* if it can be written in terms of a real-valued function f of two variables as

$$\frac{dx}{dt} = f(t, x). \tag{1.3}$$

Otherwise, the ODE is called *implicit*. Before we try to understand the general case, we consider some examples where solutions can be found by elementary methods.

1.2.2 Solution by integration

The simplest kind of differential equation can be written in the form

$$\frac{dx}{dt} = f(t),$$

where f is a real-valued function of one variable, which we assume to be continuous. By the fundamental theorem of calculus we can find solutions by integration

$$x(t) = \int f(t)dt.$$

The right hand side is an indefinite integral, which contains an arbitrary constant. As we shall see, solutions of first order differential equations are typically only determined up to an arbitrary constant. Solutions to first order ODEs which contain an arbitrary constant are called *general solutions*.

Exercise 1.1 Revise your integration by finding general solutions of the following ODEs. Where are these solutions valid?

$$\text{(i)}\ \frac{dx}{dt} = \sin(4t-3), \qquad \text{(ii)}\ \frac{dx}{dt} = \frac{1}{t^2-1}.$$

1.2.3 Separable equations

Let us find the general solution of the following slightly more complicated equation:

$$\frac{dx}{dt} = 2tx^2. \tag{1.4}$$

It can be solved by separating the variables, i.e., by bringing all the t-dependence to one side and all the x-dependence to the other, and integrating:

$$\int \frac{1}{x^2} dx = \int 2t dt,$$

where we need to assume that $x \neq 0$. Integrating once yields

$$-\frac{1}{x} = t^2 + c,$$

with an arbitrary real constant c. Hence, we obtain a one-parameter family of solutions of (1.4):

$$x(t) = -\frac{1}{t^2+c}. \tag{1.5}$$

In finding this family of solutions we had to assume that $x \neq 0$. However, it is easy to check that the constant function $x(t) = 0$ for all t also solves (1.4). It turns out that the example is typical of the general situation: in separating variables we may lose constant solutions, but these can be recovered easily by inspection. A precise formulation of this statement is given in Exercise 1.8, where you are asked to prove it using an existence and uniqueness theorem which we discuss in Section 1.3.

The general solution of a first order ODE is really the family of all solutions of the ODE, usually parametrised by a real number. A first order ODE by itself is therefore, in general, not a well-posed problem in the sense of Section 1.1 because it does not have a unique solution. In the elementary examples of ODEs in Exercise

1.1 and also in the ODE (1.4) we can obtain a well-posed problem if we impose an initial condition on the solution. If we demand that a solution of (1.4) satisfies $x(0) = 1$ we obtain the unique answer $x(t) = \dfrac{1}{1-t^2}$. If, on the other hand, we demand that a solution satisfies $x(0) = 0$ then the constant function $x(t) = 0$ for all t is the only possibility. The combination of a first order ODE with a specification of the value of the solution at one point is called an *initial value problem*.

The method of solving a first order ODE by separating variables works, at least in principle, for any equation of the form

$$\frac{dx}{dt} = f(t)g(x),$$

where f and g are continuous functions. The solution $x(t)$ is determined implicitly by

$$\int \frac{1}{g(x)} dx = \int f(t) dt. \tag{1.6}$$

In general, it may be impossible to express the integrals in terms of elementary functions and to solve explicitly for x.

1.2.4 Linear first order differential equations

If the function f in equation (1.3) is a sum of terms which are either independent of x or linear in x, we call the equation linear. Consider the following example of an initial value problem for a linear first order ODE:

$$\frac{dx}{dt} + 2tx = t \qquad x(0) = 1. \tag{1.7}$$

First order linear equations can always be solved by using an *integrating factor*. In the above example, we multiply both sides of the differential equation by $\exp(t^2)$ to obtain

$$e^{t^2}\frac{dx}{dt} + 2te^{t^2}x = te^{t^2}. \tag{1.8}$$

Now the left hand side has become a derivative, and the equation (1.8) can be written as

$$\frac{d}{dt}\left(e^{t^2}x\right) = te^{t^2}.$$

Integrating once yields $xe^{t^2} = \frac{1}{2}e^{t^2} + c$, and hence the general solution

$$x(t) = \frac{1}{2} + ce^{-t^2}.$$

Imposing $x(0) = 1$ implies $c = \frac{1}{2}$ so that the solution of (1.7) is

$$x(t) = \frac{1}{2}\left(1 + e^{-t^2}\right).$$

General linear equations of the form

$$\frac{dx}{dt} + a(t)x = b(t), \tag{1.9}$$

where a and b are continuous functions of t, can be solved using the integrating factor

$$I(t) = e^{\int a(t)dt}. \tag{1.10}$$

Since the indefinite integral $\int a(t)dt$ is only determined up to an additive constant, the integrating factor is only determined up to a multiplicative constant: if $I(t)$ is an integrating factor, so is $C \cdot I(t)$ for any non-zero real number C. In practice, we make a convenient choice. In the example above we had $a(t) = 2t$ and we picked $I(t) = \exp(t^2)$. Multiplying the general linear equation (1.9) by $I(t)$ we obtain

$$\frac{d}{dt}\left(I(t)x(t)\right) = I(t)b(t).$$

Now we integrate and solve for $x(t)$ to find the general solution. In Section 2.5 we will revisit this method as a special case of the method of variation of the parameters.

