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SUPPLEMENT TO "FIXED-SMOOTHING ASYMPTOTICS FOR TIME SERIES"*

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1. Proofs of the main results.

1.1. *Proof of the main results in Section 2.1.* Consider the $K+1$ dimensional multivariate normal density function which takes the form

$$f(y,\Sigma) = (2\pi)^{-\frac{K+1}{2}}|\Sigma|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}y'\Sigma^{-1}y\right).$$

We assume the (i,j)th element and the (j,i)th element of Σ are functionally unrelated. The results can be extended to the case where symmetric matrix elements are considered functionally equal (see e.g., McCulloch [5]). In the following, we use $\otimes$ to denote the Kronecker product in matrix algebra and use vec to denote the operator that transforms a matrix into a column vector by stacking the columns of the matrix one underneath the other. For a vector $y \in \mathbb{R}^{l\times 1}$ whose elements are differential functions of a vector $x \in \mathbb{R}^{k\times 1}$, we define $\frac{\partial y}{\partial x}$ to be a $k \times l$ matrix with the (i,j)th element being $\frac{\partial y_j}{\partial x_i}$. The notation $u \asymp v$ represents $u = O(v)$ and $v = O(u)$. Lemma 1.1 and Lemma 1.2 below are straightforward consequences of matrix calculus (see e.g., Vetter [9], Brewer [1] and Turkington [8]).

LEMMA 1.1.

$$\frac{\partial f}{\partial vec(\Sigma)}(y,\Sigma) = \frac{f(y,\Sigma)}{2}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - vec(\Sigma^{-1})\}.$$

PROOF. By matrix calculus, we get

$$\begin{aligned}
\frac{\partial f}{\partial \mathrm{vec}(\Sigma)}(y,\Sigma) =&(2\pi)^{-\frac{K+1}{2}}\left\{\exp\left(-\frac{1}{2}y'\Sigma^{-1}y\right)\frac{\partial|\Sigma|^{-\frac{1}{2}}}{\partial \mathrm{vec}(\Sigma)} + |\Sigma|^{-\frac{1}{2}}\frac{\partial}{\partial \mathrm{vec}(\Sigma)}\exp\left(-\frac{1}{2}y'\Sigma^{-1}y\right)\right\} \\
=&(2\pi)^{-\frac{K+1}{2}}\left\{-\frac{1}{2}|\Sigma|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}y'\Sigma^{-1}y\right)\mathrm{vec}(\Sigma^{-1})\right. \\
&\left.+\frac{1}{2}|\Sigma|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}y'\Sigma^{-1}y\right)(\Sigma^{-1}y)\otimes(\Sigma^{-1}y)\right\} \\
=&\frac{f(y,\Sigma)}{2}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - \mathrm{vec}(\Sigma^{-1})\},
\end{aligned}$$

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where we have used the formulas $\frac{\partial a' X^{-1} b}{\partial \mathrm{vec}(X)} = -X^{-1}b \otimes (X^{-1})'a$ and $\frac{\partial |X|^m}{\partial \mathrm{vec}(X)} = m|X|^{m-1}\frac{\partial |X|}{\partial \mathrm{vec}(X)} = m|X|^m \mathrm{vec}((X^{-1})')$ (see Theorem 4.3 and Theorem 4.19 in Turkington [8]). ♢

Lemma 1.2.

$$
\begin{aligned}
&\frac{\partial^2 f}{\partial vec(\Sigma) vec(\Sigma)}(y,\Sigma)\\
=&\frac{1}{4}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - vec(\Sigma^{-1})\}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - vec(\Sigma^{-1})\}' f(y,\Sigma)\\
&-\frac{1}{2}\{(\Sigma^{-1}yy'\Sigma^{-1})\otimes\Sigma^{-1} + \Sigma^{-1}\otimes(\Sigma^{-1}yy'\Sigma^{-1}) - \Sigma^{-1}\otimes\Sigma^{-1}\} f(y,\Sigma).
\end{aligned}
$$

Proof. From Lemma 1.1, we have

$$
\begin{aligned}
\frac{\partial^2 f}{\partial \mathrm{vec}(\Sigma)\mathrm{vec}(\Sigma)}(y,\Sigma) =&\frac{\partial}{\partial \mathrm{vec}(\Sigma)}\left(\frac{f(y,\Sigma)}{2}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - \mathrm{vec}(\Sigma^{-1})\}\right)\\
=&\left(\frac{\partial}{\partial \mathrm{vec}(\Sigma)}\frac{f(y,\Sigma)}{2}\right)\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - \mathrm{vec}(\Sigma^{-1})\}'\\
&+\frac{f(y,\Sigma)}{2}\frac{\partial}{\partial \mathrm{vec}(\Sigma)}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - \mathrm{vec}(\Sigma^{-1})\} = I_1 + I_2.
\end{aligned}
$$

Again from Lemma 1.1, it is not hard to see that

$$
I_1 = \frac{f(y,\Sigma)}{4}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - \mathrm{vec}(\Sigma^{-1})\}\{(\Sigma^{-1}y)\otimes(\Sigma^{-1}y) - \mathrm{vec}(\Sigma^{-1})\}'.
$$

In view of Lemma 4.3 in Turkington [8], we have

$$
\frac{\partial \mathrm{vec}(\Sigma^{-1}yy'\Sigma^{-1})}{\partial \mathrm{vec}(\Sigma)} = \frac{\partial \mathrm{vec}(\Sigma^{-1}y)}{\partial \mathrm{vec}(\Sigma)}(y'\Sigma^{-1}\otimes I_{K+1}) + \frac{\partial \mathrm{vec}(y'\Sigma^{-1})}{\partial \mathrm{vec}(\Sigma)}(I_{K+1}\otimes y'\Sigma^{-1}).
$$

Also by Theorem 4.3 in Turkington [8], we get

$$
\frac{\partial \mathrm{vec}(\Sigma^{-1}y)}{\partial \mathrm{vec}(\Sigma)} = -\Sigma^{-1}y\otimes\Sigma^{-1}; \quad \frac{\partial \mathrm{vec}(y'\Sigma^{-1})}{\partial \mathrm{vec}(\Sigma)} = -\Sigma^{-1}\otimes\Sigma^{-1}y,
$$

which implies that

$$
\frac{\partial \mathrm{vec}(\Sigma^{-1}yy'\Sigma^{-1})}{\partial \mathrm{vec}(\Sigma)} = -(\Sigma^{-1}yy'\Sigma^{-1})\otimes\Sigma^{-1} - \Sigma^{-1}\otimes(\Sigma^{-1}yy'\Sigma^{-1}).
$$

Further by Theorem 4.2 in Turkington [8], we obtain $\frac{\partial \mathrm{vec}(\Sigma^{-1})}{\partial \mathrm{vec}(\Sigma)} = -\Sigma^{-1}\otimes\Sigma^{-1}$. The conclusion thus follows directly from the above derivation. ♢

Lemma 1.3. *Let $\{\Sigma_T\}\subset\mathbb{R}^{(K+1)\times(K+1)}$ be a sequence of positive definite matrices with $K+1\le T$. If K is fixed with respect to T and $||\Sigma_T-\Sigma||_2 = O(1/T)$ for a positive definite matrix Σ, then we have*

$$
||\Sigma_T^{-1} - \Sigma^{-1}||_2 = O(1/T).
$$

PROOF. Let $\Sigma_T = \Sigma + R_T$ with $||R_T||_2 = O(1/T)$. For sufficiently large T, we have $||\Sigma^{-1}R_T||_2 \leq ||\Sigma^{-1}||_2||R_T||_2 < 1$. By the last equation at p. 355 of Horn and Johnson [4], we have

$$||\Sigma_T^{-1} - \Sigma^{-1}||_2 \leq \frac{||\Sigma^{-1}||_2^2||R_T||_2}{1 - ||\Sigma^{-1}R_T||_2} = O(1/T).$$

$\diamondsuit$

LEMMA 1.4. *Let $\tilde{\Sigma}_T(y)$ be a $(K+1)\times(K+1)$ positive symmetric matrix which depends on $y \in \mathbb{R}^{K+1}$. Assume that $\sup_{y\in\mathbb{R}^{K+1}} ||\tilde{\Sigma}_T(y) - \Sigma||_2 \leq ||\Sigma_T - \Sigma||_2 = O(1/T)$ for a positive definite matrix Σ. Let $R_T = \Sigma_T - \Sigma$. If K is fixed with respect to T, we have*

$$\int_{y\in\mathbb{R}^{K+1}} \left| vec(R_T)' \frac{\partial^2 f}{\partial vec(\Sigma) vec(\Sigma)}(y, \tilde{\Sigma}_T(y)) vec(R_T)\right| dy = O(1/T^2).$$

PROOF. Let $\tilde{R}_T(y) = \tilde{\Sigma}_T(y) - \Sigma$. Note that

$$\sup_{y\in\mathbb{R}^{K+1}} ||\Sigma^{-1}\tilde{R}_T(y)||_2 \leq ||\Sigma^{-1}||_2 \sup_{y\in\mathbb{R}^{K+1}} ||\tilde{R}_T(y)||_2 \leq ||\Sigma^{-1}||_2||\Sigma_T - \Sigma||_2 < 1,$$

for large enough T. By the same arguments in Lemma 1.3, we have $\sup_{y\in\mathbb{R}^{K+1}} ||\tilde{\Sigma}_T^{-1}(y) - \Sigma^{-1}||_2 = O(1/T)$. Therefore, when T is sufficiently large, we have $y'(\tilde{\Sigma}_T^{-1}(y) - \Sigma^{-1}/2)y = y'(\tilde{\Sigma}_T^{-1}(y) - \Sigma^{-1})y + y'\Sigma^{-1}y/2 \geq (\lambda_{\min}(\Sigma^{-1})/2 - ||\tilde{\Sigma}_T^{-1}(y) - \Sigma^{-1}||_2)||y||^2 \geq 0$ for all y, where $\lambda_{\min}(\Sigma^{-1})$ denotes the smallest eigenvalue of Σ^{-1}. On the other hand, for sufficiently large T, we have $\sup_{y\in\mathbb{R}^{K+1}} |\tilde{\Sigma}_T(y)|^{-1} = \sup_{y\in\mathbb{R}^{K+1}} |\tilde{\Sigma}_T^{-1}(y)| \leq \sup_{y\in\mathbb{R}^{K+1}} ||\tilde{\Sigma}_T^{-1}(y)||_2^{K+1} \leq (||\Sigma^{-1}||_2 + \sup_{y\in\mathbb{R}^{K+1}} ||\tilde{\Sigma}_T^{-1}(y) - \Sigma^{-1}||_2)^{K+1} \leq C|\Sigma|^{-1}$ with $C > 0$. Combining the above arguments, we get $f(y, \tilde{\Sigma}_T(y)) \leq C|\Sigma|^{-1/2}\exp(-y'\Sigma^{-1}y/4) \leq Cf(y, 2\Sigma)$ for all y. When K is fixed, $||\cdot||_2$ and $||\cdot||_\infty$ are equivalent, which implies $\sup_{y\in\mathbb{R}^{K+1}} ||\tilde{\Sigma}_T(y)^{-1} - \Sigma^{-1}||_\infty = O(1/T)$. Since the elements of $\tilde{\Sigma}_T^{-1}(y)$ are uniformly bounded for all y, in view of Lemma 1.2, it is straightforward to see

$$\left|\text{vec}(R_T)' \frac{\partial^2 f}{\partial \text{vec}(\Sigma)\text{vec}(\Sigma)}(y, \tilde{\Sigma}_T(y))\text{vec}(R_T)\right| \leq Cp(y)f(y, 2\Sigma)/T^2,$$

where $p(y)$ is a polynomial of degree 4. The conclusion follows by noting that $\int p(y)f(y, 2\Sigma)dy < \infty$. $\diamondsuit$

PROOF OF PROPOSITION 2.1. Note first that

$$\Upsilon(x; K)/K = -KP(|t_{K-1}| \leq x) + \frac{K+1}{K} E\left[\chi^2_{K-1} G_1\left(\frac{\chi^2_{K-1}}{K-1}x^2\right)\right] + O(1/K).$$

