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Supplement to "Two Sample Inference for the Second-order Property of Temporally Dependent Functional Data"

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This supplement contains proofs of the results in Section 3

Let $\mathbb{H}_{HS}$ be the space of Hilbert-Schmidt operator which is a separable Hilbert space with respect to the inner product $< S_1, S_2 >_{HS} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < S_1(g_i), h_j >< S_2(g_i), h_j >$ for $S_1, S_2 \in \mathbb{H}_{HS}$, where $\{g_i\}$ and $\{h_i\}$ are two arbitrary orthonormal basis of $\mathbb{H}$. Throughout the appendix, we let c be a generic constant which varies from line to line. To maximize readability, we provide a list of notation that will be used in the appendix. Recall the following:
(1) $\hat{C}_{X,m}$ is the sample covariance operator based on $\{X_i(t)\}_{i=1}^m$, and $\hat{\phi}^j_{X,m}$ and $\hat{\lambda}^j_{X,m}$ are the jth eigenfunction and eigenvalue of $\hat{C}_{X,m}$; $\hat{C}_{XY}$ is the empirical covariance operator based on the pooled sample with $\{\hat{\phi}^j_{XY}\}$ and $\{\hat{\lambda}^j_{XY}\}$ being the corresponding eigenfunctions and eigenvalues; The population counterpart of $\hat{C}_{XY}$ is given by $\tilde{C}_{XY} = \gamma_1 C_X + \gamma_2 C_Y$ whose eigenvalues and eigenfunctions are denoted by $\{\tilde{\lambda}^j\}$ and $\{\tilde{\phi}^j\}$ respectively. The notation with a subscript Y is used to denote the quantities associated with the second sample.
(2) $\hat{\alpha}_k = \text{vech}(\mathbf{C}_k)$ with $\mathbf{C}_k = (c_k^{i,j})_{i,j=1}^K$; $\hat{\theta}_k^j = \hat{\lambda}^j_{X,\lfloor kN_1/N\rfloor} - \hat{\lambda}^j_{Y,\lfloor kN_2/N\rfloor}$ and $\hat{\theta}_k = (\hat{\theta}_k^1,\dots,\hat{\theta}_k^M)'$ with $\lfloor N\epsilon\rfloor \le k \le N$ for some $\epsilon > 0$; $\hat{\eta}_k^j = (< \hat{\phi}^j_{X,\lfloor kN_1/N\rfloor} - \hat{\phi}^j_{Y,\lfloor kN_2/N\rfloor}, \hat{\phi}^{j+1}_{XY} >, \dots, < \hat{\phi}^j_{X,\lfloor kN_1/N\rfloor} - \hat{\phi}^j_{Y,\lfloor kN_2/N\rfloor}, \hat{\phi}^p_{XY} >)$ with $1 \le j \le M$ and $\lfloor N\epsilon\rfloor \le k \le N$; $\hat{\eta}_k = (\hat{\eta}_k^1, \hat{\eta}_k^2, \dots, \hat{\eta}_k^M)' \in \mathbb{R}^{M_0}$ with $M_0 = \frac{M(2p-M-1)}{2}$.
(3) $\beta_{X_i,j} = \int_{\mathcal{I}}(X_i(t)-\mu_X(t))\phi_X^j(t)dt$, $\omega^{jk}_{X_i} = \beta_{X_i,j}\beta_{X_i,k}$, $r_X^{jk,j'k'}(h) = E[(\omega^{jk}_{X_i} - \delta_{jk}\lambda_j)(\omega^{j'k'}_{X_{i+h}} - \delta_{j'k'}\lambda_{j'})]$ and $v^{jk}_{X_i} = \omega^{jk}_{X_i} - E[\omega^{jk}_{X_i}] = \omega^{jk}_{X_i} - \delta_{jk}\lambda_j$.

Lemma 0.1. *Under Assumption 3.2, $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ are both L^2-m-approximable sequences in $\mathbb{H}_{HS}$.*

Proof. Let $\mathcal{X}_1^{(m)} = X_1^{(m)} \otimes X_1^{(m)}$, where $X_1^{(m)}$ is the m-dependent approximation of X_1.

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We have

$$||\mathcal{X}_1-\mathcal{X}_1^{(m)}||_{HS}=\left(\int_{\mathcal{I}}\int_{\mathcal{I}}(X_1(t)X_1(s)-X_1^{(m)}(t)X_1^{(m)}(s))^2dtds\right)^{1/2}$$
$$\leq\sqrt{2}||X_1-X_1^{(m)}||(||X_1||+||X_1^{(m)}||),$$

which implies that $(E||\mathcal{X}_1-\mathcal{X}_1^{(m)}||_{HS}^2)^{1/2}\leq c(E||X_1-X_1^{(m)}||^4)^{1/4}$. The same arguments apply to $\{\mathcal{Y}_i\}$. The conclusion follows by noting the fact that $\{X_i(t)\}$ and $\{Y_i(t)\}$ are both L^4-m-approximable. □

Lemma 0.2. *Suppose Assumptions 3.2-3.3 hold. Then we have* $\sup_N NE||\hat{C}_{XY}-\tilde{C}||_{HS}^2<c$ *and*

$$\limsup_{N\to+\infty} NE|\hat{\lambda}_{XY}^j-\tilde{\lambda}^j|^2<\infty,\quad \limsup_{N\to+\infty} NE||\hat{c}_{XY}^j\hat{\phi}_{XY}^j-\tilde{\phi}^j||^2<\infty,$$

for $1\leq j\leq K$, *where* c *is a finite constant that does not depend on* N, *and* $\hat{c}_{XY}^j=sign(\hat{\phi}_{XY}^j,\tilde{\phi}^j)$.

Proof. Note that $\{X_i(t)\}$ and $\{Y_i(t)\}$ are both L^4-m-approximable. The conclusion follows from Theorem 3.1 and Theorem 3.2 in Hörmann and Kokoszka (2010). □

Let $\tilde{R}_{X,k}=(<(\mathcal{X}_k-C_X)\tilde{\phi}^i,\tilde{\phi}^j>)_{1\leq i,j\leq K}$ and $\tilde{R}_{Y,k}=(<(\mathcal{Y}_k-C_Y)\tilde{\phi}^i,\tilde{\phi}^j>)_{1\leq i,j\leq K}$ with $1\leq k\leq N$. Define the empirical estimates $\hat{R}_{X,k}$ and $\hat{R}_{Y,k}$ by replacing $\tilde{\phi}^j$ with $\hat{c}_{XY}^j\hat{\phi}_{XY}^j$ in the definition of $\tilde{R}_{X,k}$ and $\tilde{R}_{Y,k}$. In Lemmas 0.3 and 0.4 below, we prove an invariance principle for the partial sum process of $\{\hat{R}_{X,k},\hat{R}_{Y,k}\}$ and show that the estimation effect caused by replacing $\tilde{\phi}^j$ with $\hat{c}_{XY}^j\hat{\phi}_{XY}^j$ is asymptotically negligible.