1.2.5 Exact equations

Depending on the context, the independent variable in an ODE is often called t (particularly when it physically represents time), sometimes x (for example when it is a spatial coordinate) and sometimes another letter of the Roman or Greek alphabet. It is best not to get too attached to any particular convention. In the following example, the independent real variable is called x, and the real-valued function that we are looking for is called y. The differential equation governing y as a function of x is

$$(x + \cos y)\frac{dy}{dx} + y = 0. \tag{1.11}$$

Re-arranging this as

$$(x\frac{dy}{dx} + y) + \cos y\frac{dy}{dx} = 0,$$

we note that

$$(x\frac{dy}{dx} + y) = \frac{d}{dx}(xy), \quad \text{and} \quad \cos y\frac{dy}{dx} = \frac{d}{dx}\sin y.$$

If we define $\psi(x, y) = xy + \sin y$ then (1.11) can be written

$$\frac{d}{dx}\psi(x, y(x)) = 0 \tag{1.12}$$

and is thus solved by

$$\psi(x, y(x)) = c, \tag{1.13}$$

for some constant c.

Equations which can be written in the form (1.12) for some function ψ are called *exact*. It is possible to determine whether a general equation of the form

$$a(x, y)\frac{dy}{dx}(x) + b(x, y) = 0, \tag{1.14}$$

for differentiable functions $a, b : \mathbb{R}^2 \to \mathbb{R}$, is exact as follows. Suppose equation (1.14) were exact. Then we should be able to write it in the form (1.12) for a twice-differentiable function $\psi : \mathbb{R}^2 \to \mathbb{R}$. However, by the chain rule,

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial\psi}{\partial y}\frac{dy}{dx}(x) + \frac{\partial\psi}{\partial x},$$

so that (1.14) is exact if there exists a twice-differentiable function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with

$$\frac{\partial\psi}{\partial y}(x, y) = a(x, y) \quad \text{and} \quad \frac{\partial\psi}{\partial x}(x, y) = b(x, y).$$

Since $\dfrac{\partial^2\psi}{\partial x\partial y} = \dfrac{\partial^2\psi}{\partial y\partial x}$ for twice-differentiable functions we obtain a necessary condition for the existence of the function ψ:

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}. \tag{1.15}$$

The equation (1.11) is exact because $a(x,y) = x + \cos y$ and $b(x,y) = y$ satisfy (1.15):

$$\frac{\partial}{\partial x}(x + \cos y) = 1 = \frac{\partial y}{\partial y}.$$

To find the function ψ systematically, we need to solve

$$\frac{\partial \psi}{\partial y} = x + \cos y, \qquad \frac{\partial \psi}{\partial x} = y. \tag{1.16}$$

As always in solving simultaneous equations, we start with the easier of the two equations; in this case this is the second equation in (1.16), which we integrate with respect to x to find $\psi(x,y) = xy + f(y)$, where f is an unknown function. To determine it, we use the first equation in (1.16) to derive $f'(y) = \cos y$, which is solved by $f(y) = \sin y$. Thus

$$\psi(x,y) = xy + \sin y,$$

leading to the general solution given in (1.13).

It is sometimes possible to make a non-exact equation exact by multiplying with a suitable integrating factor. However, it is only possible to give a recipe for computing the integrating factor in the linear case. In general one has to rely on inspired guesswork.

1.2.6 Changing variables

Some ODEs can be simplified and solved by changing variables. We illustrate how this works by considering two important classes. **Homogeneous ODEs** are equations of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{1.17}$$

such as

$$\frac{dy}{dx} = \frac{2xy}{x^2 + y^2} = \frac{2(\frac{y}{x})}{1 + (\frac{y}{x})^2}. \tag{1.18}$$

If we define $u = \frac{y}{x}$, then $y = xu$ and thus

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Hence the equation (1.17) can be rewritten as

$$x\frac{du}{dx} = f(u) - u, \tag{1.19}$$

which is separable. In our example (1.18) we obtain the following equation for u:

$$x\frac{du}{dx} = \frac{2u}{1+u^2} - u = \frac{u(1-u^2)}{1+u^2}.$$

Provided $x \neq 0, u \neq 0$ and $u \neq \pm 1$ we can separate variables to obtain

$$\int \frac{1+u^2}{u(1-u)(1+u)} du = \int \frac{1}{x} dx.$$

Using partial fractions

$$\frac{1+u^2}{u(1-u)(1+u)} = \frac{1}{u} + \frac{1}{1-u} - \frac{1}{1+u}$$

we integrate to find

$$\ln\left|\frac{u}{1-u^2}\right| = \ln|x| + \tilde{c} \;\Leftrightarrow\; \frac{u}{1-u^2} = \pm cx, \tag{1.20}$$

where $c = \exp(\tilde{c})$ is a non-zero constant. By inspection we find that $u(x) = 0$ for all $x \in \mathbb{R}$ and $u(x) = \pm 1$ for all $x \in \mathbb{R}$ are also solutions. We can include the zero solution and absorb the sign ambiguity in (1.20) by allowing the constant c to take arbitrary real values. However, the solutions $u(x) = \pm 1$ have to be added to obtain the most general solution (compare again Exercise 1.8). The general solution of (1.19) is thus given by

$$y(x) = c(x^2 - y^2) \quad \text{or} \quad y(x) = x \quad \text{or} \quad y(x) = -x. \tag{1.21}$$