Using the fact that $P(|t_{K-1}| \le x) = G_1(x^2) + \frac{1}{K-1}x^4 G_1''(x^2) + O(1/K^2)$, we get

$$\begin{aligned}\Upsilon(x;K)/K =& -KG_1(x^2) - \frac{K}{K-1}x^4 G_1''(x^2) + \frac{K+1}{K}E\Bigg[\chi^2_{K-1}\Bigg\{G_1(x^2) \\ & \left(\frac{\chi^2_{K-1}}{K-1}-1\right)x^2G_1'(x^2) + \frac{1}{2}\left(\frac{\chi^2_{K-1}}{K-1}-1\right)^2 x^4 G_1''(x^2)\Bigg\}\Bigg] + O(1/K) \\ =& 2x^2G_1'(x^2) + O(1/K).\end{aligned}$$

$\diamondsuit$

Proof of Proposition 2.2. Recall that $q = T/K$ is assumed to be an integer. Using the notation in the proof of Theorem 2.1, let $S_Y^2 = \frac{1}{K-1}\sum_{i=1}^K (Y_i - \bar{Y})^2 = \frac{1}{K-1}\{\sum_{i=1}^K Y_i^2 - K(\bar{Y})^2\}$. Notice that

$$\begin{aligned}\operatorname{cov}(Y) =& \begin{pmatrix} \sigma^2 - B/q & B/(2q) & 0 & \dots & 0 \\ B/(2q) & \sigma^2 - B/q & B/(2q) & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & B/(2q) & \sigma^2 - B/q \end{pmatrix}_{K\times K} \\ & + O(1/q^2) l_K l_K' \\ =& \sigma^2 I_K + \frac{B}{2q}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 1 & -2 \end{pmatrix}_{K\times K} + O(1/q^2) l_K l_K' \\ =& \sigma^2 I_K + \frac{B}{2q}M + O(1/q^2) l_K l_K',\end{aligned}$$

where $l_K' = (1,1,\dots,1)_{1\times K}$ and the summation of all the $O(1/q^2)$ is of order $O(K/q^2)$. Because

$$E[Y_i^2] = \sum_{h=1-q}^{q-1}\left(\frac{q-|h|}{q}\right)\gamma_X(h) = \sigma^2 - B/q + O(1/q^2),$$

and

$$\begin{aligned}E[\bar{Y}^2] &= \frac{1}{K^2}\sum_{i,j=1}^K E[Y_iY_j] = \frac{1}{K^2}\{K\sigma^2 - B/q + O(K/q^2)\} \\ &= \sigma^2/K + O(1/(K^2q)) + O(1/(Kq^2)),\end{aligned}$$

we obtain

$$E[S_Y^2] - \sigma^2 = \frac{K}{K-1}\{\sigma^2 - B/q - \sigma^2/K + o(1/T)\} - \sigma^2 = -B/q + O(1/T).$$

Consider the covariance matrix of $\tilde{Y}' = (Y_1 - \bar{Y}, Y_2 - \bar{Y}, \dots, Y_K - \bar{Y})$. It is easy to see that $\tilde{Y} = (I_K - l_K l_K'/K)Y = H_K Y$, where $H_K = I_K - l_K l_K'/K$ is an idempotent matrix. Ignoring the $O(1/q^2)$ order term in $\mathrm{cov}(Y)$, we have

$$\begin{aligned}(c_{ij})_{i,j=1}^K := \mathrm{cov}(\tilde{Y}) =& H_K \mathrm{cov}(Y) H_K \approx H_K\{\sigma^2 I_K + BM/(2q)\}H_K \\ =& \sigma^2 H_K + \frac{B}{2q} H_K M H_K = \sigma^2 H_K + \frac{B}{2q}\left(M - \frac{1}{K}A - \frac{2}{K^2} l_K l_K'\right),\end{aligned}$$

where

$$A = \begin{pmatrix} -2 & -1 & -1 & \dots & -2 \\ -1 & 0 & 0 & \dots & -1 \\ & & \cdots & & \\ -1 & 0 & 0 & \dots & -1 \\ -2 & -1 & -1 & \dots & -2 \end{pmatrix}_{K\times K}.$$

Since $\tilde{Y}$ is Gaussian, we get

$$E[S_Y^4] = \frac{1}{(K-1)^2}\sum_{i,j=1}^K E[(Y_i - \bar{Y})^2 (Y_j - \bar{Y})^2] = \frac{1}{(K-1)^2}\sum_{i,j=1}^K (c_{ii}c_{jj} + 2c_{ij}^2),$$

where $c_{ii} = \left(1 - \frac{1}{K}\right)\sigma^2 - \frac{B}{q} + O(1/T)$ and $c_{ij} = -\frac{1}{K}\sigma^2 + \frac{B}{2q}\mathbf{I}\{|i-j| = 1\} + O(1/T)$, for $i \neq j$. It implies that

$$\begin{aligned}\sum_{i,j=1}^K c_{ij}^2 =& \sum_{i=1}^K c_{ii}^2 + \sum_{|i-j|=1} c_{ij}^2 + \sum_{|i-j|>1} c_{ij}^2 = K\left(1 - \frac{1}{K}\right)^2 \sigma^4 + \frac{KB^2}{q^2} - \frac{2(K-1)B}{q}\sigma^2 \\ &+ 2(K-1)\left(\frac{\sigma^4}{K^2} + \frac{B^2}{4q^2} - \frac{\sigma^2 B}{Kq}\right) + \frac{(K-1)(K-2)}{K^2}\sigma^4 + O(1/q) \\ =& (K-1)\sigma^4 + O(K/q),\end{aligned}$$

and

$$\sum_{i,j=1}^K c_{ii}c_{jj} = K^2 c_{11}^2 + O(K/q) = (K-1)^2\sigma^4 - \frac{2BK(K-1)\sigma^2}{q} + O(K/q).$$

Therefore we get

$$E[S_Y^4] = \frac{K+1}{K-1}\sigma^4 - \frac{2BK\sigma^2}{(K-1)q} + O(1/T),$$

which implies

$$\mathrm{var}(S_Y^2) = \frac{K+1}{K-1}\sigma^4 - \frac{2BK\sigma^2}{(K-1)q} - (\sigma^2 - B/q)^2 + O(1/T) = \frac{2\sigma^4}{K-1} + O(1/T).$$

Let $\mathbf{X} = (X_1, X_2, \dots, X_T)'$, $\hat{\mu}_{GLS} = (l_T'\mathrm{cov}(\mathbf{X})^{-1} l_T)^{-1} l_T' \mathrm{cov}(\mathbf{X})^{-1}\mathbf{X}$ and $\sigma_{GLS}^2 = T\mathrm{var}(\hat{\mu}_{GLS}) = T(l_T'\mathrm{cov}(\mathbf{X})^{-1} l_T)^{-1}$. Note that $\hat{\mu}_{GLS} - \mu_0$ is independent of S_Y and

$\sigma^2_{GLS} = \sigma^2 + O(1/T)$ (see Grenander and Rosenblatt [3]). Using similar arguments in Lemma 1 of Sun [6], we have

$$\begin{aligned} P(|T_K| \le x) &= P\left(\frac{T(\hat{\mu}_{GLS} - \mu_0)^2/\sigma^2_{GLS}}{S_Y^2/\sigma^2_{GLS}} \le x^2\right) + O(1/T) \\ =&E[G_1(S_Y^2 x^2/\sigma^2)] + O(1/T) \\ =&G_1(x^2) + \frac{x^2}{\sigma^2}G_1'(x^2)E[S_Y^2 - \sigma^2] + \frac{x^4 G_1''(x^2)}{2\sigma^4}E[(S_Y^2 - \sigma^2)^2] + O(1/T) \\ =&G_1(x^2) - \frac{BK}{T\sigma^2}x^2 G_1'(x^2) + \frac{1}{K-1}x^4 G_1''(x^2) + O(1/T). \end{aligned}$$

$\diamondsuit$

1.2. *Proof of the main results in Section 2.2.* We first establish a high order expansion for Wald statistic based on the kernel $\mathcal{G}_{k,1}(r,t) = \sum_{j=1}^k \lambda_j \phi_j(r)\phi_j(t)$ in Lemma 1.6 below. Let $\xi = (\xi_0, \xi_1, \ldots, \xi_K)$ with $\xi_0 = \frac{1}{\sqrt{T}}\sum_{i=1}^T (X_i - \mu_0)$ and $\xi_j = \frac{1}{\sqrt{T}}\sum_{i=1}^T \phi_j^0(i/T)X_i$ for $j = 1, 2, \ldots, K$, and Σ_ξ be the covariance matrix of ξ. Define $Q_J(x) = P(\mathcal{F}(J) \le x)$ for $1 \le J \le \infty$. We present the following lemma regarding the convergence rate of Σ_ξ for the basis functions $\{\phi_j(t)\}_{j=1}^K$ without the mean zero and orthogonality assumption. Define $R = (R_{ij})_{i,j=1}^K$ with $R_{ij} = \int_0^1 \tilde{\phi}_i(t)\tilde{\phi}_j(t)dt$, where $\tilde{\phi}_j(t) = \phi_j(t) - \int_0^1 \phi_j(t)dt$, and $\tilde{R} = \text{diag}(1, R) = (\tilde{R}_{i,j})_{i,j=0}^K$.

LEMMA 1.5. *Assume the basis functions $\{\phi_j(t)\}_{j=1}^K$ are bounded with finite discontinuous points and satisfy* $\sup_{\alpha \in (0,1]}\left\{\left|\frac{1}{\alpha}\int_0^{1-\alpha}\tilde{\phi}_s(x)\{\tilde{\phi}_r(x+\alpha) - \tilde{\phi}_r(x)\}dx\right| + \left|\frac{1}{\alpha}\int_\alpha^1 \tilde{\phi}_s(x)\{\tilde{\phi}_r(x-\alpha) - \tilde{\phi}_r(x)\}dx\right|\right\} < \infty$, *for* $1 \le s, r \le K$. *If* $\sum_{h=-\infty}^{+\infty} h^2|\gamma_X(h)| < \infty$ *and K is fixed, then we have* $||\Sigma_\xi - \sigma^2\tilde{R}||_\infty = O(1/T)$.

PROOF OF LEMMA 1.5. For $s = 1, 2, \ldots, K$, we have

$$\text{cov}(\xi_0, \xi_s) = \frac{1}{T}\sum_{i=1}^T\sum_{j=1}^T \gamma_X(j-i)\phi_s^0\left(\frac{j}{T}\right) = \frac{1}{T}\sum_{h=1-T}^{T-1}\gamma_X(h)\sum_{1\le i,h+i\le T}\phi_s^0\left(\frac{h+i}{T}\right).$$

Simple algebra gives us

$$\frac{1}{T}\sum_{1\le i,h+i\le T}\phi_s^0\left(\frac{h+i}{T}\right) = \begin{cases} \frac{h}{T^2}\sum_{i=1}^T \phi_s(i/T) - \frac{1}{T}\sum_{i=1}^h \phi_s(i/T), & h > 0; \\ \frac{|h|}{T^2}\sum_{i=1}^T \phi_s(i/T) - \frac{1}{T}\sum_{i=T-|h|+1}^T \phi_s(i/T), & h < 0. \end{cases}$$

It implies that

$$\begin{aligned} \text{cov}(\xi_0, \xi_s) =& \frac{1}{T}\int_0^1 \phi_s(t)dt \sum_{h=-\infty}^{+\infty}|h|\gamma_X(h) \\ & - \frac{1}{T}\sum_{0<h<T}\gamma_X(h)\left\{\sum_{i=1}^h \phi_s(i/T) + \sum_{i=T-h+1}^T \phi_s(i/T)\right\} + O(1/T^2). \end{aligned} \quad (1)$$

