

THE UNIVERSITY OF ILLINOIS

Department of Statistics

STATISTICS 510

Mathematical Statistics I

Fall 2016 (Homework 1: Due Aug 31th 2016, in class)

1. Suppose \underline{X} is a 3×1 random vector with $\text{cov}(\underline{X}) = \sigma^2 \mathbf{I}_3$. Find the matrix A so that

$$A\underline{X} = \begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ X_3 - \bar{X} \end{pmatrix},$$

the vector of deviations. Here $\bar{X} = (X_1 + X_2 + X_3)/3$. The covariance matrix of the deviations is

$$\text{Cov}(A\underline{X}) = \sigma^2 B$$

What is the matrix B ?

2. The double exponential random variable X has space \mathbb{R} and pdf $f(x) = (1/2)e^{-|x|}$.
- (a) Show that the mgf (moment generate function) of a double exponential is $M_X(t) = 1/(1 - t^2)$. [Break the integral into two parts according to the sign of x .] For which t is it finite?
- (b) Suppose U and V are independent Exponential(1), and let $Y = U - V$. Find the mgf of Y . What is the distribution of Y ?
- (c) The mean of a double exponential is 0 and the variance is 2. Suppose X_1, \dots, X_n are independent and identically distributed double exponentials, and let W_n be the standardized mean, $W_n = \frac{\bar{X}}{\sqrt{2/n}}$. Show that the mgf of W_n is $M_n(t) = \frac{1}{(1-t^2/(2n))^n}$.
- (d) What is the limit of $M_n(t)$ as $n \rightarrow \infty$? The limit is the mgf for what distribution?
3. Suppose $Z \sim N(0, 1)$ and $U \sim \text{Uniform}(0, 1)$, and Z and U are independent. Let $\underline{Y} = (Y_1, Y_2) = g(Z, U)$ be given by

$$Y_1 = \frac{Z}{U} \text{ and } Y_2 = U.$$

Y_1 is said to have the “slash” distribution.

- (a) What is the space of \underline{Y} ?
- (b) Find $g^{-1}(\underline{y})$.
- (c) Find the pdf of \underline{Y} ?

(d) Are Y_1 and Y_2 independent? Why or why not?

(e) Show that the marginal pdf of Y_1 is

$$f_1(y_1) = c \frac{1 - e^{-y_1^2/2}}{y_1^2}$$

What is the constant c ?

4. Suppose $X \sim \text{Beta}(\alpha, \beta)$. Find (a) $\mathbb{E}(X(1 - X))$; (b) $\mathbb{E}(X^a(1 - X)^b)$ for nonnegative integers a and b .