Lemma 0.3. *Under Assumption 3.1, we have*

$$\frac{1}{\sqrt{N}}\left(\frac{1}{\gamma_1}\sum_{k=1}^{\lfloor N_1r\rfloor}vech(\tilde{R}_{X,k})-\frac{1}{\gamma_2}\sum_{k=1}^{\lfloor N_2r\rfloor}vech(\tilde{R}_{Y,k})\right)\Rightarrow\Lambda_d^*B_d(r). \tag{0.1}$$

When $\gamma_1=\gamma_2$, *(0.1) holds under Assumption 3.2.*

Proof. Define $\tilde{R}_{X,k}^{(m)}$ and $\tilde{R}_{Y,k}^{(m)}$ by replacing X_i and Y_i with the $m-$dependent approximations $X_i^{(m)}$ and $Y_i^{(m)}$ in the definition of $\tilde{R}_{X,k}$ and $\tilde{R}_{Y,k}$ respectively. Consider the joint process $(\text{vech}(\tilde{R}_{X,k})',\text{vech}(\tilde{R}_{Y,k})')'$ and observe that

$$E\left|(\text{vech}(\tilde{R}_{X,k})',\text{vech}(\tilde{R}_{Y,k})')'-(\text{vech}(\tilde{R}_{X,k}^{(m)})',\text{vech}(\tilde{R}_{Y,k}^{(m)})')'\right|^2$$
$$\leq c\sum_{1\leq i\leq j\leq K}\left\{E(<(\mathcal{X}_1-\mathcal{X}_1^{(m)})\tilde{\phi}^i,\tilde{\phi}^j>)^2+E(<(\mathcal{Y}_1-\mathcal{Y}_1^{(m)})\tilde{\phi}^i,\tilde{\phi}^j>)^2\right\}$$
$$\leq c(E||\mathcal{X}_1-\mathcal{X}_1^{(m)}||_{HS}^2+E||\mathcal{Y}_1-\mathcal{Y}_1^{(m)}||_{HS}^2),$$

which implies that the joint process is L^2-m-approximable. By Theorem A.1 of Aue et al. (2009), we can establish the functional central limit theorem for the joint processes, i.e.,

$$\frac{1}{\sqrt{N_0}}\sum_{k=1}^{\lfloor N_0 r\rfloor}\begin{pmatrix}\mathrm{vech}(\tilde{R}_{X,k})\\ \mathrm{vech}(\tilde{R}_{Y,k})\end{pmatrix} \Rightarrow \begin{pmatrix}\Sigma_{11}^* & \mathbf{0}\\ \Sigma_{21}^* & \Sigma_{22}^*\end{pmatrix}\begin{pmatrix}B_d^{(1)}(r)\\ B_d^{(2)}(r)\end{pmatrix}, \quad N_0\to+\infty,$$

where $B_d^{(1)}(r)$ and $B_d^{(2)}(r)$ are two independent $d-$dimensional Brownian motions. It is easy to see the conclusion holds when $\{X_i(t)\}$ and $\{Y_i(t)\}$ are independent. If $\gamma_1=\gamma_2$, the continuous mapping theorem from $D^{2d}[0,1]$ to $D^d[0,1]$ yields that,

$$\begin{aligned}&\frac{1}{\sqrt{N}}\left(\frac{1}{\gamma_1}\sum_{k=1}^{\lfloor N_1 r\rfloor}\mathrm{vech}(\tilde{R}_{X,k})-\frac{1}{\gamma_2}\sum_{k=1}^{\lfloor N_2 r\rfloor}\mathrm{vech}(\tilde{R}_{Y,k})\right)\\ \Rightarrow&\frac{1}{\gamma_1}(\Sigma_{11}^*-\Sigma_{21}^*)B_d^{(1)}(\gamma_1 r)-\frac{1}{\gamma_1}\Sigma_{22}^*B_d^{(2)}(\gamma_1 r)\\ =^d&\Lambda_d^*B_d(r),\end{aligned}$$

where Λ_d^* is a lower triangular matrix such that $\Lambda_d^*(\Lambda_d^*)'=\frac{1}{\gamma_1}\{(\Sigma_{11}^*-\Sigma_{21}^*)(\Sigma_{11}^*-\Sigma_{21}^*)'+\Sigma_{22}^*(\Sigma_{22}^*)'\}$. □

Lemma 0.4. *Suppose Assumptions 3.2-3.3 hold, we have*

$$\frac{1}{\sqrt{N}}\sup_{r\in[0,1]}\left\{\left|\sum_{k=1}^{\lfloor N_1 r\rfloor}\{vech(\hat{R}_{X,k})-vech(\tilde{R}_{X,k})\}\right|+\left|\sum_{k=1}^{\lfloor N_2 r\rfloor}\{vech((\hat{R}_{Y,k})-vech(\tilde{R}_{Y,k})\}\right|\right\}=o_p(1). \tag{0.2}$$

Proof. Define $Z_{X,k}(t,s)=X_k(t)X_k(s)-C_X(t,s)$ and $Z_{X,k}^{(m)}(t,s)=X_k^{(m)}(t)X_k^{(m)}(s)-C_X(t,s)$. We aim to show that for each $1\le i,j\le K$,

$$\frac{1}{\sqrt{N}}\sup_{r\in[0,1]}\left|\sum_{k=1}^{\lfloor N_1 r\rfloor}\int_{\mathcal{I}}\int_{\mathcal{I}}Z_{X,k}(t,s)\left(\tilde{\phi}^i(t)\tilde{\phi}^j(s)-\hat{c}_{XY}^i\hat{c}_{XY}^j\hat{\phi}_{XY}^i(t)\hat{\phi}_{XY}^j(s)\right)dtds\right|=o_p(1).$$

From Lemma 0.2 and the proof of Theorem 3.4 in Zhang et al. (2011), the result follows from the fact that

$$\frac{1}{N^2}\max_{1\le k\le N_1}\left[\int_{\mathcal{I}}\int_{\mathcal{I}}\left\{\sum_{i=1}^{k}Z_{X,i}(t,s)\right\}^2dtds\right]=o_p(1). \tag{0.3}$$

Under Assumption 3.2, it is straightforward to show that $\{Z_{X,k}(t,s)\}$ is L^2-m-approximable. Let $g(t,s)=E|Z_{X,1}(t,s)|^2+2(E|Z_{X,1}(t,s)|^2)^{1/2}\sum_{r=1}^{+\infty}(E|Z_{X,r+1}(t,s)-Z_{X,r+1}^{(r)}(t,s)|^2)^{1/2}$.