Note that the second term on the right hand side of (1) is of order $O(1/T)$ because the basis functions $\{\phi_s(t)\}$ are bounded. Consider the covariance between ξ_s and ξ_r with $1 \leq s, r \leq K$. Straightforward calculation yields

$$\begin{aligned}
\operatorname{cov}(\xi_s, \xi_r) =& \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T} \phi_s^0\left(\frac{i}{T}\right)\phi_r^0\left(\frac{j}{T}\right)\gamma_X(i-j) \\
=& \frac{1}{T}\sum_{h=1}^{T-1}\sum_{1\leq j, j+h\leq T} \phi_s^0\left(\frac{j+h}{T}\right)\phi_r^0\left(\frac{j}{T}\right)\gamma_X(h) \\
&+ \frac{1}{T}\sum_{h=1-T}^{-1}\sum_{1\leq j, j+h\leq T} \phi_s^0\left(\frac{j+h}{T}\right)\phi_r^0\left(\frac{j}{T}\right)\gamma_X(h) \\
&+ \gamma_X(0)\frac{1}{T}\sum_{j=1}^{T}\phi_s^0\left(\frac{j}{T}\right)\phi_r^0\left(\frac{j}{T}\right) = I_1 + I_2 + I_3.
\end{aligned}$$

Notice that

(2)
$$\frac{1}{T}\sum_{1\leq j\leq T}\phi_s^0\left(\frac{j}{T}\right)\phi_r^0\left(\frac{j}{T}\right) = \int_0^1 \tilde{\phi}_s(t)\tilde{\phi}_r(t)dt + C_T(\tilde{\phi}_s(t), \tilde{\phi}_r(t)) = R_{sr} + C_T(\tilde{\phi}_s(t), \tilde{\phi}_r(t)),$$

where $C_T(\tilde{\phi}_s(t), \tilde{\phi}_r(t))$ is of order $O(1/T)$. It is not hard to see that

$$\begin{aligned}
I_1 =& \frac{1}{T}\sum_{h=1}^{T-1}\gamma_X(h)\left[\sum_{j=1}^{T-h}\phi_r^0\left(\frac{j}{T}\right)\left\{\phi_s^0\left(\frac{j+h}{T}\right) - \phi_s^0\left(\frac{j}{T}\right)\right\}\right], \quad \text{say } J_{1,T} \\
&+ \frac{1}{T}\sum_{h=1}^{T-1}\gamma_X(h)\left\{\sum_{j=1}^{T}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right) - \sum_{j=T-h+1}^{T}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right)\right\},
\end{aligned}$$

and

$$\begin{aligned}
I_2 =& \frac{1}{T}\sum_{h=1-T}^{-1}\gamma_X(h)\left[\sum_{j=1+|h|}^{T}\phi_r^0\left(\frac{j}{T}\right)\left\{\phi_s^0\left(\frac{j+h}{T}\right) - \phi_s^0\left(\frac{j}{T}\right)\right\}\right], \quad \text{say } J_{2,T} \\
&+ \frac{1}{T}\sum_{h=1-T}^{-1}\gamma_X(h)\left\{\sum_{j=1}^{T}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right) - \sum_{j=1}^{|h|}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right)\right\}.
\end{aligned}$$

Using (2), we have

$$\begin{aligned}\operatorname{cov}(\xi_s,\xi_r)=&\left\{\sigma^2-\sum_{|h|\geq T}\gamma_X(h)\right\}\left\{R_{sr}+C_T(\tilde\phi_s(t),\tilde\phi_r(t))\right\}\\ &-\frac{1}{T}\sum_{h=1}^{T-1}\gamma_X(h)\bigg\{\sum_{j=1}^{h}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right)\\ &+\sum_{j=T-h+1}^{T}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right)\bigg\}+J_{1,T}+J_{2,T}.\end{aligned}\tag{3}$$

Under the assumption that $\sup_{\alpha\in[0,1]}|\frac{1}{\alpha}\int_0^{1-\alpha}\tilde\phi_r(x)(\tilde\phi_s(x+\alpha)-\tilde\phi_s(x))dx|<\infty$, it is straightforward to see that

$$\begin{aligned}|J_{1,T}|\leq&\frac{1}{T}\sum_{h=1}^{T-1}|h\gamma_X(h)|\sup_{1\leq h\leq T}\left|\frac{1}{h}\sum_{j=1}^{T-h}\phi_r^0\left(\frac{j}{T}\right)\left\{\phi_s^0\left(\frac{j+h}{T}\right)-\phi_s^0\left(\frac{j}{T}\right)\right\}\right|\\ \leq&\frac{C}{T}\sum_{h=1}^{T-1}|h\gamma_X(h)|\left\{\sup_{\alpha\in[0,1]}\left|\frac{1}{\alpha}\int_0^{1-\alpha}\tilde\phi_r(x)(\tilde\phi_s(x+\alpha)-\tilde\phi_s(x))dx\right|\right\},\end{aligned}$$

which implies that $J_{1,T}=O(1/T)$. The same argument applies to $J_{2,T}$. The proof is then complete. $\diamondsuit$

The assumption regarding the basis functions in Lemma 1.5 is mild. If $\{\phi_j(t)\}_{j=1}^K$ are lipschitz continuous of order one, then the assumption is satisfied.

LEMMA 1.6. *Suppose $\sigma^2>0$ and the basis functions $\{\phi_j(t)\}_{j=1}^K$ are mean zero and orthogonal. Under the assumptions in Lemma 1.5 and H_0, we have $\sup_{x\in[0,+\infty)}|\aleph_T(x;K)|=O(1/T)$ and*

$$\sup_{x\in[0,+\infty)}|P(F_T(K)\leq x)-Q_K(x)-\aleph_T(x;K)|=O(1/T^2),\tag{4}$$

with K fixed and $T\to\infty$.

PROOF OF LEMMA 1.6. Note that when $\{\phi_j(t)\}_{j=1}^K$ are mean zero and orthogonal, we have $\tilde R=I_{K+1}$. It follows directly from Lemma 1.5 that $\sup_{x\in[0,+\infty)}|\aleph_T(x;K)|=O(1/T)$. To show the second part, we first note that under the Gaussian assumption, the density function of ξ is given by $f(u,\Sigma_\xi)=(2\pi)^{-(K+1)/2}|\Sigma_\xi|^{-1/2}\exp\left(-\frac{1}{2}u'\Sigma_\xi^{-1}u\right)$. Taking a Taylor expansion of the density function $f(u,\Sigma_\xi)$ around the covariance matrix $\sigma^2 I_{k+1}$, we get

$$f(u,\Sigma_\xi)=f(u,\sigma^2I_{K+1})+\frac{\partial f}{\partial\operatorname{vec}(\Sigma)}(u,\sigma^2I_{K+1})\operatorname{vec}(\Sigma_\xi-\sigma^2I_{k+1})+R_T(u).$$

By Lemma 1.4, the remainder term $R_T(u)$ satisfies that $\int_{\mathbb{R}^{K+1}} |R_T(u)|dv = O(1/T^2)$. Following Lemma 1.1, we have $\frac{\partial f}{\partial \mathrm{vec}(\Sigma)}(u, \sigma^2 I_{K+1}) = f(u, \sigma^2 I_{K+1})\left\{\frac{1}{2\sigma^4} u \otimes u - \frac{1}{2\sigma^2}\mathrm{vec}(I_{K+1})\right\}$, which implies that

$$P(F_T(K) \le x) = Q_K(x)\left\{1 - \frac{1}{2\sigma^2}\sum_{i=0}^{K}(\mathrm{var}(\xi_i) - \sigma^2)\right\} + \zeta_T(x),$$

where $\zeta_T(x) = \frac{1}{2\sigma^4}\int_{\{\mathcal{F}(u;K)\le x\}} f(u, \sigma^2 I_{K+1}) u' \otimes u' \mathrm{vec}(\Sigma_\xi - \sigma^2 I_{K+1})du + \int_{\{\mathcal{F}(u;K)\le x\}} R_T(u)du$ and $\mathcal{F}(u;K) = \frac{u_0^2}{\sum_{j=1}^K \lambda_j u_j^2}$. By letting $v = u/\sigma$ and noting that $E[\mathbf{I}\{\mathcal{F}(v;K) \le x\} v_s v_r] = 0$ for $s \ne r$, we obtain

$$\begin{aligned}\zeta_T(x) &= \frac{1}{2\sigma^2} E[\mathbf{I}\{\mathcal{F}(v;K) \le x\}(v \otimes v)']\mathrm{vec}(\Sigma_\xi - \sigma^2 I_{K+1}) + \int_{\{\mathcal{F}(u;K)\le x\}} R_T(u)du \\ &= \frac{1}{2\sigma^2}\sum_{i=0}^{K} E[\mathbf{I}\{\mathcal{F}(v;K) \le x\} v_i^2](\mathrm{var}(\xi_i) - \sigma^2) + \int_{\{\mathcal{F}(u;K)\le x\}} R_T(u)du,\end{aligned}$$

where $v = (v_0, v_1, \ldots, v_K)$ is a $(K+1)$-dimensional vector of i.i.d. standard normal random variables. Therefore, we get

$$\begin{aligned}\sup_{x\in[0,+\infty)} |P(F_T(K) \le x) - Q_K(x) - \aleph_T(x;K)| &= \sup_{x\in[0,+\infty)}\left|\int_{\{\mathcal{F}(u;K)\le x\}} R_T(u)du\right| \\ &\le \int_{\mathbb{R}^{K+1}} |R_T(u)|du = O(1/T^2),\end{aligned}$$

which completes the proof. ♢

Lemma 1.7. *Let $\{\Sigma_{T,J+1}\} \subset \mathbb{R}^{(J+1)\times(J+1)}$ be an array of positive definite matrices with $J + 1 \le T$. Assume that $||\Sigma_{T,J+1} - \Sigma_{J+1}||_\infty = O(J/T)$ for a sequence of positive definite matrices $\{\Sigma_j\}_{j=1}^\infty$ with $\sup_j ||\Sigma_j^{-1}||_2 < \infty$. If J satisfies that $1/J + J^2/T \to 0$, then we have $||\Sigma_{T,J+1}^{-1} - \Sigma_{J+1}^{-1}||_\infty = O(J^2/T)$.*

Proof. Let $\Sigma_{T,J+1} = \Sigma_{J+1} + R_{T,J+1}$. For sufficiently large T, we have $||\Sigma_{J+1}^{-1} R_{T,J+1}||_2 \le (J+1)||\Sigma_{J+1}^{-1}||_2 ||R_{T,J+1}||_\infty < 1$, where we are using the fact that $||R_{T,J+1}||_2 \le (J+1)||R_{T,J+1}||_\infty$. It follows that

$$||\Sigma_{T,J+1}^{-1} - \Sigma_{J+1}^{-1}||_\infty \le ||\Sigma_{T,J+1}^{-1} - \Sigma_{J+1}^{-1}||_2 \le (J+1)\frac{||\Sigma_{J+1}^{-1}||_2^2 ||R_{T,J+1}||_\infty}{1 - ||\Sigma_{J+1}^{-1} R_{T,J+1}||_2} = O(J^2/T).$$

♢

LEMMA 1.8. *Let $\tilde{\Sigma}_{T,J+1}(y)$ be a $(J+1)\times(J+1)$ positive definite matrix which depends on $y\in\mathbb{R}^{J+1}$, and $\Sigma_{T,J+1}$ and $\Sigma_j=\sigma^2 I_j$ satisfy the assumptions in Lemma 1.7. Assume that $\sup_{y\in\mathbb{R}^{J+1}}||\tilde{\Sigma}_{T,J+1}(y)-\sigma^2 I_{J+1}||_\infty \le ||\Sigma_{T,J+1}-\sigma^2 I_{J+1}||_\infty = O(J/T)$. Let $R_{T,J+1}=\Sigma_{T,J+1}-\sigma^2 I_{J+1}$. If $J=o(T^{1/6})$, we have*

$$\int_{y\in\mathbb{R}^{J+1}}\left|vec(R_{T,J+1})'\frac{\partial^2 f}{\partial vec(\Sigma)vec(\Sigma)}(y,\tilde{\Sigma}_{T,J+1}(y))vec(R_{T,J+1})\right|dy=o(1/T).$$