Note that these are well-defined for $x = 0$, and that they satisfy the original differential equation (1.18) there.

Bernoulli equations are of the form

$$\frac{dy}{dx} + a(x)y = b(x)y^{\alpha},$$

where α is a real number not equal to 1. They are named after Jacob Bernoulli (1654–1705) who also invented the method for solving separable equations. The 'difficult' term here is the right hand side, so we divide by y^α (making a mental note that y is assumed to be non-vanishing) to find

$$y^{-\alpha}\frac{dy}{dx} + a(x)y^{1-\alpha} = b(x). \tag{1.22}$$

The equation simplifies when written in terms of

$$u = y^{1-\alpha}. \tag{1.23}$$

Using

$$\frac{du}{dx} = (1-\alpha)y^{-\alpha}\frac{dy}{dx},$$

the equation (1.22) becomes

$$\frac{1}{1-\alpha}\frac{du}{dx} + a(x)u = b(x).$$

This is a linear ODE and can be solved by finding an integrating factor. For example the equation

$$\frac{dy}{dx} + y = y^4$$

turns into a linear equation for $u = y^{-3}$

$$\frac{du}{dx} - 3u = -3,$$

which we can solve using the integrating factor $\exp(-3x)$. After integrating, do not forget to find the function y by inverting the relation (1.23).

Exercise 1.2 Name the type and find the general solution for each of the following first order equations:

(i) $\displaystyle\frac{dy}{dx} = \frac{e^x}{3+6e^x}$, (ii) $\displaystyle\frac{dy}{dx} = \frac{x^2}{y}$,

(iii) $\displaystyle\frac{dy}{dx} + 3y = x + e^{-2x}$, (iv) $\displaystyle x\frac{dy}{dx} = x\cos(2x) - y$,

(v) $\displaystyle\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$, (vi) $\displaystyle xy^2 - x + (x^2y+y)\frac{dy}{dx} = 0$.

Exercise 1.3 Solve the following initial value problems.

$$\text{(i)}\ (\sin x + x^2 e^y - 1)\frac{dy}{dx} + y\cos x + 2xe^y = 0,\ \ y(0) = 0,$$

$$\text{(ii)}\ \frac{dx}{dt} + x = x^4,\ x(0) = 1,\ \text{(iii)}\ x\frac{dy}{dx} + y = x^4 y^3,\ y(1) = 1.$$

Exercise 1.4 Give equations for an ellipse, parabola and hyperbola in the (x, y)-plane and derive an exact differential equation for each.

Exercise 1.5 Consider the following mathematical model of epidemics. Assume that there is a community of N members with I infected and U uninfected individuals, so $U + I = N$. Define the ratios $x = I/N$ and $y = U/N$ and assume that N is constant and so large that x and y may be considered as continuous variables. Then we have $x, y \in [0, 1]$ and

$$x + y = 1. \tag{1.24}$$

Denoting time by t, the rate at which the disease spreads is $\dfrac{dx}{dt}$. If we make the assumption that the disease spreads by contact between sick and healthy members of the community, and if we further assume that both groups move freely among each other, we arrive at the differential equation

$$\frac{dx}{dt} = \beta xy, \tag{1.25}$$

where β is a real and positive constant of proportionality.

(i) Combine equations (1.24) and (1.25) to derive a differential equation for $x(t)$.
(ii) Find the solution of this differential equation for $x(0) = x_0$.
(iii) Show that $\lim_{t\to\infty} x(t) = 1$ if $x_0 > 0$ and interpret this result.
(iv) Is the model of epidemics studied here realistic? If not, what is missing in it?

Exercise 1.6 Two friends sit down to enjoy a cup of coffee. As soon as the coffee is poured, one of them adds cold milk to her coffee. The friends then chat without drinking any coffee. After five minutes the second friend also adds milk to her coffee, and both begin to drink. Which of them has got the hotter coffee? *Hint*: You may assume that the milk is colder than room temperature. If you do not know how to start this question look up Newton's law of cooling!