By the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
&\int_{\mathcal{I}}\int_{\mathcal{I}} g(t,s)dtds\\
=&E||Z_{X,1}(t,s)||^2+2\sum_{r=1}^{+\infty}\int_{\mathcal{I}}\int_{\mathcal{I}}(E|Z_{X,1}(t,s)|^2)^{1/2}(E|Z_{X,r+1}(t,s)-Z_{X,r+1}^{(r)}(t,s)|^2)^{1/2}dtds\\
\leq&E||Z_{X,1}(t,s)||^2+2(E||Z_{X,1}(t,s)||^2)^{1/2}\sum_{r=1}^{+\infty}(E||Z_{X,r+1}(t,s)-Z_{X,r+1}^{(r)}(t,s)||^2)^{1/2}<\infty.
\end{aligned}
$$

Since $Z_{X,1}$ and $Z_{X,r+1}^{(r)}$ are independent, we have

$$
\begin{aligned}
&E\left(\sum_{i=1}^{N_1}Z_{X,i}(t,s)\right)^2=\sum_{|r|<N_1}(N_1-|r|)E[Z_{X,1}(t,s)Z_{X,r+1}(t,s)]\\
\leq&N_1E[Z_{X,1}(t,s)^2]+2\sum_{0<r<N_1}(N_1-|r|)E[Z_{X,1}(t,s)(Z_{X,r+1}(t,s)-Z_{X,r+1}^{(r)}(t,s))]\\
\leq&N_1E[Z_{X,1}(t,s)^2]+2N_1\sum_{r=1}^{+\infty}(E|Z_{X,1}(t,s)|^2)^{1/2}(E|Z_{X,r+1}(t,s)-Z_{X,r+1}^{(r)}(t,s)|^2)^{1/2}=N_1g(t,s).
\end{aligned}
$$

Hence by Menshov's inequality [see Billingsley (1999)], we obtain that

$$
E\max_{1\leq k\leq N_1}\left\{\sum_{i=1}^{k}Z_{X,i}(t,s)\right\}^2\leq(\log\log 4N_1)^2N_1g(t,s),
$$

which implies (0.3). The proof is completed by applying the same argument to the second sample. □

Proof of Theorem 3.1. Define α_t^* by replacing $c_k^{i,j}$ with $\frac{1}{\lfloor kN_1/N\rfloor}\sum_{l=1}^{\lfloor kN_1/N\rfloor}\hat{R}_{X,l}^{i,j}-\frac{1}{\lfloor kN_2/N\rfloor}\sum_{l=1}^{\lfloor kN_2/N\rfloor}\hat{R}_{Y,l}^{i,j}$ in $\hat{\alpha}_t$ where $\hat{R}_{X,l}^{i,j}$ and $\hat{R}_{Y,l}^{i,j}$ are the (i,j)th elements of $\hat{R}_{X,l}$ and $\hat{R}_{Y,l}$. Notice that the SN-based test statistic can be written as,

$$
\begin{aligned}
G_{SN,N}^{(1)}(d)=&N\hat{\alpha}_N'V_{SN,N}^{-1}(d)\hat{\alpha}_N\\
=&N(\alpha_N^*+\mathrm{vech}(\hat{\Delta}))'\left\{\frac{1}{N^2}\sum_{k=1}^{N}k^2(\alpha_k^*-\alpha_N^*)(\alpha_k^*-\alpha_N^*)'\right\}^{-1}(\alpha_N^*+\mathrm{vech}(\hat{\Delta})),
\end{aligned}
$$

where $\hat{\Delta}=(<(C_X-C_Y)\hat{c}_{XY}^i\hat{\phi}_{XY}^i,\hat{c}_{XY}^j\hat{\phi}_{XY}^j>)_{i,j=1}^K$ which is a matrix of zeros under the null. Using the fact that $||\hat{c}_{XY}^i\hat{\phi}_{XY}^i-\tilde{\phi}^i||=o_p(1)$, we have $\sqrt{N}\hat{\Delta}=L\Delta+o_p(L)$ under the local alternative. By Lemma 0.3, Lemma 0.4 and the continuous mapping theorem, we have $G_{SN,N}^{(1)}(d)\to^d W_d$ under $H_{1,0}$ and $\lim_{|L|\to+\infty}\lim_{N\to+\infty}G_{SN,N}^{(1)}(d)=+\infty$ under $H_{1,a}$. In view of Lemma 0.3, the conclusion also holds under Assumption 3.2 provided that $\gamma_1=\gamma_2$. □

Define the $m-$dimensional linear operator $\Pi_{X,m}x = \sum_{i=1}^{m} < \phi_X^j, x > \phi_X^j$. Consider the symmetric operator $\Pi_{X,m}\hat{C}_{X,k}\Pi_{X,m}$ as well as its eigenvalues and eigenfunctions denoted by $\hat{\tau}_{X,k}^{m,1} \geq \hat{\tau}_{X,k}^{m,2} \geq \cdots$, and $\hat{\gamma}_{X,k}^{m,1}(t)$, $\hat{\gamma}_{X,k}^{m,2}(t), \dots$, respectively. By the arguments in Benko et al. (2009), we have $\lim_{m\to\infty} \hat{\tau}_{X,k}^{m,j} = \hat{\lambda}_{X,k}^j$ and $\lim_{m\to\infty} ||\hat{\gamma}_{X,k}^{m,j} - \hat{\phi}_{X,k}^j|| = 0$, if $\hat{\lambda}_{X,k}^{j-1} > \hat{\lambda}_{X,k}^j > \hat{\lambda}_{X,k}^{j+1}$. Let $\hat{\Sigma}_{X,k}^m$ be the $m \times m$ matrix with (j,l)th entry being $\frac{1}{k}\sum_{i=1}^{k} w_{X_i}^{jl}$, where $\omega_{X_i}^{jl} = \beta_{X_i,j}\beta_{X_i,l}$. Suppose the eigenvalues and eigenvectors of $\hat{\Sigma}_{X,k}^m$ are denoted correspondingly by $\lambda^a(\hat{\Sigma}_{X,k}^m)$ and $\xi^a(\hat{\Sigma}_{X,k}^m)$ with $1 \leq a \leq m$. It is not hard to see that

$$\hat{\tau}_{X,k}^{m,a} = \lambda^a(\hat{\Sigma}_{X,k}^m), \quad \hat{\gamma}_{X,k}^{m,a}(t) = \xi^a(\hat{\Sigma}_{X,k}^m)'\Phi_X^m(t),$$

where $\Phi_X^m(t) = (\phi_X^1(t), \phi_X^2(t), \dots, \phi_X^m(t))$. By Lemma A.1 of Kneip and Utikal (2001), we obtain for every $1 \leq a \leq m$,

$$\hat{\tau}_{X,k}^{m,a} - \lambda_X^a = \frac{1}{k}\sum_{i=1}^{k}(\beta_{X_i,a}^2 - \lambda_X^a) + R_{X,k}^{m,a},$$

where

$$|R_{X,k}^{m,a}| \leq \frac{6\sup_{||e||=1} e'(\hat{\Sigma}_{X,k}^m - \Sigma_X^m)^2 e}{\min_{s\neq a}|\lambda_X^s - \lambda_X^a|}$$

and $\Sigma_X^m = \mathrm{diag}(\lambda_X^1, \dots, \lambda_X^m)$. The same arguments can be applied directly to the second sample. Let $\hat{\tau}_k^m = ((\hat{\tau}_{X,k}^m)', (\hat{\tau}_{Y,k}^m)')' = (\hat{\tau}_{X,k}^{m,1}, \dots, \hat{\tau}_{X,k}^{m,M}, \hat{\tau}_{Y,k}^{m,1}, \dots, \hat{\tau}_{Y,k}^{m,M})'$, $\lambda = (\lambda_X^{1:M}, \lambda_Y^{1:M})'$ and $U^i = ((U_X^i)', (U_Y^i)')' = (\beta_{X_i,1}^2 - \lambda_X^1, \dots, \beta_{X_i,M}^2 - \lambda_X^M, \beta_{Y_i,1}^2 - \lambda_Y^1, \dots, \beta_{Y_i,M}^2 - \lambda_Y^M)'$. Then we have

$$\hat{\tau}_k^m - \lambda = \frac{1}{k}\sum_{i=1}^{k}\begin{pmatrix} U_X^i \\ U_Y^i \end{pmatrix} + R_k^m, \tag{0.4}$$

with $R_k^m = ((R_{X,k}^m)', (R_{Y,k}^m)')' = (R_{X,k}^{m,1}, \dots, R_{X,k}^{m,M}, R_{Y,k}^{m,1}, \dots, R_{Y,k}^{m,M})'$.