PROOF. Let $\tilde{R}_{T,J+1}(y)=\tilde{\Sigma}_{T,J+1}(y)-\sigma^2 I_{J+1}$. Note first that $\sup_{y\in\mathbb{R}^{J+1}}||\tilde{R}_{T,J+1}(y)/\sigma^2||_2\le (J+1)\sup_{y\in\mathbb{R}^{J+1}}||\tilde{R}_{T,J+1}(y)||_\infty/\sigma^2\le (J+1)||\Sigma_{T,J+1}-\sigma^2 I_{J+1}||_\infty/\sigma^2<1$, for large enough T. Following the arguments in Lemma 1.7, we know that

$$\sup_{y\in\mathbb{R}^{J+1}}||\tilde{\Sigma}^{-1}_{T,J+1}(y)-\sigma^{-2}I_{J+1}||_2\le\frac{C(J+1)||\Sigma_{T,J+1}-\sigma^2 I_{J+1}||_\infty}{1-(J+1)||\Sigma_{T,J+1}-\sigma^2 I_{J+1}||_\infty/\sigma^2}=O(J^2/T).$$

Choose $r=J^3/T$. Then we have

$$\begin{aligned}y'\left(\tilde{\Sigma}^{-1}_{T,J+1}(y)-\frac{1}{(1+r)\sigma^2}I_{J+1}\right)y=&y'\left(\tilde{\Sigma}^{-1}_{T,J+1}(y)-\frac{1}{\sigma^2}I_{J+1}\right)y+\frac{r}{(r+1)\sigma^2}||y||^2\\ &\ge\left(\frac{r}{(r+1)\sigma^2}-||\tilde{\Sigma}^{-1}_{T,J+1}(y)-I_{J+1}/\sigma^2||_2\right)||y||^2\ge 0,\end{aligned}$$

when T is sufficiently large. On the other hand, we have

$$\begin{aligned}\sup_{y\in\mathbb{R}^{J+1}}|\tilde{\Sigma}^{-1}_{T,J+1}(y)|&\le\sup_{y\in\mathbb{R}^{J+1}}||\tilde{\Sigma}^{-1}_{T,J+1}(y)||_2^{J+1}\le\left(\frac{1}{\sigma^2}+\frac{CJ^2}{T}\right)^{J+1}\\ &\le\left|\frac{1}{(r+1)\sigma^2}I_{J+1}\right|\left(1+r+\frac{C(r+1)J^2\sigma^2}{T}\right)^{J+1}\\ &\le\left|\frac{1}{(r+1)\sigma^2}I_{J+1}\right|(1+Cr)^{(1/r)(J+1)r}\le C\left|\frac{1}{(r+1)\sigma^2}I_{J+1}\right|.\end{aligned}$$

The above arguments imply that $f(y,\tilde{\Sigma}_{T,J+1}(y))\le Cf(y,(1+r)\sigma^2 I_{J+1})$ for all y. Therefore we get

$$\begin{aligned}&\int_{y\in\mathbb{R}^{J+1}}\left|\mathrm{vec}(R_{T,J+1})'\frac{\partial^2 f}{\partial\mathrm{vec}(\Sigma)\mathrm{vec}(\Sigma)}(y,\tilde{\Sigma}_{J+1}(y))\mathrm{vec}(R_{T,J+1})\right|dy\\ \le&C\int_{y\in\mathbb{R}^{J+1}}\Big|\mathrm{vec}(R_{T,J+1})'\{(\tilde{\Sigma}^{-1}_{J+1}(y)y)\otimes(\tilde{\Sigma}^{-1}_{J+1}(y)y)-\mathrm{vec}(\tilde{\Sigma}^{-1}_{J+1}(y))\}\{(\tilde{\Sigma}^{-1}_{J+1}(y)y)\otimes(\tilde{\Sigma}^{-1}_{J+1}(y)y)\\ &-\mathrm{vec}(\tilde{\Sigma}^{-1}_{J+1}(y))\}'\mathrm{vec}(R_{T,J+1})\Big|f(y,(1+r)\sigma^2 I_{J+1})dy\\ &+C\int_{y\in\mathbb{R}^{J+1}}\Big|\mathrm{vec}(R_{T,J+1})'\{(\tilde{\Sigma}^{-1}_{J+1}(y)yy'\tilde{\Sigma}^{-1}_{J+1}(y))\otimes\tilde{\Sigma}^{-1}_{J+1}(y)+\tilde{\Sigma}^{-1}_{J+1}(y)\otimes(\tilde{\Sigma}^{-1}_{J+1}(y)yy'\tilde{\Sigma}^{-1}_{J+1}(y))\\ &-\tilde{\Sigma}^{-1}_{J+1}(y)\otimes\tilde{\Sigma}^{-1}_{J+1}(y)\}\mathrm{vec}(R_{T,J+1})\Big|f(y,(1+r)\sigma^2 I_{J+1})dy\le CJ^6/T^2=o(1/T),\end{aligned}$$

where the first inequality in the last row comes from the fact that $\sup_{y\in\mathbb{R}^{J+1}}||\tilde{\Sigma}_{J+1}^{-1}(y)-\sigma^{-2}I_{J+1}||_\infty \le \sup_{y\in\mathbb{R}^{J+1}}||\tilde{\Sigma}_{J+1}^{-1}(y)-\sigma^{-2}I_{J+1}||_2 = O(J^2/T)$. ♢

LEMMA 1.9. *Recall that $Q_J(x) = P(\mathcal{F}(J)\le x)$ for $1\le J\le\infty$. We have*

$$\sup_{x\in[0,+\infty)}|Q_J(x)-Q_\infty(x)| = O\left(\sum_{j=J+1}^{\infty}\lambda_j\right). \tag{5}$$

PROOF. Let $\{v_j\}_{j=0}^{+\infty}$ be a sequence of i.i.d. standard normal random variables. Define $U(J)=\sum_{j=1}^J\lambda_j v_j^2$, $V(J)=\sum_{j=J+1}^\infty\lambda_j v_j^2$ and $Q_J = v_0^2/U(J)$ for $2\le J\le\infty$. For any $x\in[0,+\infty)$ and large enough J with $J\ge 3$, we have,

$$\begin{aligned}|Q_J(x)-Q_\infty(x)| =&|E[E[\mathbf{I}\{Q_J\le x\}|U(J)]]-E[E[\mathbf{I}\{Q_\infty\le x\}|U(\infty)]]|\\ =&|E[G_1(xU(J))]-E[G_1(xU(\infty))]|\\ =&|E[G_1(xU(J)+xV(J))]-E[G_1(xU(J))]|\\ =&|E[xV(J)G_1'(x\tilde{U}(J))]| = \left|E\left[\frac{V(J)}{\tilde{U}(J)}x\tilde{U}(J)G_1'(x\tilde{U}(J))\right]\right|\\ \le& CE\left[\frac{V(J)}{\tilde{U}(J)}\right]\le CE[V(J)]E\left[\frac{1}{U(J)}\right]\le C\sum_{j=J+1}^{\infty}\lambda_j,\end{aligned}$$

where $U(J)\le\tilde{U}(J)\le U(J)+V(J)$ and C does not depend on x. Note that we are using the mean value theorem, and the facts that $E[1/U(J)]\le E\left[1/(\lambda_3\chi_3^2)\right]<\infty$ and $\sup_{x\in[0,+\infty)}|xG_1'(x)|<\infty$. ♢

LEMMA 1.10. *Let $V_T(J)=\sum_{j=J+1}^\infty\lambda_j\xi_j^2$. Assume that $\sup_{1\le i\le\infty}\sup_{t\in[0,1]}\phi_i(t)<\infty$ and $\{X_i\}$ is a stationary Gaussian time series. Then we have $EV_T^2(J)=O((\sum_{j=J+1}^\infty\lambda_j)^2)$.*

PROOF. Let $\sigma_{ij}=\gamma_X(i-j)$. For $i,j\ge J+1$, we have

$$\begin{aligned}E[\xi_i^2\xi_j^2] =&\frac{1}{T^2}\sum_{i_1,i_2=1}^{T}\sum_{j_1,j_2=1}^{T}\phi_i^0(i_1/T)\phi_i^0(i_2/T)\phi_j^0(j_1/T)\phi_j^0(j_2/T)E[(X_{i_1}-\mu)(X_{i_2}-\mu)(X_{j_1}-\mu)(X_{j_2}-\mu)]\\ =&\frac{1}{T^2}\sum_{i_1,i_2=1}^{T}\sum_{j_1,j_2=1}^{T}\phi_i^0(i_1/T)\phi_i^0(i_2/T)\phi_j^0(j_1/T)\phi_j^0(j_2/T)(\sigma_{i_1i_2}\sigma_{j_1j_2}+\sigma_{i_1j_1}\sigma_{i_2j_1}+\sigma_{i_1j_2}\sigma_{i_2j_1})\\ =&I_{1,T}+I_{2,T}+I_{3,T}.\end{aligned}$$

For the first term, we have

$$I_{1,T}=\left(\frac{1}{T}\sum_{i_1,i_2=1}^{T}\phi_i^0(i_1/T)\phi_i^0(i_2/T)\sigma_{i_1i_2}\right)\left(\frac{1}{T}\sum_{j_1,j_2=1}^{T}\phi_j^0(j_1/T)\phi_j^0(j_2/T)\sigma_{j_1j_2}\right)=L_{1,T}L_{2,T}.$$

Note that

$$|L_{1,T}| = \left| \frac{1}{T} \sum_{h=1-T}^{T-1} \sum_{1\le i_1, i_1+h \le T} \phi_i^0(i_1/T)\phi_i^0(h/T)\gamma_X(h) \right|$$
$$\le C \sum_{h=-\infty}^{+\infty} |\gamma_X(h)| \frac{1}{T} \sum_{1\le i_1\le T} |\phi_i^0(i_1/T)| \le C \sum_{h=-\infty}^{+\infty} |\gamma_X(h)|,$$

which implies that $|I_{1,T}| \le C(\sum_{h=-\infty}^{+\infty} |\gamma_X(h)|)^2$. Similar arguments apply to the other terms $I_{2,T}$ and $I_{3,T}$. We then have $\sup_{J+1\le i,j\le\infty} E[\xi_i^2\xi_j^2] < C$. Therefore, we obtain $E[V_T(J)^2] = \sum_{i=J+1}^{\infty}\sum_{j=J+1}^{\infty} \lambda_i\lambda_j E[\xi_i^2\xi_j^2] \le C\left(\sum_{i=J+1}^{\infty}\lambda_i\right)^2$. ♢

LEMMA 1.11. *Assume the eigenfunctions are continuously differentiable, mean zero and uniformly bounded, and* $\sum_{j=1}^{\infty}\lambda_j < \infty$. *Suppose that* $\{X_i\}$ *is a stationary Gaussian time series with* $\sum_{h=-\infty}^{+\infty} h^2|\gamma_X(h)| < \infty$. *When* $1/J + J/T \to 0$, *we have*

$$\sup_{x\in[0,+\infty)} |P(F_T(J)\le x) - P(F_T(\infty)\le x)| = O\left(\left(\sum_{j=J+1}^{\infty}\lambda_j\right)^{1/3}\right).$$

Recall that $F_T(J) = \frac{\xi_0^2}{\sum_{j=1}^{J}\lambda_j\xi_j^2}$ *for* $J = 1, 2, \ldots, \infty$.