1.3 Existence and uniqueness of solutions

Most differential equations cannot be solved by the elementary methods described so far. There exist further techniques and tricks for finding explicit solutions of first order ODEs, but if you write down an example at random your chances of finding an explicit solution in terms of elementary functions are slim - even if you use all the tricks known to mathematicians. However, the mere absence of an explicit formula does not imply that no solution exists. Therefore, one would still like to have a general criterion to determine if a given ODE has a solution and also what additional conditions are needed to make sure that the solution is unique. Our experience so far suggests that solutions of first order ODEs are unique once we specify the value of the unknown function at one point. We could therefore ask when the general initial value problem

$$\frac{dx}{dt} = f(t,x), \qquad x(t_0) = x_0, \tag{1.26}$$

has a unique solution. An important theorem by Charles Emile Picard (1856–1941) and Ernst Lindelöf (1870–1946) says that, under fairly mild assumption on f, initial value problems have unique solutions, at least locally. Before we state the theorem we specify what we mean by a solution.

Definition 1.1 Let I be an interval and $t_0 \in I$. We say that a differentiable function $x : I \to \mathbb{R}$ is a solution of (1.26) in the interval I if $\dfrac{dx}{dt} = f(t,x)$ for all $t \in I$ and $x(t_0) = x_0$.

In this book we will use the following version of the Picard–Lindelöf Theorem:

Theorem 1.1 *Consider the intervals* $I_T = [t_0 - T, t_0 + T]$ *and* $B_d = [x_0 - d, x_0 + d]$ *for positive, real numbers* T, d. *Suppose that* $f : I_T \times B_d \to \mathbb{R}$ *is continuous and that its partial derivative*

$\dfrac{\partial f}{\partial x} : I \times B \to \mathbb{R}$ *is also continuous. Then there is a* $\delta > 0$ *so that the initial value problem* (1.26) *has a unique solution in the interval* $I_\delta = [t_0 - \delta, t_0 + \delta]$.

This version of the Picard–Lindelöf Theorem is useful in practice, but not the most general formulation. One can replace the requirement that the partial derivative $\frac{\partial f}{\partial x}$ exists and is continuous by the weaker condition that f satisfies a Lipschitz condition with respect to its second argument (see Definition 5.3 in the Project 5.5). The proof of the theorem proceeds via the re-formulation of the initial value problem (1.26) in terms of an integral equation, and the application of a contraction mapping theorem. We will not discuss the proof in the main part of this book, but in Project 5.5 you are guided through the main steps.

We would like to gain some understanding of how the Picard–Lindelöf Theorem works. To achieve this, we first sketch the various intervals and points mentioned in the theorem in the (t, x)-plane, leading to a picture like the one shown in Fig. 1.2. The diagonally shaded rectangle is the region $I_T \times B_d$ where the functions f and $\frac{\partial f}{\partial x}$ are continuous, i.e., where the conditions of the theorem are satisfied. The point (t_0, x_0) is contained in this region, and the graph of the solution we seek has to pass through this point (since $x(t_0) = x_0$). The smaller interval I_δ cuts a slice out of this rectangle, shown as the cross-hatched region. This slice is the region where we expect to have a unique solution of the initial value problem. The graph of such a solution is also sketched in the picture.

Now that we have a picture in mind, we are going to discuss the kind of questions that a mathematician might ask when confronted with the statement of an unfamiliar theorem. Is the meaning of each of the terms in the theorem clear? What happens when conditions of the theorem are violated? Can we give examples of initial value problems which do not satisfy the conditions of the Picard–Lindelöf Theorem and for which there are no or perhaps many solutions? These questions are addressed in the following remarks.

(i) Although the Picard–Lindelöf Theorem guarantees the existence of a solution for a large class of equations, it is not always

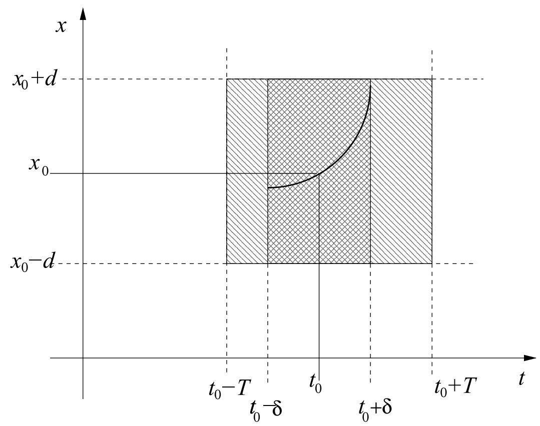

Fig. 1.2. A picture to illustrate the contents of the Picard–Lindelöf Theorem. Explanation in the main text

possible to find an explicit form of the solution in terms of elementary functions. For example

$$\frac{dx}{dt} = \sin(xt), \qquad x(0) = 1,$$

satisfies the condition of the theorem but no explicit formula is known for the solution.
(ii) If f is not continuous, then (1.26) may not have a solution. Consider, for example,

$$f(t,x) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases} \qquad \text{and} \quad x(0) = 0.$$