Lemma 0.5. *Under Assumption 3.1 or Assumption 3.2 with $\gamma_1 = \gamma_2$, we have*

$$\frac{1}{\sqrt{N}}\left(\frac{1}{\gamma_1}\sum_{i=1}^{\lfloor N_1 r\rfloor} U_X^i - \frac{1}{\gamma_2}\sum_{i=1}^{\lfloor N_2 r\rfloor} U_Y^i\right) \Rightarrow^D \tilde{\Lambda}_M B_M(r). \tag{0.5}$$

Proof. Suppose $\beta_{X_i,k}^{(v)}$ is the principle component associated with the $v-$dependent approximation sequence $\{X_i^{(v)}(t)\}$. For every $1 \leq k \leq M$,

$$
\begin{aligned}
E\left|\beta^2_{X_i,k} - \left(\beta^{(v)}_{X_i,k}\right)^2\right|^2 =& E\left|\left(\int X_i(t)\phi^k_X(t)dt\right)^2 - \left(\int X_i^{(v)}(t)\phi^k_X(t)dt\right)^2\right|^2 \\
=& E\left|\left(\int (X_i(t) - X_i^{(v)}(t))\phi^k_X(t)dt\right)\left(\int (X_i(t) + X_i^{(v)}(t))\phi^k_X(t)dt\right)\right|^2 \\
\leq& E||X_1 - X_1^{(v)}||^2||X_1 + X_1^{(v)}||^2 \\
\leq& (E||X_1 - X_1^{(v)}||^4)^{1/2}(E||X_1 + X_1^{(v)}||^4)^{1/2} \\
\leq& c(E||X_1 - X_1^{(v)}||^4)^{1/2}.
\end{aligned}
$$

The same argument can be applied to $\{\beta^2_{Y_i,k}\}$ which implies that the process $\{(U^i_X, U^i_Y)\}$ is L^2-m-approximable and hence satisfies the functional central limit theorem. The rest of the proof is analogous to the proof of Lemma 0.3. □

Lemma 0.6. *Under Assumptions 3.2-3.4, we have*

$$
N^{-1}\sum_{l=1}^{N_1} \sup_{m\in\mathbb{N}} |lR^m_{X,l}|^2 = O_p(1), \tag{0.6}
$$

and $N^{1/2}\sup_{m\in\mathbb{N}}|R^m_{X,N_1}| = o_p(1)$ *as* $N \to \infty$. *The same conclusion also holds for the remainder term* $\{R^m_{Y,l}\}$.

Proof. Recall that $v^{jk}_{X_i} = \omega^{jk}_{X_i} - E[\omega^{jk}_{X_i}]$. Note first that for any $1 \leq a \leq M$ with M fixed and $1 \leq k \leq N$, we have

$$
\begin{aligned}
E\sup_{m\in\mathbb{N}}|lR^{m,a}_{X,l}|^2 \leq& l^2 E\sup_{m\in\mathbb{N}}\left\{\frac{6\sup_{||e||=1} e'(\hat{\Sigma}^m_{X,l} - \Sigma^m_X)^2 e}{\min_{s\neq a}|\lambda^s_X - \lambda^a_X|}\right\}^2 \\
\leq& cl^2 E\sup_{m\in\mathbb{N}}\left\{\sum_{j,k=1}^m\left(\frac{1}{l}\sum_{i=1}^l \beta_{X_i,j}\beta_{X_i,k} - \delta_{jk}\lambda^j_X\right)^2\right\}^2 \\
\leq& cl^2 E\left\{\sum_{j,k=1}^\infty\left(\frac{1}{l}\sum_{i=1}^l \beta_{X_i,j}\beta_{X_i,k} - \delta_{jk}\lambda^j_X\right)^2\right\}^2 \\
=& \frac{c}{l^2}\sum_{j,k}\sum_{j',k'}\sum_{i_1,i_2,i_3,i_4=1}^l E[v^{jk}_{X_{i_1}}v^{jk}_{X_{i_2}}v^{j'k'}_{X_{i_3}}v^{j'k'}_{X_{i_4}}],
\end{aligned}
$$

where we have used the inequality that $\sup_{||e||=1} e'B^2e \leq \mathrm{tr}(BB') = \sum_{j,k=1}^m b^2_{jk}$ for a symmetric matrix $B = (b_{jk})^m_{j,k=1}$. Using the fact that

$$
\begin{aligned}
&E[v^{jk}_{X_{i_1}}v^{jk}_{X_{i_2}}v^{j'k'}_{X_{i_3}}v^{j'k'}_{X_{i_4}}] = \mathrm{cum}(v^{jk}_{X_0}, v^{jk}_{X_{i_2-i_1}}, v^{j'k'}_{X_{i_3-i_1}}, v^{j'k'}_{X_{i_4-i_1}}) \\
&+ r^{jk}_X(i_2-i_1)r^{j'k'}_X(i_4-i_3) + r^{jk,j'k'}_X(i_3-i_1)r^{jk,j'k'}_X(i_4-i_2) + r^{jk,j'k'}_X(i_3-i_2)r^{jk,j'k'}_X(i_4-i_1),
\end{aligned}
$$

with $r_X^{jk}(h) = r_X^{jk,jk}(h)$, we obtain $\sup_{m\in\mathbb{N}} |lR_{X,l}^{m,a}|^2 \le c(I_1 + I_2 + I_3 + I_4)$. Under Assumption 3.4, we have

$$|I_1| \le \frac{1}{l^2} \sum_{j,k} \sum_{j',k'} \sum_{i_1=1}^{l} \sum_{i_2,i_3,i_4\in\mathbb{Z}} |\mathrm{cum}(v_{X_0}^{jk}, v_{X_{i_2}}^{jk}, v_{X_{i_3}}^{j'k'}, v_{X_{i_4}}^{j'k'})| < \frac{c}{l}.$$

Further, it is not hard to see that

$$|I_2| \le c \left(\sum_{j,k} \sum_{h=-\infty}^{+\infty} |r_X^{jk}(h)| \right)^2 < +\infty, \quad |I_s| \le c \sum_{j,k} \sum_{j',k'} \left(\sum_{h=-\infty}^{+\infty} |r_X^{jk,j'k'}(h)| \right)^2 < +\infty,$$

for $s = 3, 4$. Therefore, we get $N^{-1} \sum_{l=1}^{N_1} \sup_{m\in\mathbb{N}} |lR_{X,l}^{m}|^2 = O_p(1)$. Using similar arguments above, we have

$$E[N^{1/2} \sup_{m\in\mathbb{N}} |R_{X,N_1}^{m,a}|] \le cN^{1/2} E \sum_{j,k} \left(\frac{1}{N_1} \sum_{i=1}^{N_1} v_{X_i}^{jk} \right)^2 \le c \frac{N^{1/2}}{N_1} \sum_{j,k} \sum_{h=-\infty}^{+\infty} |r_X^{jk}(h)| \to 0.$$