PROOF. Let $R_T(J) = F_T(J) - F_T(\infty) = \frac{\xi_0^2 V_T(J)}{(\sum_{j=1}^{\infty}\lambda_j\xi_j^2)(\sum_{j=1}^{J}\lambda_j\xi_j^2)}$. For any $\delta > 0$, we have

(6)
$$P(F_T(\infty)\le x-\delta) - P(|R_T(J)|\ge\delta) \le P(F_T(J)\le x) \le P(F_T(\infty)\le x+\delta) + P(|R_T(J)|\ge\delta).$$

Observe that

$$P(|R_T(J)|\ge\delta) \le \frac{E|R_T(J)|}{\delta} \le \frac{(E[V_T^2(J)])^{1/2}}{\delta}\left(E\left[\frac{\xi_0^4}{(\sum_{j=1}^{J}\lambda_j\xi_j^2)^4}\right]\right)^{1/2}.$$

Choose a fixed $J_0 \ge 9$, denote by $\hat{\Sigma}_{T,J_0+1}$ the covariance matrix of $(\xi_0,\ldots,\xi_{J_0})$. By Lemma 1.5, we know that $||\hat{\Sigma}_{T,J_0+1} - \sigma^2 I_{J_0+1}||_2 \le (J_0+1)||\hat{\Sigma}_{T,J_0+1} - \sigma^2 I_{J_0+1}||_\infty = O(1/T)$. For large enough T, we have $||\hat{\Sigma}_{T,J_0+1}||_2 \le 2\sigma^2$. Let $\tilde{\lambda} = \min(1, \frac{1}{2\sigma^2}) > 0$, we know that $\hat{\Sigma}_{T,J_0+1}^{-1} - \tilde{\lambda} I_{J+1}$ is semi-positive definite, i.e., for any $x \in \mathbb{R}^{J+1}$, $x'\hat{\Sigma}_{T,J_0+1}^{-1}x \ge \tilde{\lambda}x'x$. Using similar arguments in Lemma 1.3, we know that $|\hat{\Sigma}_{T,J_0+1}|^{-1} \le$

$||\hat{\Sigma}_{T,J_0+1}^{-1}||_2^{J_0+1} \le (2/\sigma^2)^{J_0+1}$ for large enough T. For any $J \ge J_0$, we have

$$\begin{aligned}E\left[\frac{\xi_0^4}{(\sum_{j=1}^{J}\lambda_j\xi_j^2)^4}\right] \le & E\left[\frac{\xi_0^4}{\lambda_{J_0}^4(\sum_{j=1}^{J_0}\xi_j^2)^4}\right] \\ \le & \frac{1}{(2\pi)^{(J_0+1)/2}|\hat{\Sigma}_{T,J_0+1}|^{1/2}}\int_{w\in\mathbb{R}^{J_0+1}}\frac{w_0^4}{\lambda_{J_0}^4(\sum_{j=1}^{J_0}w_j^2)^4}\exp(-\tilde{\lambda}w'w/2)dw \\ \le & CE[(\chi_1^2)^2]E[(1/\chi_{J_0}^2)^4] < \infty,\end{aligned}$$

where $w=(w_0,w_1,\ldots,w_{J_0})$ and χ_m^2 denotes a chi-square random variable with m degrees of freedom. By Lemma 1.10, we obtain

$$P(|R_T(J)|\ge\delta)\le C\left(\sum_{j=J+1}^{\infty}\lambda_j\right)/\delta. \tag{7}$$

In what follows, we show that $\sup_{x\in[0,\infty)}|P(F_T(\infty)\le x\pm\delta)-P(F_T(\infty)\le x)|\le C\sqrt{\delta}$ for any $\delta>0$. Let $X=(X_1,X_2,\ldots,X_T)'$, $l_T=(1,1,\ldots,1)'$, $X^*=X-l_T\mu_0$ and $\Omega_T=\mathrm{cov}(X)$. Then the GLS estimate of μ is given by $\hat{\mu}_{GLS}=(l_T'\Omega_T^{-1}l_T)^{-1}l_T'\Omega_T^{-1}X$ and $\hat{\mu}_{OLS}-\mu_0=\hat{\mu}_{GLS}-\mu_0+\frac{1}{T}l_T'\tilde{X}$, where $\tilde{X}=(I_T-l_T(l_T'\Omega_T^{-1}l_T)^{-1}l_T'\Omega_T^{-1})X^*$. The following facts which can be found in Sun et al. [7] play an important role in the proof presented below: (1) $\hat{\mu}_{GLS}-\mu_0$ is independent of $\tilde{X}$; (2) $\hat{\mu}_{GLS}-\mu_0$ is independent of $X-l_T\hat{\mu}_{OLS}$. Notice that $\hat{D}_T=\sum_{j=1}^{\infty}\lambda_j\xi_j^2=\frac{1}{T}(X-l_T\hat{\mu}_{OLS})'\mathcal{G}_T(X-l_T\hat{\mu}_{OLS})$ with $\mathcal{G}_T=(\mathcal{G}(i/T,j/T))_{i,j=1}^T$. Then $\hat{\mu}_{GLS}-\mu_0$ is also independent of $\hat{D}_T$. Define $\sigma_{GLS}^2=T\mathrm{var}(\hat{\mu}_{GLS})=T(l_T'\Omega_T^{-1}l_T)^{-1}$. Denote by Φ_{norm} and ϕ_{norm} the cumulative distribution function and density function of the standard normal distribution. Therefore, we get

$$\begin{aligned}P(F_T(\infty)\le x)=&2P\left(\frac{\sqrt{T}(\hat{\mu}_{OLS}-\mu_0)}{\sqrt{\hat{D}_T}}\le\sqrt{x}\right)-1=2P\left(\sqrt{T}(\hat{\mu}_{OLS}-\mu_0)\le\sqrt{x\hat{D}_T}\right)-1\\ =&2P\left(\sqrt{T}(\hat{\mu}_{GLS}-\mu_0)/\sigma_{GLS}\le\sqrt{x\hat{D}_T}/\sigma_{GLS}-l_T'\tilde{X}/(\sqrt{T}\sigma_{GLS})\right)-1\\ =&2E\left[\Phi_{norm}\left(\sqrt{x\hat{D}_T}/\sigma_{GLS}-l_T'\tilde{X}/(\sqrt{T}\sigma_{GLS})\right)\right]-1,\end{aligned}$$

which implies that for $x,\delta\ge0$ with $x-\delta\ge0$,

$$\begin{aligned}&|P(F_T(\infty)\le x\pm\delta)-P(F_T(\infty)\le x)|\\ =&\Bigg|2E\left[\Phi_{norm}\left(\sqrt{(x\pm\delta)\hat{D}_T}/\sigma_{GLS}-l_T'\tilde{X}/(\sqrt{T}\sigma_{GLS})\right)\right]\\ &-2E\left[\Phi_{norm}\left(\sqrt{x\hat{D}_T}/\sigma_{GLS}-l_T'\tilde{X}/(\sqrt{T}\sigma_{GLS})\right)\right]\Bigg|\\ =&2\left|(\sqrt{x\pm\delta}-\sqrt{x})E\left[\sqrt{\hat{D}_T}\phi_{norm}(a^*-l_T'\tilde{X}/(\sqrt{T}\sigma_{GLS}))/\sigma_{GLS}\right]\right|\\ \le&C\sqrt{\delta}E[\sqrt{\hat{D}_T/\sigma_{GLS}}]<C\sqrt{\delta}(E[\hat{D}_T])^{1/2}/\sigma_{GLS}<C\sqrt{\delta},\end{aligned} \tag{8}$$

with $\sqrt{x\hat{D}_T}/\sigma_{GLS} \leq a^* \leq \sqrt{(x+\delta)\hat{D}_T}/\sigma_{GLS}$ or $\sqrt{(x-\delta)\hat{D}_T}/\sigma_{GLS} \leq a^* \leq \sqrt{x\hat{D}_T}/\sigma_{GLS}$. Here we are using the fact that $\sigma_{GLS}^2 = \sigma^2 + O(1/T)$ and $E[\hat{D}_T]$ is uniformly bounded for all T. Choosing $\delta = (\sum_{j=J+1}^{\infty} \lambda_j)^{2/3}$, the conclusion follows in view of (6), (7) and (8). ♢

Lemma 1.12. *Under the assumptions in Theorem 2.2, we have* $||\Sigma_{\xi,J+1} - \sigma^2 I_{J+1}||_\infty = O(J/T)$ *with* $J \leq T$, *where* $\Sigma_{\xi,J+1}$ *denotes the covariance matrix of* $(\xi_0, \xi_1, \dots, \xi_J)$.

Proof of Lemma 1.12. Using the arguments in Lemma 1.5, we have for any $1 \leq s \leq J$,

$$|\text{cov}(\xi_0, \xi_s)| \leq C\left|\frac{1}{T^2}\sum_{i=1}^{T}\phi_s(i/T)\right| + \frac{1}{T}\sum_{0<h<T}\left|\gamma_X(h)\left\{\sum_{i=1}^{h}\phi_s(i/T) + \sum_{i=T-h+1}^{T}\phi_s(i/T)\right\}\right| \leq C/T,$$

where C is a generic constant which does not depend on s. Again by the arguments in Lemma 1.5, we have

$$\begin{aligned} |\text{cov}(\xi_s, \xi_r) - \sigma^2\delta_{sr}| \leq &\sum_{h=1-T}^{T-1}|\gamma_X(h)C_T(\tilde{\phi}_s(t), \tilde{\phi}_r(t))| + \sum_{|h|\geq T}|\gamma_X(h)|\delta_{sr} \\ &+ \frac{1}{T}\sum_{h=1}^{T-1}\left|\gamma_X(h)\left\{\sum_{j=1}^{h}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right) + \sum_{j=T-h+1}^{T}\phi_r^0\left(\frac{j}{T}\right)\phi_s^0\left(\frac{j}{T}\right)\right\}\right| \\ &+ |J_{1,T}| + |J_{2,T}|, \quad 1 \leq s, r \leq J, \end{aligned}$$

where $J_{1,T}$, $J_{2,T}$ and $C_T(\tilde{\phi}_s(t), \tilde{\phi}_r(t))$ are defined in the proof of Lemma 1.5. By the Trapezoidal rule and the assumption that $\sup_{1\leq i\leq J}\sup_{t\in[0,1]}|\phi_i''(t)| < CJ^2$, we have

$$|C_T(\tilde{\phi}_s(t), \tilde{\phi}_r(t))| \leq C(J^2/T^2 + 1/T), \tag{9}$$

which implies that $|\text{cov}(\xi_s, \xi_r) - \sigma^2\delta_{sr}| \leq CJ/T + |J_{1,T}| + |J_{2,T}|$ for $J \leq T$. By the mean value theorem and the assumption that $\sup_{1\leq i\leq J}\sup_{t\in[0,1]}|\phi_i'(t)| < CJ$, we get

$$\begin{aligned} |J_{1,T}| \leq &\frac{1}{T}\sum_{h=1}^{T-1}|\gamma_X(h)|\left|\sum_{j=1}^{T-h}\phi_r^0\left(\frac{j}{T}\right)\left\{\phi_s^0\left(\frac{j+h}{T}\right) - \phi_s^0\left(\frac{j}{T}\right)\right\}\right| \\ \leq &\frac{CJ}{T}\sum_{h=1}^{T-1}|h\gamma_X(h)|\frac{1}{T}\sum_{j=1}^{T-h}\left|\phi_r^0\left(\frac{j}{T}\right)\right| \leq \frac{CJ}{T}. \end{aligned} \tag{10}$$

Using the same argument for $J_{2,T}$, we get $|\text{cov}(\xi_s, \xi_r) - \sigma^2\delta_{sr}| \leq CJ/T$, which completes the proof. ♢