This would imply that $x(t) = t$ for $t \geq 0$ and $x(t) = 0$ for $t < 0$, which is not differentiable at $t = 0$. This is illustrated on the left in Fig. 1.3.
(iii) If f does not have a continuous first order partial derivative, (1.26) may have more than one solution. For example

$$\frac{dx}{dt} = x^{\frac{1}{3}}, \qquad x(0) = 0 \tag{1.27}$$

is solved by

$$x(t) = \begin{cases} (\frac{2}{3}(t-c))^{\frac{3}{2}} & \text{for} \quad t \geq c \\ 0 & \text{for} \quad t < c \end{cases}$$

for any $c \geq 0$. An example of such a solution is sketched in the middle in Fig. 1.3.
(iv) The Picard–Lindelöf Theorem guarantees a solution for t close to t_0, but this solution may not exist for all $t \in \mathbb{R}$. As we saw after (1.4), the initial value problem

$$\frac{dx}{dt} = 2tx^2, \qquad x(0) = 1$$

has the solution

$$x(t) = \frac{1}{1-t^2},$$

which tends to infinity as $t \to \pm 1$. Therefore, a solution in the sense of Definition 1.3 only exists in the interval $(-1, 1)$. This solution is shown on the right in Fig. 1.3. It is possible to determine the smallest interval in which a solution is guaranteed to exist - see the discussion in Section 5.5.
(v) Very straightforward-looking initial value problems may have no solution:

$$t\frac{dx}{dt} + x = t, \qquad x(0) = 1.$$

Setting $t = 0$ on both sides of the ODE we deduce that $x(0) = 0$, which is not consistent with the initial condition $x(0) = 1$. Why does the Picard–Lindelöf Theorem not apply?

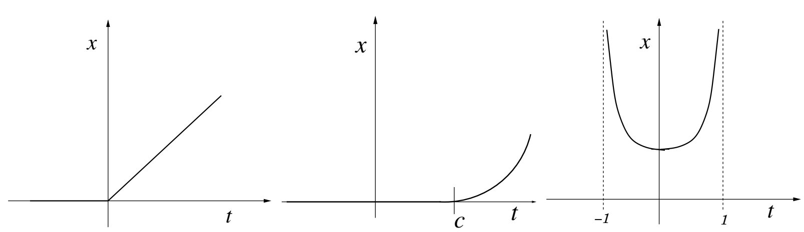

Fig. 1.3. Graphs illustrating the remarks (ii) (left), (iii) (middle) and (iv) (right) about the Picard–Lindelöf Theorem

Exercise 1.7 Show that each of the following initial value problems has two distinct solutions.
(i) $\frac{dx}{dt} = |x|^{1/2}$, $x(0) = 0$, (ii) $x\frac{dx}{dt} = t$, $x(0) = 0$.
Explain, in each case, why the hypotheses of the Picard–Lindelöf Theorem are not satisfied.

Exercise 1.8 Justify the method of separating variables as follows. Consider the first order differential equation

$$\frac{dx}{dt} = f(t)g(x), \tag{1.28}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions. Show that if $g(\eta) = 0$, then the constant function $x(t) = \eta$ for all $t \in \mathbb{R}$ is a solution of (1.28). Use the Picard–Lindelöf Theorem to deduce that any solution x of (1.28) which satisfies $g(x(t_0)) \neq 0$ at some point t_0 must satisfy $g(x(t)) \neq 0$ for all t for which x is defined. Explain in which sense this result justifies the method of separation of variables.

Exercise 1.9 Curve sketching revision: sketch graphs of the following functions:
(i) $f(x) = \frac{1}{2}x^2 - 3x + \frac{9}{2}$, (ii) $f(x) = \sqrt{x+2}$,
(iii) $f(x) = 3\cos(\pi x + 1)$, (iv) $f(x) = -e^{(x-1)}$,
(v) $f(x) = \ln(x+4)$ (vi) The solutions (1.21), for a range of values for c.

Exercise 1.10 Find solutions of each of the equations

$$\text{(i)}\ \frac{dx}{dt} = x^{\frac{1}{2}}, \quad \text{(ii)}\ \frac{dx}{dt} = x, \quad \text{(iii)}\ \frac{dx}{dt} = x^2,$$

satisfying the initial condition $x(0) = 1$, and sketch them. Comment on the way solutions of $\dfrac{dx}{dt} = x^\alpha$ grow in each of the cases (i) $0 < \alpha < 1$, (ii) $\alpha = 1$, (iii) $\alpha > 1$.

1.4 Geometric methods: direction fields

In this section we explain how to sketch the graph of solutions of first order ODEs *without* solving the ODE. The tool that allows one to do this is called the *direction field*. To understand how it works we first assume that we have a family of solutions and compute the associated direction field. Then we reverse the process and sketch the solutions directly from the direction field.