The conclusion follows by using the same arguments for the second sample. □

Proof of Theorem 3.2. Let $\hat{\lambda}_k = (\hat{\lambda}'_{X,k}, \hat{\lambda}'_{Y,k})' = (\hat{\lambda}^1_{X,k}, \dots, \hat{\lambda}^M_{X,k}, \hat{\lambda}^1_{Y,k}, \dots, \hat{\lambda}^M_{Y,k})'$. Using the Bahadur representation of $\hat{\tau}^m_{\lfloor Nr \rfloor}$ with $r \in [\epsilon, 1]$, we have

$$\begin{aligned} \frac{\lfloor Nr \rfloor}{\sqrt{N}} (\hat{\lambda}_{\lfloor Nr \rfloor} - \lambda) =& \frac{\lfloor Nr \rfloor}{\sqrt{N}} (\hat{\tau}^m_{\lfloor Nr \rfloor} - \lambda) + \frac{\lfloor Nr \rfloor}{\sqrt{N}} (\hat{\lambda}_{\lfloor Nr \rfloor} - \hat{\tau}^m_{\lfloor Nr \rfloor}) \\ =& \frac{1}{\sqrt{N}} \sum_{i=1}^{\lfloor Nr \rfloor} \begin{pmatrix} U_X^i \\ U_Y^i \end{pmatrix} + \frac{\lfloor Nr \rfloor}{\sqrt{N}} R^m_{\lfloor Nr \rfloor} + \frac{\lfloor Nr \rfloor}{\sqrt{N}} (\hat{\lambda}_{\lfloor Nr \rfloor} - \hat{\tau}^m_{\lfloor Nr \rfloor}). \end{aligned} \tag{0.7}$$

By Lemma 3.2 in Hörmann and Kokoszka (2010), we know that

$$\sup_{\lfloor N\epsilon \rfloor \le k \le N} |\hat{\lambda}^j_{X,k} - \lambda^j_X| \le c \sup_{\lfloor N\epsilon \rfloor \le k \le N} ||\hat{C}_{X,k} - C_X||_{HS},$$

for $1 \le j \le M$. Recall that $Z_{X,k}(t,s) = X_k(t)X_k(s) - C_X(t,s)$. By the arguments in the proof of Lemma 0.4, we can show that

$$\begin{aligned} E \sup_{\lfloor N\epsilon \rfloor \le k \le N} ||\hat{C}_{X,k} - C_X||^2_{HS} \le& E \max_{\lfloor N\epsilon \rfloor \le k \le N} \int_{\mathcal{I}} \int_{\mathcal{I}} \left\{ \frac{1}{k} \sum_{i=1}^{k} Z_{X,i}(t,s) \right\}^2 dtds \\ \le& \frac{1}{\epsilon^2 N^2} E \max_{1 \le k \le N} \int_{\mathcal{I}} \int_{\mathcal{I}} \left\{ \sum_{i=1}^{k} Z_{X,i}(t,s) \right\}^2 dtds = o(1). \end{aligned} \tag{0.8}$$

Hence we know that $\sup_{\lfloor N\epsilon\rfloor\le k\le N}|\hat{\lambda}^j_{X,k}-\lambda^j_X|=o_p(1)$, which implies that the event $E^j_X:=\{\hat{\lambda}^{j+1}_{X,k}>\hat{\lambda}^j_{X,k}>\hat{\lambda}^{j-1}_{X,k}$ for all $\lfloor N\epsilon\rfloor\le k\le N\}$ occurs with probability tending to 1. The same arguments apply to the second sample. Let $\Pi_{X,\infty}=\sum_{j=1}^{+\infty}<\phi^j_X,\cdot>\phi^j_X$. By the properties of Hilbert-Schmidt norm, we have

$$\begin{aligned}
&\sup_{r\in[\epsilon,1]}||\Pi_{X,m}\hat{C}_{X,\lfloor Nr\rfloor}\Pi_{X,m}-\hat{C}_{X,\lfloor Nr\rfloor}||_{HS}\\
&\le\sup_{r\in[\epsilon,1]}\left(||(\Pi_{X,m}-\Pi_{X,\infty})\hat{C}_{X,\lfloor Nr\rfloor}\Pi_{X,m}||_{HS}+||\hat{C}_{X,\lfloor Nr\rfloor}(\Pi_{X,m}-\Pi_{X,\infty})||_{HS}\right)\\
&\le 2\sup_{r\in[\epsilon,1]}||\hat{C}_{X,\lfloor Nr\rfloor}(\Pi_{X,m}-\Pi_{X,\infty})||_{HS}=2\left(\sup_{r\in[\epsilon,1]}||(\Pi_{X,m}-\Pi_{X,\infty})\hat{C}_{X,\lfloor Nr\rfloor}||^2_{HS}\right)^{1/2}\\
&=2\left(\sup_{r\in[\epsilon,1]}\sum_{j=1}^{+\infty}(\hat{\lambda}^j_{X,\lfloor Nr\rfloor})^2||(\Pi_{X,m}-\Pi_{X,\infty})\hat{\phi}^j_{X,\lfloor Nr\rfloor}||^2_{HS}\right)^{1/2}\\
&=2\left(\sup_{r\in[\epsilon,1]}\sum_{i=m+1}^{+\infty}\sum_{j=1}^{+\infty}(\hat{\lambda}^j_{X,\lfloor Nr\rfloor})^2<\phi^i_X,\hat{\phi}^j_{X,\lfloor Nr\rfloor}>^2\right)^{1/2},
\end{aligned}$$

From (0.8), we know $P(\sup_{\lfloor N\epsilon\rfloor\le k\le N}||\hat{C}_{X,k}||_{HS}\le 2||C_X||_{HS})\to 1$. On the event $\{\sup_{\lfloor N\epsilon\rfloor\le k\le N}||\hat{C}_{X,k}||_{HS}\le 2||C_X||_{HS}\}$, we get

$$\sup_{r\in[\epsilon,1]}\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}(\hat{\lambda}^j_{X,\lfloor Nr\rfloor})^2<\phi^i_X,\hat{\phi}^j_{X,\lfloor Nr\rfloor}>^2=\sup_{r\in[\epsilon,1]}\sum_{j=1}^{+\infty}(\hat{\lambda}^j_{X,\lfloor Nr\rfloor})^2\le 2||C_X||^2_{HS},$$

which implies that $\sup_{r\in[\epsilon,1]}||\Pi_{X,m}\hat{C}_{X,\lfloor Nr\rfloor}\Pi_{X,m}-\hat{C}_{X,\lfloor Nr\rfloor}||_{HS}\to 0$ as $m\to+\infty$ for any fixed N. Then on the event $\cap_{1\le j\le M}(E^j_X\cap E^j_Y)\cap\{\sup_{\lfloor N\epsilon\rfloor\le k\le N}||\hat{C}_{X,k}||_{HS}\le 2||C_X||_{HS}\}$, we have $\lim_{m\to+\infty}\sup_{r\in[\epsilon,1]}|\hat{\lambda}_{\lfloor Nr\rfloor}-\hat{\tau}^m_{\lfloor Nr\rfloor}|=0$. Recall that $\hat{\theta}_k=(\hat{\theta}^1_k,\dots,\hat{\theta}^M_k)'$ with $\hat{\theta}^j_k=\hat{\lambda}^j_{X,\lfloor kN_1/N\rfloor}-\hat{\lambda}^j_{Y,\lfloor kN_2/N\rfloor}$. Take $\limsup_{m\to+\infty}$ elementwise in (0.7), we get