PROOF OF THEOREM 2.2. Suppose $J = o(T^{1/6})$. By Lemma 1.12, we know $||\Sigma_{\xi,J+1} - \sigma^2 I_{J+1}||_\infty = O(J/T)$. Using Lemma 1.8 and similar arguments in the proof of Lemma 1.6, we can show that

$$\sup_{x\in\mathbb{R}} |P(F_T(J) \le x) - Q_J(x) - \aleph_T(x;J)| = o(1/T),$$

where $\aleph_T(x;J) = \frac{1}{2\sigma^2}\sum_{i=0}^{J}(\mathrm{var}(\xi_i) - \sigma^2)E[(v_i^2 - 1)\mathbf{I}\{\mathcal{F}(v;J) \le x\}]$ with $v = (v_0, v_1, \ldots, v_J) \sim N(0, I_{J+1})$. Next, we show that $\aleph_T(x;J)$ converges uniformly as $J \to +\infty$. Note first that

$$\begin{aligned}
\sup_{x\in[0,+\infty)} |\aleph_T(x;J+p) - \aleph_T(x;J)| \le & \sup_{x\in[0,+\infty)} \left|\frac{1}{2\sigma^2}\sum_{i=J+1}^{J+p}(\mathrm{var}(\xi_i) - \sigma^2)E[(v_i^2 - 1)\mathbf{I}\{\mathcal{F}(v;J+p) \le x\}]\right| \\
& + \sup_{x\in[0,+\infty)} \left|\frac{1}{2\sigma^2}\sum_{i=1}^{J}(\mathrm{var}(\xi_i) - \sigma^2)E[(v_i^2 - 1)(\mathbf{I}\{\mathcal{F}(v;J+p) \le x\} - \mathbf{I}\{\mathcal{F}(v;J) \le x\})]\right| \\
& + \sup_{x\in[0,+\infty)} \left|\frac{1}{2\sigma^2}(\mathrm{var}(\xi_0) - \sigma^2)E[(v_0^2 - 1)(\mathbf{I}\{\mathcal{F}(v;J+p) \le x\} - \mathbf{I}\{\mathcal{F}(v;J) \le x\})]\right| = I_1 + I_2 + I_3,
\end{aligned}$$

for any $J, p \in \mathbb{Z}^+$. In view of (9) and (10), we have

$$|\mathrm{var}(\xi_i) - \sigma^2| < C(i/T + i^2/T^2), \tag{11}$$

for $1 \le i < \infty$. Hence we get, for sufficiently large J,

(12)

$$\begin{aligned}
I_1 \le & \frac{1}{2\sigma^2}\sup_{x\in[0,+\infty)}\sum_{i=J+1}^{J+p}\left|(\mathrm{var}(\xi_i) - \sigma^2)E\left[(v_i^2 - 1)G_1\left(x\sum_{j=1}^{J+p}\lambda_j v_j^2\right)\right]\right| \\
\le & \frac{C}{T}\sup_{x\in[0,+\infty)}\sum_{i=J+1}^{J+p}(i + i^2/T)\left|E\left[(v_i^2 - 1)G_1\left(x\sum_{j=1}^{J+p}\lambda_j v_j^2\right)\right]\right| \\
\le & \frac{C}{T}\sup_{x\in[0,+\infty)}\sum_{i=J+1}^{J+p}(i + i^2/T)\left|E\left[(v_i^2 - 1)\left\{G_1\left(x\sum_{j\ne i}\lambda_j v_j^2\right) + \lambda_i v_i^2 x G_1'(y_i)\right\}\right]\right| \\
\le & \frac{C}{T}\sup_{x\in[0,+\infty)}\sum_{i=J+1}^{J+p}(i + i^2/T)\lambda_i E\left|v_i^2(v_i^2 + 1)xG_1'(y_i)\right| \\
\le & \frac{C}{T}\sum_{i=J+1}^{J+p}(i + i^2/T)\lambda_i E\left[\frac{v_i^2(v_i^2 + 1)}{\sum_{j\ne i}\lambda_j v_j^2 + \alpha_i\lambda_i^2 v_i^2}\right] \\
\le & \frac{C}{T}\sum_{i=J+1}^{J+p}(i + i^2/T)\lambda_i E\left[v_i^2(v_i^2 + 1)\right] E\left[\frac{1}{\sum_{j\ne i}\lambda_j v_j^2}\right] \\
\le & \frac{C}{T}\left\{\sum_{i=J+1}^{+\infty} i\lambda_i + \frac{1}{T}\sum_{i=J+1}^{+\infty} i^2\lambda_i\right\} = O\left(\frac{J^{-a+2}}{T}\right),
\end{aligned}$$

where $y_i = x(\sum_{j\neq i}\lambda_j v_j^2 + \alpha_i\lambda_i^2 v_i^2)$ for some $0 \le \alpha_i \le 1$. On the other hand, we get

$$\begin{aligned}
I_2 \le& \frac{CJ}{T}\sup_{x\in[0,+\infty)}\sum_{i=1}^{J}\left|E\left[(v_i^2-1)\left\{G_1\left(x\sum_{j=1}^{J+p}\lambda_j v_j^2\right)-G_1\left(x\sum_{j=1}^{J}\lambda_j v_j^2\right)\right\}\right]\right| \\
\le& \frac{CJ}{T}\sup_{x\in[0,+\infty)}\sum_{i=1}^{J}\left|E\left[x(v_i^2-1)\left(\sum_{j=J+1}^{J+p}\lambda_j v_j^2\right)\left\{G_1'\left(x\sum_{j=1}^{J}\lambda_j v_j^2 + x\alpha\sum_{j=J+1}^{J+p}\lambda_j v_j^2\right)\right\}\right]\right| \\
\le& \frac{CJ}{T}\left(\sum_{j=J+1}^{J+p}\lambda_j\right)E\left[\frac{\sum_{i=1}^{J}(v_i^2+1)}{\sum_{j=1}^{J}\lambda_j v_j^2}\right] \le \frac{CJ^2}{T}\left(\sum_{j=J+1}^{\infty}\lambda_j\right) = O\left(\frac{J^{-a+3}}{T}\right).
\end{aligned}$$

Finally using the Cauchy-Schwarz inequality and similar arguments in Lemma 1.9, we know

$$\begin{aligned}
I_3 \le& \frac{C}{T}\left\{E[(v_0^2-1)^2]\right\}^{1/2}\sup_{x\in[0,+\infty)}\left\{E[(\mathbf{I}\{\mathcal{F}(v;J+p)\le x\}-\mathbf{I}\{\mathcal{F}(v;J)\le x\})^2]\right\}^{1/2} \\
\le& \frac{C}{T}\sup_{x\in[0,+\infty)}\{E[|\mathbf{I}\{\mathcal{F}(v;J+p)\le x\}-\mathbf{I}\{\mathcal{F}(v;J)\le x\}|]\}^{1/2} \le \frac{C}{T}\left(\sum_{j=J}^{+\infty}\lambda_j\right)^{1/2} = O\left(\frac{J^{(-a+1)/2}}{T}\right).
\end{aligned}$$

Therefore, it is straightforward to see that $\sup_{x\in[0,\infty)}|\aleph_T(x;J)-\aleph_T(x;\infty)| = O\left(J^{(-a+1)/2}/T\right)$ and $\sup_{x\in[0,\infty)}|\aleph_T(x;\infty)| = O(1/T)$, which imply that

$$\sup_{x\in[0,\infty)}|P(F_T(J)\le x)-Q_J(x)-\aleph_T(x;\infty)| = o(1/T),$$

for $J = o(T^{1/6})$. Let $J = T^{1/6}/\log(T)$ and note that $(\sum_{j=J+1}^{\infty}\lambda_j)^{1/3} = o(1/T)$. The proof is completed in view of Lemma 1.9 and Lemma 1.11. ♢

Proof of Proposition 2.3. Under the assumption that $\sup_{x\in\mathbb{R}}|\mathcal{K}(x)| \le 1$ and $\int_{-\infty}^{+\infty}|\mathcal{K}(x)|dx < \infty$, we have

$$\begin{aligned}
\sum_{j=1}^{+\infty}(\tilde{\lambda}_{j,b})^2 =& \int_0^1\int_0^1\tilde{\mathcal{G}}_b^2(r,t)drdt \le \sup_{t\in[0,1]}\int_0^1\tilde{\mathcal{G}}_b^2(r,t)dr \le 4\sup_{t\in[0,1]}\int_0^1|\tilde{\mathcal{G}}_b(r,t)|dr \\
\le& 16\sup_{t\in[0,1]}\int_{-t}^{1-t}|\mathcal{K}_b(r)|dr \le 16\int_{-\infty}^{+\infty}|\mathcal{K}_b(r)|dr \le Cb,
\end{aligned}$$

and $\tilde{\lambda}_{1,b} \le (\int_0^1\int_0^1\tilde{\mathcal{G}}_b^2(r,t)drdt)^{1/2} \le C\sqrt{b}$. Suppose $\{\tilde{a}_i\}$ is a sequence of random variables such that $0 \le \tilde{a}_i \le 1$. Using the fact that $\sum_{j=1}^{+\infty}\tilde{\lambda}_{j,b} = \int_0^1\tilde{\mathcal{G}}_b(r,r)dr = 1 + O(b)$, we get

(13)

$$\sup_i E\left(\sum_{j\neq i}^{+\infty}\tilde{\lambda}_{j,b}v_j^2 + \tilde{a}_i\tilde{\lambda}_{i,b}v_i^2 - 1\right)^2 = \sup_i\left\{2\sum_{j\neq i}^{+\infty}(\tilde{\lambda}_{j,b})^2 + (\tilde{\lambda}_{i,b})^2E(\tilde{a}_iv_i^2-1)^2\right\} + O(b) \le Cb.$$

By the Talyor expansion, we have

$$\begin{aligned}\aleph_{T,b}(x;\infty) =& \frac{1}{2\sigma^2}\sum_{i=1}^{+\infty}(\mathrm{var}(\tilde{\xi}_{i,b})-\sigma^2)E[(v_i^2-1)\mathbf{I}\{\mathcal{F}(v;\infty)\le x\}]+O(1/T)\\ =& \frac{1}{2\sigma^2}\sum_{i=1}^{+\infty}(\mathrm{var}(\tilde{\xi}_{i,b})-\sigma^2)E\left[(v_i^2-1)G_1\left(x\sum_{j=1}^{+\infty}\tilde{\lambda}_{j,b}v_j^2\right)\right]+O(1/T)\\ =& \frac{x}{\sigma^2}\sum_{i=1}^{+\infty}\tilde{\lambda}_{i,b}(\mathrm{var}(\tilde{\xi}_{i,b})-\sigma^2)E\left[G_1'\left(x\left(\sum_{j\neq i}^{+\infty}\tilde{\lambda}_{j,b}v_j^2\right)\right)\right]\\ &+\frac{x^2}{4\sigma^2}\sum_{i=1}^{+\infty}(\tilde{\lambda}_{i,b})^2(\mathrm{var}(\tilde{\xi}_{i,b})-\sigma^2)E\left[v_i^4(v_i^2-1)G_1''\left(x\left(\sum_{j\neq i}^{+\infty}\tilde{\lambda}_{j,b}v_j^2+a_i\tilde{\lambda}_{i,b}v_i^2\right)\right)\right]+O(1/T)\\ =& I_{1T,b}+I_{2T,b}+O(1/T),\end{aligned}$$

where $0\le a_i\le 1$. Let $A_{i,b}=E\left[G_1'\left(x\left(\sum_{j\neq i}^{+\infty}\tilde{\lambda}_{j,b}v_j^2\right)\right)\right]$, $B_{i,b}=\tilde{\lambda}_{i,b}(\mathrm{var}(\tilde{\xi}_{i,b})-\sigma^2)$, $C_{i,b}=\sum_{j=1}^{i}B_{j,b}$ and $S_{N,b}=\sum_{i=1}^{N}A_{i,b}B_{i,b}$. Using summation by parts, we have $S_{N,b}=A_{N,b}C_{N,b}-\sum_{i=1}^{N-1}(A_{i+1,b}-A_{i,b})C_{i,b}$. Note that $\{A_{i,b}\}_{i=1}^{+\infty}$ is a nonincreasing sequence and $\lim_{b\to 0}\sup_i A_{i,b}=G_1'(x)$ as seen from (13). Let $\hat{D}_{T,b}$ be defined by replacing ϕ_j and λ_j with $\tilde{\phi}_{j,b}$ and $\tilde{\lambda}_{j,b}$ in the definition of $\hat{D}_T$. It is not hard to see that as $b+1/(bT)\to 0$,

$$\begin{aligned}\lim_{N\to+\infty}A_{N,b}C_{N,b}=&\sigma^2G'(x)\left(E[\hat{D}_{T,b}]/\sigma^2-\sum_{j=1}^{+\infty}\tilde{\lambda}_{j,b}\right)(1+o(1))\\ =&-\frac{G'(x)g_q\sum_{h=-\infty}^{+\infty}|h|^q\gamma_X(h)}{(bT)^q}(1+o(1))+O(1/T),\end{aligned}$$

where we have used the fact $E[\hat{D}_{T,b}]/\sigma^2-\sum_{j=1}^{+\infty}\tilde{\lambda}_{j,b}=-\frac{g_q\sum_{h=-\infty}^{+\infty}|h|^q\gamma_X(h)}{\sigma^2(bT)^q}(1+o(1))+O(1/T)$, which can be proved by using similar arguments in the proof of Lemma 2 in Sun et al. [7]. On the other hand, observe that $|\sum_{i=1}^{N-1}(A_{i+1,b}-A_{i,b})C_{i,b}|\le\sup_{i\in\mathbb{N}}|C_{i,b}|\sum_{i=1}^{N-1}(A_{i,b}-A_{i+1,b})\le\sup_{i\in\mathbb{N}}|C_{i,b}|(A_{1,b}-\lim_{N\to+\infty}A_{N,b})=o(|\lim_{N\to+\infty}C_{N,b}|)$ as $b+1/(bT)\to 0$, for all N. Hence we get

$$I_{1T,b}=-\frac{xG'(x)g_q\sum_{h=-\infty}^{+\infty}|h|^q\gamma_X(h)}{\sigma^2(bT)^q}(1+o(1))+O(1/T).$$