Consider the general form of an explicit first order ODE

$$\frac{dx}{dt}(t) = f(t, x(t)), \tag{1.29}$$

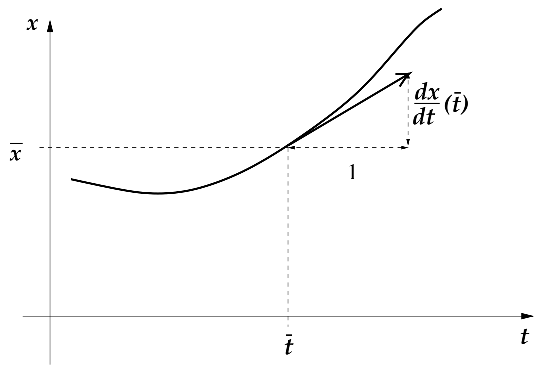

Fig. 1.4. Tangent to the graph of a function $x(t)$

where we have exhibited the independent variable t for the sake of the following discussion. Suppose we have a solution $x(t)$ of (1.29) and draw its graph in the (t, x)-plane, as shown in Fig. 1.4. Suppose that $x(\bar{t}) = \bar{x}$, i.e., the point $(\bar{t}, \bar{x})$ lies on the graph of $x(t)$. Then the slope of the graph at $(\bar{t}, \bar{x})$ is $\dfrac{dx}{dt}(\bar{t})$. Geometrically, this means that if we start at $(\bar{t}, \bar{x})$ and move one step in the t-direction we have to move $\dfrac{dx}{dt}(\bar{t})$ steps in the x-direction in order to stay on the tangent to the curve through $(\bar{t}, \bar{x})$. In other words, the direction of the tangent at (t, x) is given by the vector

$$\begin{pmatrix} 1 \\ \dfrac{dx}{dt}(\bar{t}) \end{pmatrix}.$$

Now note that (1.29) allows us to replace $\dfrac{dx}{dt}(\bar{t})$ by $f(\bar{t}, \bar{x})$. Thus we can write down the tangent vector to the solution curve - without knowing the solution! This motivates the definition of the *direction field* of the differential equations (1.29) as the map

$$\boldsymbol{V} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \boldsymbol{V}(t, x) = \begin{pmatrix} 1 \\ f(t, x) \end{pmatrix}. \tag{1.30}$$

It is a collection of all tangent vectors to graphs of all possible solutions of the original ODE (1.29). The direction field is an example of a *vector field*: a map which assigns a vector (in our case in $\mathbb{R}^2$) to every point in a set (in our case this happens to be $\mathbb{R}^2$, too).

The graph of any solution of (1.29) through a point (t, x) must have the tangent vector $\boldsymbol{V}(t, x)$ given in (1.30). To understand the general nature of all solutions it is therefore helpful to sketch the vector field $\boldsymbol{V}$, i.e., to pick a few points $(t_1, x_1), (t_2, x_2), \ldots$ and to draw the vector $\boldsymbol{V}(t_1, x_1)$ with base at (t_1, x_1), then the vector $\boldsymbol{V}(t_2, x_2)$ with base at (t_2, x_2) and so on. It is useful to think of the direction field as 'signpostings' for someone walking in the (t, x)-plane: by following the arrows at every point in the plane the walker's path ends up being the graph of a solution of the differential equation (1.29). This point of view also provides a very important way of thinking about a solution of an ODE: it is a global path which is determined by lots of local instructions.

Example 1.1 Sketch the direction field for the ODE

$$\frac{dx}{dt} = t^2 + x^2. \tag{1.31}$$

Hence sketch the solutions satisfying the initial conditions
(i) $x(-1) = -1$, (ii) $x(-1) = 0$.

In order to sketch the direction field we evaluate it at a couple of points, e.g.,

$$\boldsymbol{V}(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{V}(1,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{V}(0,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\boldsymbol{V}(1,1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \boldsymbol{V}(-1,1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{V}(2,0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

We draw a horizontal arrow representing $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ starting at the origin, arrows representing $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ starting at the points $(1, 0)$ and $(0, 1)$ in the (t, x)-plane, the arrow for $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ at $(1, 1)$ and so on, leading to the picture shown in Fig. 1.5.

From the formula for the direction field it is clear that vectors attached to all points on a circle $t^2 + x^2 =$ const. are the same, and that the arrows point up more steeply the further away we are

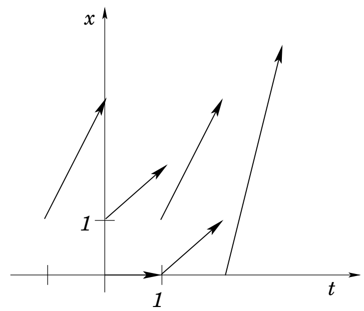

Fig. 1.5. Sketching the direction field for the differential equation (1.31)

from the origin. Generally, curves on which the direction field is constant are called *isoclines* and are helpful in sketching direction fields. Recalling that our interest in the direction field stems from the fact that it is tangent to any solution we notice that we can rescale the length of the vectors by a constant without affecting this property. It is therefore often better to make all the vectors shorter by a suitably chosen constant in order to avoid a crowded picture. This was done in producing Fig. 1.6, which also shows the required solution curves. □

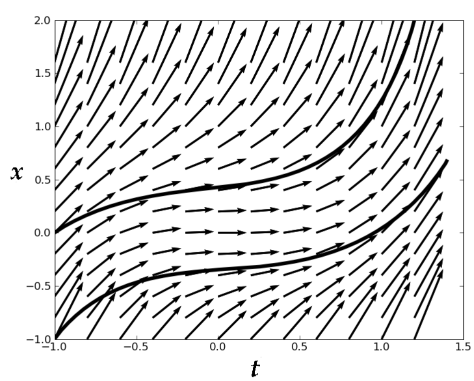

Fig. 1.6. Direction field and graphs of solutions for the differential equation (1.31)

It is not always possible to figure out the behaviour of all solutions from the direction field. However, the following rule is often very helpful.