$$\frac{\lfloor Nr\rfloor}{\sqrt{N}}(\hat{\lambda}_{\lfloor Nr\rfloor}-\lambda)=\frac{1}{\sqrt{N}}\sum_{i=1}^{\lfloor Nr\rfloor}\begin{pmatrix}U^i_X\\U^i_Y\end{pmatrix}+\frac{\lfloor Nr\rfloor}{\sqrt{N}}\limsup_{m\to\infty}R^m_{\lfloor Nr\rfloor},$$

which implies that

$$\begin{aligned}
\frac{k}{\sqrt{N}}\{\hat{\theta}_k-(\lambda^{1:M}_X-\lambda^{1:M}_Y)\}=&\frac{1}{\sqrt{N}}\left\{\frac{k}{\lfloor kN_1/N\rfloor}\sum_{i=1}^{\lfloor kN_1/N\rfloor}U^i_X-\frac{k}{\lfloor kN_2/N\rfloor}\sum_{i=1}^{\lfloor kN_2/N\rfloor}U^i_Y\right\}\\
&+\frac{k}{\sqrt{N}}\left(\limsup_{m\to\infty}R^m_{X,\lfloor kN_1/N\rfloor}-\limsup_{m\to\infty}R^m_{Y,\lfloor kN_2/N\rfloor}\right),
\end{aligned}$$

with $N\epsilon \le k \le N$. Notice that

$$\left|\limsup_{m\to\infty} R^m_{X,\lfloor kN_1/N\rfloor} - \limsup_{m\to\infty} R^m_{Y,\lfloor kN_2/N\rfloor}\right| \le \sup_m |R^m_{X,\lfloor kN_1/N\rfloor}| + \sup_m |R^m_{Y,\lfloor kN_2/N\rfloor}|.$$

By Lemma 0.5 and Lemma 0.6, the assumptions in Theorem 2.1 of Shao (2010) are satisfied. Thus by similar arguments, we get $G^{(2)}_{SN,N}(M) \to^d W_M(\epsilon)$ under $H_{2,0}$ and $\lim_{|L|\to+\infty}\lim_{N\to+\infty} G^{(2)}_{SN,N}(M) = +\infty$ under $H_{2,a}$. □

***Remark* 0.1.** Suppose $\{X_i(t)\}$ are on a finite dimensional space spanned by $\{\phi^1_X,\dots,\phi^{m_0}_X\}$. Then it is easy to see that $\hat{C}_{X,k} = \Pi_{X,m}\hat{C}_{X,k}\Pi_{X,m}$ for $2\le k\le N_1$ and $m \ge m_0$, which implies that $\hat{\lambda}_{\lfloor Nr\rfloor} = \hat{\tau}^m_{\lfloor Nr\rfloor}$ for $r\in(0,1]$. As seen from the proof of Theorem 3.2, trimming is not required in this case as the arguments go through without approximating the compact operator $\hat{C}_{X,k}$ with a sequence of finite rank operators.

Proof of Proposition 3.1. Note that $X_m(t) = \sum_{j=0}^{\infty} b_j\varepsilon_{m-j}(t) = \sum_{i=1}^{+\infty}\sqrt{\lambda_i}\left(\sum_{j=0}^{+\infty} b_j z_{i,m-j}\right)\phi_i(t)$, where $z_{k,j}$'s are independent standard normal random variables. Hence we get

$$w^{jk}_{X_m} = \sqrt{\lambda_j\lambda_k}\left(\sum_{i=0}^{\infty} b_i z_{j,m-i}\right)\left(\sum_{i=0}^{+\infty} b_i z_{k,m-i}\right) = \sqrt{\lambda_j\lambda_k}\sum_{i,i'=0}^{\infty} b_i b_{i'} z_{j,m-i}z_{k,m-i'},$$

which implies that $r^{jk,j'k'}_X(h) = \sqrt{\lambda_j\lambda_k\lambda_{j'}\lambda_{k'}}\sum_{i_1,i_2,i_3,i_4} b_{i_1}b_{i_2}b_{i_3}b_{i_4}\mathrm{cov}(z_{j,m-i_1}z_{k,m-i_2}, z_{j',m+h-i_3}z_{k',m+h-i_4})$. We then have

$$\begin{aligned}\sum_{j,k}\sum_{j',k'}\left(\sum_{h=-\infty}^{+\infty}|r^{jk,j'k'}_X(h)|\right)^2 \le& \sum_{j,k}\left\{2\lambda_j\lambda_k\sum_h\left(\sum_i b_ib_{i+h}\right)^2\right\}^2\\ =&4\left(\sum_j\lambda_j^2\right)^2\left(\sum_h \pi(h)^2\right)^2 < \infty,\end{aligned}$$

and $\sum_{j,k}\sum_{h=-\infty}^{+\infty}|r^{jk}_X(h)| \le \left(\sum_{j,k}\lambda_j\lambda_k + \sum_j\lambda_j^2\right)\sum_h\left(\sum_i b_ib_{i+h}\right)^2 < \infty$, where we have used the fact $\mathrm{cov}(X_1X_2,X_3X_4) = \mathrm{cov}(X_1,X_3)\mathrm{cov}(X_2,X_4) + \mathrm{cov}(X_1,X_4)\mathrm{cov}(X_2,X_3)$ for 4-variate normal random variable (X_1,X_2,X_3,X_4). Notice that

$$\mathrm{cum}(v^{jk}_{X_0}, v^{jk}_{X_{i_1}}, v^{j'k'}_{X_{i_2}}, v^{j'k'}_{X_{i_3}}) = \lambda_j\lambda_k\lambda_{j'}\lambda_{k'}\sum_{l_1,\dots,l_8=0}^{+\infty}\prod_{t=1}^{8} b_{l_t}$$

$$\mathrm{cum}\big(z_{j,m-l_1}z_{k,m-l_2}, z_{j,m+i_1-l_3}z_{k,m+i_1-l_4}, z_{j',m+i_2-l_5}z_{k',m+i_2-l_6}, z_{j',m+i_3-l_7}z_{k',m+i_3-l_8}\big).$$

We may decompose the cumulant using equation (2.3.7) in Brillinger (1975). Some typical terms are the following:

$$\mathrm{cov}(z_{j,m-l_1}, z_{k,m+i_1-l_4})\mathrm{cov}(z_{k,m-l_2}, z_{j',m+i_2-l_5})\mathrm{cov}(z_{j,m+i_1-l_3}, z_{k',m+i_3-l_8})\mathrm{cov}(z_{k',m+i_2-l_6}, z_{j',m+i_3-l_7}),$$
$$\mathrm{cov}(z_{j,m-l_1}, z_{k,m+i_1-l_4})\mathrm{cov}(z_{j,m+i_1-l_3}, z_{k',m+i_2-l_6})\mathrm{cov}(z_{j',m+i_2-l_5}, z_{k',m+i_3-l_8})\mathrm{cov}(z_{j',m+i_3-l_7}, z_{k,m-l_2}).$$