Define $H_{i,b}=\tilde{\lambda}_{i,b}E\left[v_i^4(v_i^2-1)G_1''\left(x\left(\sum_{j\neq i}^{+\infty}\tilde{\lambda}_{j,b}v_j^2+a_i\tilde{\lambda}_{i,b}v_i^2\right)\right)\right]$ and $\tilde{S}_{N,b}=\sum_{i=1}^{+\infty}H_{i,b}B_{i,b}$. Again using summation by parts, we obtain $\tilde{S}_{N,b}=H_{N,b}C_{N,b}-\sum_{i=1}^{N-1}(H_{i+1,b}-H_{i,b})C_{i,b}$. By (13), we can show that $\sup_i|H_{i,b}/\tilde{\lambda}_{i,b}-12G_1''(x)|=O(\sqrt{b})$. Therefore, we get $\lim_{N\to+\infty} C_{N,b}H_{N,b} = o(\lim_{N\to+\infty} C_{N,b})$ and

$$\left|\sum_{i=1}^{N-1}(H_{i+1,b}-H_{i,b})C_{i,b}\right| \leq \sup_{i\in\mathbb{N}}|C_{i,b}|\left\{\sum_{i=1}^{N-1}(|H_{i+1,b}-12\tilde{\lambda}_{i+1,b}G_1''(x)|+12G''(x)(\tilde{\lambda}_{i,b}-\tilde{\lambda}_{i+1,b}) + |12\tilde{\lambda}_{i,b}G_1''(x)-H_{i,b}|)\right\} = O\left(\sqrt{b}\lim_{N\to+\infty} C_{N,b}\right).$$

The conclusion follows from the above arguments by noting that $I_{2T,b} = o(I_{1T,b})$. $\diamondsuit$

1.3. *Proof of the main results in Section 3.*

LEMMA 1.13. *Let $\omega_l(x) = (1-|x/l|)\mathbf{I}\{|x/l|<1\}$. Suppose that $m^3/l^2+(ml)^3/T+1/m \to 0$ and $\sum_{h=-\infty}^{+\infty} h^2|\gamma_X(h)| < \infty$. Then under the Gaussian assumption, we have*

$$\sup_{0\leq k\leq m}\left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\hat{\gamma}_X(h) - \sum_{h=1-T}^{T-1} g_{k,T}(h)\gamma_X(h)\right| = O_p(\sqrt{m^3/l^2+(ml)^3/T}),$$

where $|g_{k,T}(h)| \leq C(k|h|+|h|+1)$ for $0\leq k\leq m$ and $|h|\leq T$, and the constant C does not depend on k and h.

PROOF OF LEMMA 1.13. Note first that for any $\epsilon > 0$,

$$\begin{aligned}
&P\left(\sup_{0\leq k\leq m}\left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\hat{\gamma}_X(h) - \sum_{h=1-T}^{T-1} g_{k,T}(h)\gamma_X(h)\right| > \epsilon\right) \\
&\leq \sum_{k=0}^{m} P\left(\left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\hat{\gamma}_X(h) - \sum_{h=1-T}^{T-1} g_{k,T}(h)\gamma_X(h)\right| > \epsilon\right) \\
&\leq \frac{1}{\epsilon^2}\sum_{k=0}^{m} E\left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\hat{\gamma}_X(h) - \sum_{h=1-T}^{T-1} g_{k,T}(h)\gamma_X(h)\right|^2 \\
&\leq \frac{2}{\epsilon^2}\sum_{k=0}^{m} E\left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\{\omega_l(h)\hat{\gamma}_X(h) - \gamma_X(h)\}\right|^2 + \frac{Cm^3}{l^2\epsilon^2}.
\end{aligned}$$

Let $z_i = X_i - E[X_i]$ and $w_{i|h|} = z_i z_{i+|h|} - \gamma_X(h)$. Simple calculation yields that

$$\begin{aligned}
&\left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\{\omega_l(h)\hat{\gamma}_X(h) - \gamma_X(h)\}\right| = \left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\left\{\hat{\gamma}_X(h) - \gamma_X(h)\right\}\right| + C(k+1)/l \\
&\leq \left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\left\{\frac{1}{T}\sum_{i=1}^{T-|h|} w_{i|h|}\right\}\right| + \left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\left\{\frac{T-|h|}{T}\bar{z}^2\right\}\right| \\
&\quad + \left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\omega_l(h)\left\{\frac{\bar{z}}{T}\sum_{i=1}^{T-|h|}(z_i + z_{i+|h|})\right\}\right| + C(k+1)/l := I_{1T} + I_{2T} + I_{3T} + C(k+1)/l,
\end{aligned}$$

which implies that $E\left|\sum_{h=1-l}^{l-1} g_{k,T}(h)\{\omega_l(h)\hat{\gamma}_X(h)-\gamma_X(h)\}\right|^2 \le C(EI_{1T}^2+EI_{2T}^2+EI_{3T}^2+(k+1)^2/l^2)$. We proceed to derive the order of EI_{1T}^2. Notice that

$$
\begin{aligned}
EI_{1T}^2 =&\frac{1}{T^2}\sum_{h_1,h_2=1-l}^{l-1}\sum_{i_1=1}^{T-|h_1|}\sum_{i_2=1}^{T-|h_2|}\mathrm{cov}(w_{i_1|h_1|},w_{i_2|h_2|})g_{k,T}(h_1)g_{k,T}(h_2)\omega_l(h_1)\omega_l(h_2)\\
\le&\frac{C(k+1)^2}{T^2}\sum_{h_1,h_2=1-l}^{l-1}\sum_{i_1=1}^{T-|h_1|}\sum_{i_2=1}^{T-|h_2|}(|h_1|+1)(|h_2|+1)|\gamma_X(i_1-i_2)\gamma_X(i_1-i_2+|h_1|-|h_2|)|\\
&+\frac{C(k+1)^2}{T^2}\sum_{h_1,h_2=1-l}^{l-1}\sum_{i_1=1}^{T-|h_1|}\sum_{i_2=1}^{T-|h_2|}(|h_1|+1)(|h_2|+1)|\gamma_X(i_1-i_2-|h_2|)\gamma_X(i_1-i_2+|h_1|)|\\
\le&\frac{C(k+1)^2}{T^2}\sum_{h_1,h_2=1-l}^{l-1}\sum_{s=1-T}^{T-1}(T-|s|)(|h_1|+1)(|h_2|+1)|\gamma_X(s)\gamma_X(s+|h_1|-|h_2|)|\\
&+\frac{C(k+1)^2}{T^2}\sum_{h_1,h_2=1-l}^{l-1}\sum_{s=1-T}^{T-1}(T-|s|)(|h_1|+1)(|h_2|+1)|\gamma_X(s-|h_2|)\gamma_X(s+|h_1|)| := \mathcal{J}_{1,T}+\mathcal{J}_{2,T}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\mathcal{J}_{1,T} \le&\frac{C(k+1)^2}{T}\sum_{h_1,h_2=1-l}^{l-1}(|h_1|+1)(|h_2|+1)\sum_{s=-\infty}^{+\infty}|\gamma_X(s)\gamma_X(s+|h_1|-|h_2|)|\\
\le&\frac{C(k+1)^2l^2}{T}\sum_{s=-\infty}^{+\infty}|\gamma_X(s)|\sum_{h_1,h_2=1-l}^{l-1}|\gamma_X(s+|h_1|-|h_2|)| \le \frac{C(k+1)^2l^3}{T},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{J}_{2,T} \le&\frac{C(k+1)^2}{T}\sum_{h_1,h_2=1-l}^{l-1}(|h_1|+1)(|h_2|+1)\sum_{s=-\infty}^{+\infty}|\gamma_X(s)\gamma_X(s+|h_1|+|h_2|)|\\
\le&\frac{C(k+1)^2l}{T}\sum_{s=-\infty}^{+\infty}|\gamma_X(s)|\sum_{h_1,h_2=1-l}^{l-1}(|h_1|+|h_2|+1)|\gamma_X(s+|h_1|+|h_2|)|\\
\le&\frac{C(k+1)^2l}{T}\sum_{s=-\infty}^{+\infty}|\gamma_X(s)|\sum_{v=1}^{2l-1}v^2|\gamma_X(s+v)| \le \frac{C(k+1)^2l}{T}.
\end{aligned}
$$

It implies that $EI_{1T}^2 \le \frac{C(k+1)^2l^3}{T}$. Applying similar arguments to I_{2T} and I_{3T}, we get $EI_{2T}^2 \le C(k+1)^2l^4/T^2$ and $EI_{3T}^2 \le C(k+1)^2l^4/T^2$. Note the constant C above does not depend on m by the assumption. We then have

$$
P\left(\sup_{0\le k\le m}\left|\sum_{h=1-l}^{l-1}g_{k,T}(h)\omega_l(h)\hat{\gamma}_X(h)-\sum_{h=1-T}^{T-1}g_{k,T}(h)\gamma_X(h)\right|>\epsilon\right)\le\frac{C}{\epsilon^2}(m^3/l^2+(ml)^3/T)\to 0.
$$

$\diamondsuit$

Proof of Theorem 3.2. We choose m so that $m^3/l^2 + (ml)^3/T + 1/m \to 0$ (e.g., $l \asymp T^{1/5}$ and $m \asymp T^{2/15-\epsilon}$ for some $\epsilon > 0$). From equation (3) in Lemma 1.5, we know that

$$\mathrm{var}(\xi_i) - \sigma^2 - (\mathrm{var}^*(\xi_i^*) - \hat{\sigma}^2) = \frac{1}{T}\left\{\sum_{h=1-T}^{T-1} g_{i,T}(h)\gamma_X(h) - \sum_{h=1-l}^{l-1} g_{i,T}(h)\omega_l(h)\hat{\gamma}_X(h)\right\} - \sum_{|h|\geq T}\gamma_X(h),$$

where $\hat{\sigma}^2 = \sum_{h=1-l}^{l-1}\omega_l(h)\hat{\gamma}_X(h)$ and $g_{0,T}(h) = -|h|$,

$$\begin{aligned} g_{i,T}(h) =& TC_T(\phi_i(s),\phi_i(t)) - \left[\sum_{j=1}^{h}\left\{\phi_i^0\left(\frac{j}{T}\right)\right\}^2 + \sum_{j=T-h+1}^{T}\left\{\phi_i^0\left(\frac{j}{T}\right)\right\}^2\right]\mathbf{I}\{h\geq 1\} \\ &+ \left[\sum_{j=1}^{T-h}\phi_i^0\left(\frac{j}{T}\right)\left\{\phi_i^0\left(\frac{j+h}{T}\right) - \phi_i^0\left(\frac{j}{T}\right)\right\}\right]\mathbf{I}\{h\geq 1\} \\ &+ \left[\sum_{j=1+|h|}^{T}\phi_i^0\left(\frac{j}{T}\right)\left\{\phi_i^0\left(\frac{j+h}{T}\right) - \phi_i^0\left(\frac{j}{T}\right)\right\}\right]\mathbf{I}\{h\leq -1\}, \end{aligned}$$

for $1 \leq i \leq m$. Note that $\sup_{1\leq i\leq m}|TC_T(\phi_i(s),\phi_i(t))| \leq C$. It is not hard to see that $|g_{i,T}(h)| \leq C(|ih| + |h| + 1)$ for $0 \leq i \leq m$. By Lemma 1.13, we know

$$\sup_{0\leq i\leq m}|\mathrm{var}(\xi_i) - \sigma^2 - \mathrm{var}^*(\xi_i^*) + \hat{\sigma}^2| = O_p\left(\frac{\sqrt{m^3/l^2 + (ml)^3/T}}{T}\right).$$