Lemma 1.1 *If the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the conditions of Picard–Lindelöf Theorem in a region $U \in \mathbb{R}^2$, it is impossible for graphs of solutions of the differential equation* (1.29) *to cross in the region U.*

Proof If graphs of two solutions $x, \tilde{x}$ were to cross in a point (t_0, x_0), then the initial value problem (1.29) with $x(t_0) = x_0$ would have two solutions, which contradicts the uniqueness of the solution guaranteed by the Picard–Lindelöf Theorem. □

Example 1.2 Sketch the direction fields for the differential equation

$$\frac{dx}{dt} = x(x-1)(x-2). \tag{1.32}$$

Find all constant solutions and hence sketch the solutions with the initial conditions (i) $x(0) = 0.5$, (ii) $x(0) = 1.5$.

In this case the vectors representing (1.30) do not depend on t. They point down for $x < 0$ and $1 < x < 2$, up for $0 < x < 1$ and $x > 2$, and are horizontal for $x = 0, x = 1, x = 2$. In sketching, we again rescale by a suitable factor. Next we note that the equation has the constant solutions $x(t) = 0$ for all t, $x(t) = 1$ for all t, $x(t) = 2$ for all t. We thus draw horizontal lines at $x = 0$, $x = 1$ and $x = 2$. The graphs of other solutions must not cross these lines. Hence we obtain sketches like those shown in Fig. 1.7. □

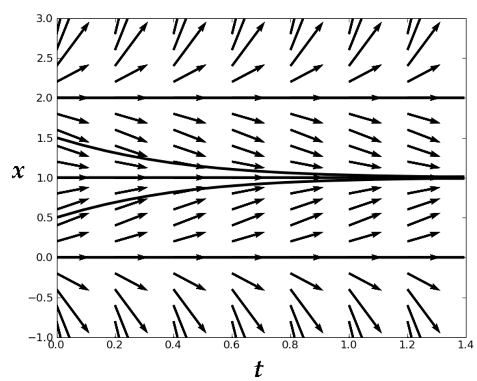

Fig. 1.7. Direction field and graphs of solutions for the differential equation (1.32)

Exercise 1.11 Find all constant solutions of $\dfrac{dx}{dt} + x^2 = 1$ and sketch the direction field for this equation. Hence sketch the solutions satisfying
(i) $x(0) = 1.1$, (ii) $x(0) = 0$, (iii) $x(0) = -1.1$.

Exercise 1.12 Consider the ODE $\dfrac{dx}{dt} = x^2 - t$. Determine the points in the (t, x)-plane at which

$$\text{(i)}\,\frac{dx}{dt} = 0, \ \text{(ii)}\,\frac{dx}{dt} > 0, \ \text{and (iii)}\,\frac{dx}{dt} < 0.$$

Hence sketch the direction field. Also sketch the solution satisfying $x(1) = 1$.

1.5 Remarks on numerical methods

At various points in this book you will be asked to use mathematical software for solving ODEs on a computer. When first studying ODEs it is not unreasonable to treat computer programmes for solving ODEs largely as a 'black box': you learn which input the programme requires, provide the input data and then take the programme's output on trust. This is the way most people use pocket calculators when evaluating $\sqrt{7}$, for example. However, as mathematicians we would like to have some general understanding of what is inside the 'black box' - just like we would have some idea of how to compute approximations to $\sqrt{7}$.

In this short section we develop a general notion of what is involved in solving an initial value problem numerically. If you use numerical ODE solvers extensively in your work you will need to consult other sources to learn more about the different algorithms for solving ODEs numerically. In particular, you will need to find out which algorithms are well adapted to solving the kind of ODEs you are interested in. However, it is worth stressing that numerical methods, even when carefully selected, do not solve all problems one encounters in studying ODEs. For example, computers are of limited use in solving ODEs with singularities or for understanding qualitative properties of a large class of solutions. Both of these points are illustrated in Project 5.4 at the end of this book.

Most methods for solving ODEs numerically are based on the geometrical way of thinking about initial value problems discussed after (1.29). We saw there that the initial value problem

$$\frac{dx}{dt} = f(t, x), \qquad x(t_0) = x_0,$$

can be thought of as a set of instructions for moving in the (t, x)-plane, starting from the point (t_0, x_0) specified by the initial condition. Numerical algorithms for initial value problems essentially construct the solution by following these instructions step by step. However, knowing a solution $x(t)$ even on a small interval I would require recording the value of $x(t)$ at infinitely many points $t \in I$. No computer can store infinitely many data points, so numerical algorithms for solving ODEs invariably require a discretisation of the problem: a finite number of points $t_i, i = 0, \ldots n$, in I is selected, and the algorithm will return values x_i which are good approximations to the values $x(t_i)$ of the true solution evaluated at the points t_i.