It is not hard to see that each term is bounded provided that

$$\sum_{i_1,i_2,i_3} |\pi(i_1)\pi(i_2)\pi(i_3-i_1)\pi(i_3-i_2)| \le c \sum_{i_1,i_2,i_3} |\pi(i_1)\pi(i_2)\pi(i_3-i_1)| \le c \left(\sum_h |\pi(h)|\right)^3 < \infty.$$

□

To prove Theorem 3.3, we begin with the Bahadur representation of the recursive estimates of the eigenfunctions. By Lemma A of Kneip and Utikal (2001), we get

$$\xi^a(\hat{\Sigma}^m_{X,k}) - e^{m,a} = -S^{m,a}_X(\hat{\Sigma}^m_{X,k} - \Sigma^m_X)e^{m,a} + \tilde{R}^{m,a}_{X,k},$$

where

$$|\tilde{R}^{m,a}_{X,k}| \le \frac{6\sup_{||e||=1} e'(\hat{\Sigma}^m_{X,k} - \Sigma^m_X)^2 e}{\min_{s\neq a} |\lambda^s_X - \lambda^a_X|^2},$$

$S^{m,a}_X = \sum_{s\neq a} \frac{1}{\lambda^s_X - \lambda^a_X} e^{m,s}(e^{m,s})'$, and $e^{m,1}, e^{m,2}, \ldots, e^{m,m}$ are $m-$dimensional unit vectors. It follows that

$$\begin{aligned}\hat{\gamma}^{m,a}_{X,k}(t) - \phi^a_X(t) =& (\xi^a(\hat{\Sigma}^m_{X,k}) - e^{m,a})'\Phi^m_X(t) \\ =& -\sum_{s\neq a} \frac{1}{\lambda^s_X - \lambda^a_X} \left\{\frac{1}{k}\sum_{i=1}^k \beta_{X_i,s}\beta_{X_i,a}\right\} \phi^s_X(t) + \Phi^m_X(t)'\tilde{R}^{m,a}_{X,k}.\end{aligned}$$

Let $\nu_i = (\tilde{\phi}^{i+1}, \ldots, \tilde{\phi}^p)$ and further define $\tilde{\eta}^a_{X,k} = (<\hat{\phi}^a_{X,k}, \tilde{\phi}^{a+1}>, \ldots, <\hat{\phi}^a_{X,k}, \tilde{\phi}^p>)'$, $\tilde{\Gamma}^{m,a}_{X,k} = (<\hat{\gamma}^{m,a}_{X,k}, \tilde{\phi}^{a+1}>, \ldots, <\hat{\gamma}^{m,a}_{X,k}, \tilde{\phi}^p>)'$, and $\tilde{\Psi}^{s,a}_X = (<\phi^s_X, \tilde{\phi}^{a+1}>, \ldots, <\phi^s_X, \tilde{\phi}^p>)'$ with $\tilde{\Psi}^{a,a}_X = \tilde{\Psi}^a_X$. We thus get

$$\tilde{\Gamma}^{m,a}_{X,k} - \tilde{\Psi}^a_X = -\sum_{1\le s\neq a\le m} \frac{1}{\lambda^s_X - \lambda^a_X} \left\{\frac{1}{k}\sum_{i=1}^k \beta_{X_i,s}\beta_{X_i,a}\right\} \tilde{\Psi}^{s,a}_X + \tilde{\Psi}_X(m,a)\tilde{R}^{m,a}_{X,k},$$

where $\tilde{\Psi}_X(m,a) = (\tilde{\Psi}^{1,a}_X, \ldots, \tilde{\Psi}^{m,a}_X) \in \mathbb{R}^{(p-a)\times m}$. Note that on the event $\cap_{1\le j\le M}(E^j_X \cap E^j_Y)$, we have

$$|\tilde{\eta}^a_{X,k} - \tilde{\Gamma}^{m,a}_{X,k}| \le c||\hat{\phi}^a_{X,k} - \hat{\gamma}^{m,a}_{X,k}|| \to 0, \quad \text{as } m\to\infty,$$

for $N\epsilon \le k \le N$. Then we have

$$\tilde{\eta}^a_{X,k} - \tilde{\Psi}^a_X = \frac{1}{k}\sum_{i=1}^k \left\{-\sum_{1\le s\neq a<\infty} \frac{\tilde{\Psi}^{s,a}_X}{\lambda^s_X - \lambda^a_X} w^{sa}_{X_i}\right\} + \limsup_{m\to\infty} \tilde{\Psi}_X(m,a)\tilde{R}^{m,a}_{X,k}.$$

Under Assumption 3.4, it is not hard to show that $N^{-1}\sum_{l=1}^{N_1} \sup_{m\in\mathbb{N}} |l\tilde{\Psi}_X(m,a)\tilde{R}^{m,a}_{X,l}|^2 = O_p(1)$ and $N^{1/2}\sup_{m\in\mathbb{N}} |\tilde{\Psi}_X(m,a)\tilde{R}^{m,a}_{X,N_1}| = o_p(1)$ for $1 \le a \le M$ by using similar arguments in the proof of Lemma 0.6. Defining analogous quantities for the second sample with the subscript Y, we can obtain the same result for the second sample.

Lemma 0.7. *Define* $\tilde{h}^a_{X,i} = -\sum_{s\neq a}\left(\frac{\tilde{\Psi}^{s,a}_X}{\lambda^s_X-\lambda^a_X}w^{sa}_{X_i}\right)$ *and* $\tilde{h}^a_{Y,i} = -\sum_{s\neq a}\left(\frac{\tilde{\Psi}^{s,a}_Y}{\lambda^s_Y-\lambda^a_Y}w^{sa}_{Y_i}\right)$. *Let* $\tilde{h}_i = (\tilde{h}'_{X,i}, \tilde{h}'_{Y,i})' = (\tilde{h}^1_{X,i}, \dots, \tilde{h}^M_{X,i}, \tilde{h}^1_{Y,i}, \dots, \tilde{h}^M_{Y,i})'$. *Suppose Assumptions 3.1 holds. Then under the null, we have*

$$\frac{1}{\sqrt{N}}\left\{\frac{1}{\gamma_1}\sum_{i=1}^{\lfloor N_1 r\rfloor}\tilde{h}_{X,i} - \frac{1}{\gamma_2}\sum_{i=1}^{\lfloor N_2 r\rfloor}\tilde{h}_{Y,i}\right\} \Rightarrow \bar{\Lambda}_{M_0}B_{M_0}(r). \tag{0.9}$$

Under Assumption 3.5 and the local alternative, (0.9) also holds. We have the same conclusion with Assumption 3.1 replaced by Assumption 3.2 and $\gamma_1 = \gamma_2$.