Suppose we want to know the values of a solution $x(t)$ of (1.29) at equally spaced points $t_0, t_1, \ldots, t_n$,with $t_{i+1} - t_i = h$, given that $x(t_0) = x_0$. Since the function f in the ODE (1.29) is given, we can compute $\frac{dx}{dt}(t_0) = f(t_0, x_0)$. Thus we have the first two terms in the Taylor expansion of the solution

$$x(t_1) = x(t_0 + h) = x(t_0) + hf(t_0, x_0) + \ldots,$$

where we neglected terms which are of quadratic or higher order in h. If h is sufficiently small, the number

$$x_1 = x_0 + hf(t_0, x_0)$$

is therefore a good approximation to $x(t_1)$. Then we compute an approximation x_2 to $x(t_2)$ by repeating the procedure, i.e.,

$$x_2 = x_1 + hf(t_1, x_1).$$

The iteration of this step is the *Euler method* for computing numerical approximations x_i to the values $x(t_i)$ of a solution of (1.29). The general formula of the Euler method is

$$x_{i+1} = x_i + hf(t_i, x_i), \quad i = 0, \ldots, n-1. \tag{1.33}$$

To illustrate this scheme, consider the simple initial value problem

$$\frac{dx}{dt} = x, \quad x(0) = 1.$$

We can solve this exactly (by many of the methods in Section 1.2, or by inspection) and have the exact solution $x(t) = e^t$. Let us apply the Euler method, with $t_0 = 0, t_n = 1$ and $h = 1/n$. Then (1.33) gives the recursion

$$x_{i+1} = \left(1 + \frac{1}{n}\right) x_i,$$

which we easily solve to obtain

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

as an approximation to the value of the exact solution at $t = 1$, i.e., to $x(1) = e$. Using the limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e,$$

we see that, in this case, the Euler method approximates the true solution evaluated at $t = 1$ arbitrarily well as $n \to \infty$ or, equivalently, as the stepsize $h = 1/n$ goes to zero.

In general, the Euler method is not the most effective way to obtain a good numerical approximation to solutions of ODEs. One way of improving the Euler method is based on the observation that the general form (1.26) of an initial value problem is equivalent to the equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{1.34}$$

This kind of equation is called an *integral equation* for x since it contains the unknown function x and an integral involving x. You are asked to prove the equivalence with (1.26) in Exercise 1.13. Suppose now that the function f satisfies the condition of the Picard–Lindelöf Theorem on $\mathbb{R} \times \mathbb{R}$, so that (1.26) and therefore also (1.34) has a unique solution. We would like to compute good approximations x_i to the values of that solution at points $t_i = t_0 + ih$, for some (small) increment h and $i = 1, \ldots, n$. From (1.34) we can deduce the (exact) recursion relation

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(s, x(s))\, ds. \tag{1.35}$$

The Euler recursion (1.33) is equivalent to approximating the integral in this formula via

$$\int_{t_i}^{t_{i+1}} f(s, x(s))\, ds \approx h f(t_i, x(t_i)),$$

which amounts to estimating the integral over each interval $[t_i, t_{i+1}]$ by the area of a rectangle of width h and height $f(t_i, x(t_i))$. A better approximation to the integral in (1.35) can be obtained by estimating the area in terms of the trapezium rule, which takes into account the value of the function $f(s, x(s))$ at both endpoints t_i and t_{i+1} of the interval. This would lead to the recursion

$$x_{i+1} = x_i + \frac{h}{2} \left(f(t_i, x_i) + f(t_{i+1}, x_{i+1}) \right).$$

The trouble with this formula is that x_{i+1} appears on both sides. To get around this we approximate x_{i+1} on the right hand side by the (old) Euler method (1.33), thus obtaining the recursion

$$x_{i+1} = x_i + \frac{h}{2} \left(f(t_i, x_i) + f(t_{i+1}, x_i + h f(t_i, x_i)) \right). \tag{1.36}$$

This scheme is called the improved Euler method or Heun's method.

Exercise 1.13 Prove the equivalence between the initial value problem (1.26) and the integral equation (1.34) on an interval $[t_0 - T, t_0 + T]$, for any $T > 0$.

Exercise 1.14 Solve the initial value problem $\frac{dx}{dt} = 0.5 - t + 2x,\ x(0) = 1$ numerically and compute approximations to the values of x at $t = 0.1$, $t = 0.5$ and $t = 1$ in three ways: (a) using the Euler method with $h = 0.05$, (b) using the improved Euler method with $h = 0.1$, (c) using the improved Euler method with $h = 0.05$. Also find the exact solution and compare the approximate with the exact values!