Proof of Lemma 0.7. We only prove the result under condition (3.5) as the derivation under (3.6) is straightforward (see Remark 0.2). Under condition (3.5), we have for $a+1 \leq j \leq p$,

$$\begin{aligned}
&\left\{E\left|\sum_{1\leq s\neq a<\infty}\frac{<\phi^s_X, \tilde{\phi}^j>}{\lambda^s_X-\lambda^a_X}(\beta_{X_1,s}\beta_{X_1,a} - \beta^{(v)}_{X_1,s}\beta^{(v)}_{X_1,a})\right|^2\right\}^{1/2}\\
\leq & c\sum_{1\leq s\neq a<\infty}|<\phi^s_X, \tilde{\phi}^j>|\left\{E\left(\beta_{X_1,s}\beta_{X_1,a} - \beta^{(v)}_{X_1,s}\beta^{(v)}_{X_1,a}\right)^2\right\}^{1/2}\\
\leq & c\sum_{1\leq s\neq a<\infty}|<\phi^s_X, \tilde{\phi}^j>|\left\{E\left(\beta_{X_1,s} - \beta^{(v)}_{X_1,s}\right)^2(\beta_{X_1,a})^2 + E\left(\beta_{X_1,a} - \beta^{(v)}_{X_1,a}\right)^2(\beta^{(v)}_{X_1,s})^2\right\}^{1/2}\\
\leq & c\sum_{1\leq s<\infty}\left\{E\left(\beta_{X_1,s} - \beta^{(v)}_{X_1,s}\right)^4\right\}^{1/4} + c\left(E||X_1 - X^{(v)}_1||^4\right)^{1/4}\sum_{1\leq s<\infty}\left(E\beta^4_{X_1,s}\right)^{1/4},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{v=1}^{\infty}\left\{E\left|\sum_{1\leq s\neq a<\infty}\frac{<\phi^s_X, \nu^j_a>}{\lambda^s_X-\lambda^a_X}(\beta_{X_1,s}\beta_{X_1,a} - \beta^{(v)}_{X_1,s}\beta^{(v)}_{X_1,a})\right|^2\right\}^{1/2}\\
\leq & c\sum_{v=1}^{\infty}\sum_{s=1}^{\infty}\left\{E\left(\beta_{X_1,s} - \beta^{(v)}_{X_1,s}\right)^4\right\}^{1/4} + c\sum_{s=1}^{\infty}\left(E\beta^4_{X_1,s}\right)^{1/4} < \infty.
\end{aligned}$$

Applying the same arguments to the second sample, we show that the sequence $\{\tilde{h}_k\}$ is L^2-m-approximable. The conclusion follows from the arguments in the proof of Lemma 0.3. □

***Remark* 0.2.** Notice that under the assumption $\phi^j_X = \phi^j_X = \tilde{\phi}^j$ for $2 \leq j \leq p$, $\tilde{h}^a_{X,i} = -\left(\frac{w^{a+1,a}_{X_i}}{\lambda^{a+1}_X-\lambda^a_X}, \dots, \frac{w^{p,a}_{X_i}}{\lambda^p_X-\lambda^a_X}\right)'$ and the result in Lemma 0.7 can be established without the summability condition (3.5). In general, if we have $\sum_{s=1}^{+\infty} | < \phi_X^s, \tilde{\phi}^j > | < +\infty$ and $\sum_{s=1}^{+\infty} | < \phi_Y^s, \tilde{\phi}^j > | < +\infty$, for $2 \le j \le p$, then condition (3.5) can be dropped by noting that

$$\left\{ E\left(\beta_{X_1,s}\beta_{X_1,a} - \beta_{X_1,s}^{(v)}\beta_{X_1,a}^{(v)}\right)^2 \right\}^{1/2} \le c\left(E||X_1 - X_1^{(v)}||^4\right)^{1/4}.$$

Proof of Theorem 3.3 and Proposition 3.2. Define $\hat{\Psi}_X^{s,a}$ and $\hat{h}_k$ by replacing ν_i with $\hat{\nu}_i$ in $\tilde{\Psi}_X^{s,a}$ and $\tilde{h}_k$. We shall first show that

$$\frac{1}{\sqrt{N}} \sup_{r\in[0,1]} \left\{ \left| \sum_{i=1}^{\lfloor N_1 r\rfloor} (\hat{h}_{X,i} - \tilde{h}_{X,i}) \right| + \left| \sum_{i=1}^{\lfloor N_2 r\rfloor} (\hat{h}_{Y,i} - \tilde{h}_{Y,i}) \right| \right\} = o_p(1), \tag{0.10}$$

which suggests that the estimation effect caused by using $\hat{\phi}_{XY}^i$ instead of $\tilde{\phi}^i$ is asymptotically negligible. Using the fact that $||\hat{\phi}_{XY}^j - \tilde{\phi}^j|| = O_p(1/\sqrt{N})$ and the Cauchy-Schwarz inequality, we have

$$\frac{1}{\sqrt{N_1}} \sup_{r\in[0,1]} \left| \sum_{i=1}^{\lfloor N_1 r\rfloor} \sum_{s\neq a} \left(\frac{\tilde{\Psi}_X^{s,a} - \hat{\Psi}_X^{s,a}}{\lambda_X^s - \lambda_X^a} w_{X_i}^{sa} \right) \right| \le \frac{c\sum_{j=a+1}^{p} ||\hat{\phi}_{XY}^j - \tilde{\phi}^j||}{\sqrt{N_1}} \left(\sum_{s\neq a} \sup_{r\in[0,1]} \left| \sum_{i=1}^{\lfloor N_1 r\rfloor} w_{X_i}^{sa} \right|^2 \right)^{1/2}. \tag{0.11}$$

Under Assumptions 3.4, we can show that $E\left(\sum_{i=1}^{N_1} w_{X_i}^{sa}\right)^2 \le N_1 \tilde{g}_s$ with $\sum_{s=1}^{+\infty} \tilde{g}_s < +\infty$. Note that

$$\sum_{s\neq a} E \max_{1\le k\le N_1} \left| \sum_{i=1}^{k} w_{X_i}^{sa} \right|^2 \le (\log\log 4N_1)^2 N_1 \sum_{s\neq a} \tilde{g}_s \le c(\log\log 4N_1)^2 N_1.$$

Then it is not hard to see the the RHS of (0.11) is of order $o_p(1)$. The same arguments for the second sample imply (0.10). The rest of the proof follows similar arguments in the proofs of Theorem 3.2 and Lemma 0.7. We omit the details. □

Proof of Proposition 3.3. Note first that the assumptions in Proposition 3.1 are satisfied provided that $\sum_{j=1}^{\infty} \sqrt{\lambda_j} < \infty$ and $\sum_{m=1}^{\infty} (\sum_{j=m}^{\infty} b_j^2)^{1/2} < \infty$. Because $\beta_{X_1,j}^{(m)} = \sqrt{\lambda_j}\left(\sum_{l=0}^{m-1} b_l z_{j,1-l} + \sum_{l=m}^{+\infty} b_l z_{j,1-l}^{(1)}\right)$, where $\{z_{j,k}^{(1)}\}$ is a sequence of independent copies of $\{z_{j,k}\}$, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty} \left\{ E\left(\beta_{X_1,j} - \beta_{X_1,j}^{(m)}\right)^4 \right\}^{1/4} &= \sum_{j=1}^{+\infty} \sqrt{\lambda_j} \sum_{m=1}^{+\infty} \left(E\left\{ \sum_{l=m}^{+\infty} b_l (z_{j,1-l} - z_{j,1-l}^{(1)}) \right\}^4 \right)^{1/4} \\ &\le \sum_{j=1}^{+\infty} \sqrt{\lambda_j} \sum_{m=1}^{+\infty} \left(\sum_{l=m}^{+\infty} b_l^2 \right)^{1/2} < +\infty.\end{aligned}$$

Using similar arguments, we can show that $\sum_{j=1}^{\infty}\left(E\beta_{X_1,j}^4\right)^{1/4} < \infty$. □

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