Since the bootstrap sample is normally distributed and $\sum_{h=l-l}^{l-1} h^2\omega_l(h)|\hat{\gamma}_X(h)|$ is bounded in probability in view of the fact that $\sum_{h=-\infty}^{+\infty} h^2\omega_l(h)E|\hat{\gamma}_X(h)| < \infty$, Theorem 2.2 is also applicable to the bootstrap sample, i.e.,

$$\sup_{x\in[0,\infty)}|P(F_T^*(\infty)\leq x) - Q_\infty(x) - \aleph_T^*(x;\infty)| = o_p(1/T),$$

where $\aleph_T^*(x;\infty) = \frac{1}{2\hat{\sigma}^2}\sum_{i=0}^{\infty}(\mathrm{var}^*(\xi_i^*) - \hat{\sigma}^2)E[(v_i^2-1)\mathbf{I}\{\mathcal{F}(v;\infty)\leq x\}]$. It is not hard to show that $\hat{\sigma}^2 - \sigma^2 = O_p(\sqrt{l/T + 1/l^2})$. Note that $\mathrm{var}^*(\xi_i^*) - \hat{\sigma}^2 = \frac{1}{T}\sum_{h=1-l}^{l-1} g_{i,T}(h)\omega_l(h)\hat{\gamma}_X(h)$, which implies that $\sup_{1\leq i<+\infty}\frac{|\mathrm{var}^*(\xi_i^*) - \hat{\sigma}^2|}{i/T + i^2/T^2} = O_p(1)$ (see e.g., (11)). Using the arguments in (12), we can show that

$$\sup_{x\in[0,+\infty)}\left|\frac{1}{2\sigma^2}\sum_{i=m+1}^{\infty}(\mathrm{var}(\xi_i) - \mathrm{var}^*(\xi_i^*) + \hat{\sigma}^2 - \sigma^2)E[(v_i^2-1)\mathbf{I}\{\mathcal{F}(v;\infty)\leq x\}]\right| = O_p\left(\frac{1}{Tm^{a-2}}\right).$$

Thus we get

$$\begin{aligned}
&\sup_{x\in[0,+\infty)} |\aleph_T(x;\infty) - \aleph_T^*(x;\infty)| \\
\leq &\sup_{x\in[0,+\infty)} \left|\frac{1}{2\sigma^2}\sum_{i=0}^{\infty}(\mathrm{var}(\xi_i) - \mathrm{var}^*(\xi_i^*) + \hat{\sigma}^2 - \sigma^2)E[(v_i^2-1)\mathbf{I}\{\mathcal{F}(v;\infty)\leq x\}]\right| \\
&+ \sup_{x\in[0,+\infty)} \left|\left(\frac{1}{2\hat{\sigma}^2} - \frac{1}{2\sigma^2}\right)\sum_{i=1}^{\infty}(\mathrm{var}^*(\xi_i^*) - \hat{\sigma}^2)E[(v_i^2-1)\mathbf{I}\{\mathcal{F}(v;\infty)\leq x\}]\right| \\
\leq &\frac{1}{2\sigma^2}\sup_{1\leq i\leq m}|\mathrm{var}(\xi_i) - \sigma^2 - \mathrm{var}^*(\xi_i^*) + \hat{\sigma}^2| \sup_{x\in[0,+\infty)}\left|\sum_{i=1}^{m}E[(v_i^2-1)\mathbf{I}\{\mathcal{F}(v;\infty)\leq x\}]\right| \\
&+ \sup_{x\in[0,+\infty)}\left|\frac{1}{2\sigma^2}\sum_{i=m+1}^{\infty}(\mathrm{var}(\xi_i) - \mathrm{var}^*(\xi_i^*) + \hat{\sigma}^2 - \sigma^2)E[(v_i^2-1)\mathbf{I}\{\mathcal{F}(v;\infty)\leq x\}]\right| + O_p\left(\frac{\sqrt{l/T + 1/l^2}}{T}\right) \\
= &O_p\left(\frac{\sqrt{m^3/l^2 + (ml)^3/T}}{T}\right) + O_p\left(\frac{\sqrt{l/T+1/l^2}}{T}\right) + O_p\left(\frac{1}{Tm^{a-2}}\right).
\end{aligned}$$

It then follows that $\sup_{x\in[0,+\infty)}|P(F_T(\infty)\leq x) - P(F_T^*(\infty)\leq x)| \leq \sup_{x\in[0,+\infty)}|\aleph_T(x;\infty) - \aleph_T^*(x;\infty)| + o_p(1/T) = o_p(1/T)$. ♢

Proof of Theorem 3.1. The proof is similar to those of Lemma 1.13 and Theorem 3.2. The details are omitted. ♢

2. Simulation study. We conduct a small simulation study to compare and contrast the finite sample performance of the small-b approximation, fixed-b approximation, MBB and Gaussian dependent bootstrap (GDB). Following the setup in Gonçalves and Vogelsang [2], we consider the AR(1) model,

$$y_t = \rho y_{t-1} + \sqrt{1-\rho^2}\varepsilon_t, \quad t = 1, 2, \ldots, T, \tag{14}$$

with $\{\varepsilon_t\}$ being a sequence of iid $N(0,1)$, $t(3)$ or $\exp(1)-1$ random variables. Consider the Wald statistic based on the HAC estimator with the Bartlett kernel and QS kernel for testing the null hypothesis $E[y_t]=0$ versus the alternative that $E[y_t]\neq 0$ at 5% nominal level. Throughout the simulation we set $T=50$ and the number of Monte Carlo replications to be 1000. The bootstrap tests are based on 1000 replications for each sample. We implement the MBB in a 'naive' fashion as described in Gonçalves and Vogelsang [2]. The simulation results for $b = 0.04, 0.06, 0.08, 0.1, 0.2, \ldots, 1$ and $\rho = -0.7, 0, 0.5, 0.9$ are summarized in Figures 1-3. We present the results for GDB with $l = 5, 10$ and MBB with block size equal to 5 and 10. It is seen from the figures that the GDB is more accurate than the small-b asymptotic approximation in most cases and improvement is often substantial especially for large b. In the dependent cases (e.g., $\rho = -0.7, 0.5$ and 0.9), the GDB tends to provide a refinement over the fixed-b approximation for a proper bandwidth which is consistent with our theoretical findings. The improvement is apparent when the dependence is strong and b is small. In addition it is interesting to note that the GDB not only provides an improvement when the innovations are Gaussian but also in the case of $t(3)$ distributed fat tailed innovations and $\exp(1) - 1$ distributed skewed innovation. The performance of GDB and MBB is in general comparable. MBB delivers slightly better size in most cases when the dependence is positive. When $\rho = 0.9$, the MBB with block size 10 apparently outperforms all the other methods for all three cases, suggesting that with a proper choice of block size, the MBB is capturing not only the asymptotic bias and variance of long run variance estimator but also the higher order moments. Since the GDB only captures the second order properties, it is not suprising that it can be inferior to MBB in some cases. Overall, the simulation results are consistent with those in Gonçalves and Vogelsang [2], and demonstrate the effectiveness of the proposed Gaussian dependent bootstrap in the Gaussian setting. The simulation results also suggest that our procedure may be useful in some non-Gaussian settings, though it can hardly be justified theoretically. The moving block bootstrap is expected to be second order accurate under the fixed-smoothing asymptotics, as seen from its empirical performance, but a rigorous theoretical justification seems very difficult.

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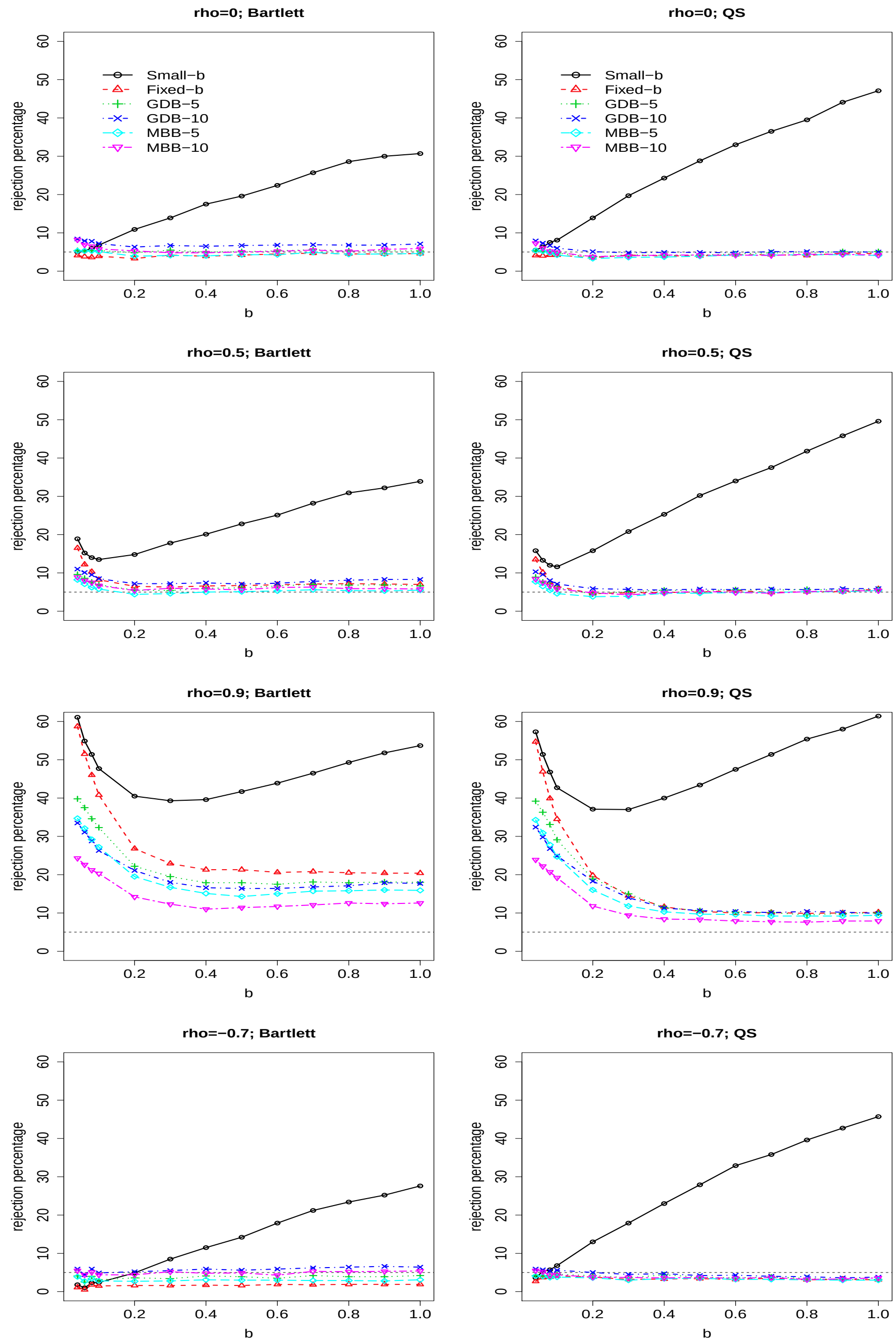

Fig 1. *Empirical rejection probabilities for the Wald statistic with the Bartlett kernel (left panel) and QS kernel (right panel) and for the AR(1) model with* $N(0,1)$ *innovations*

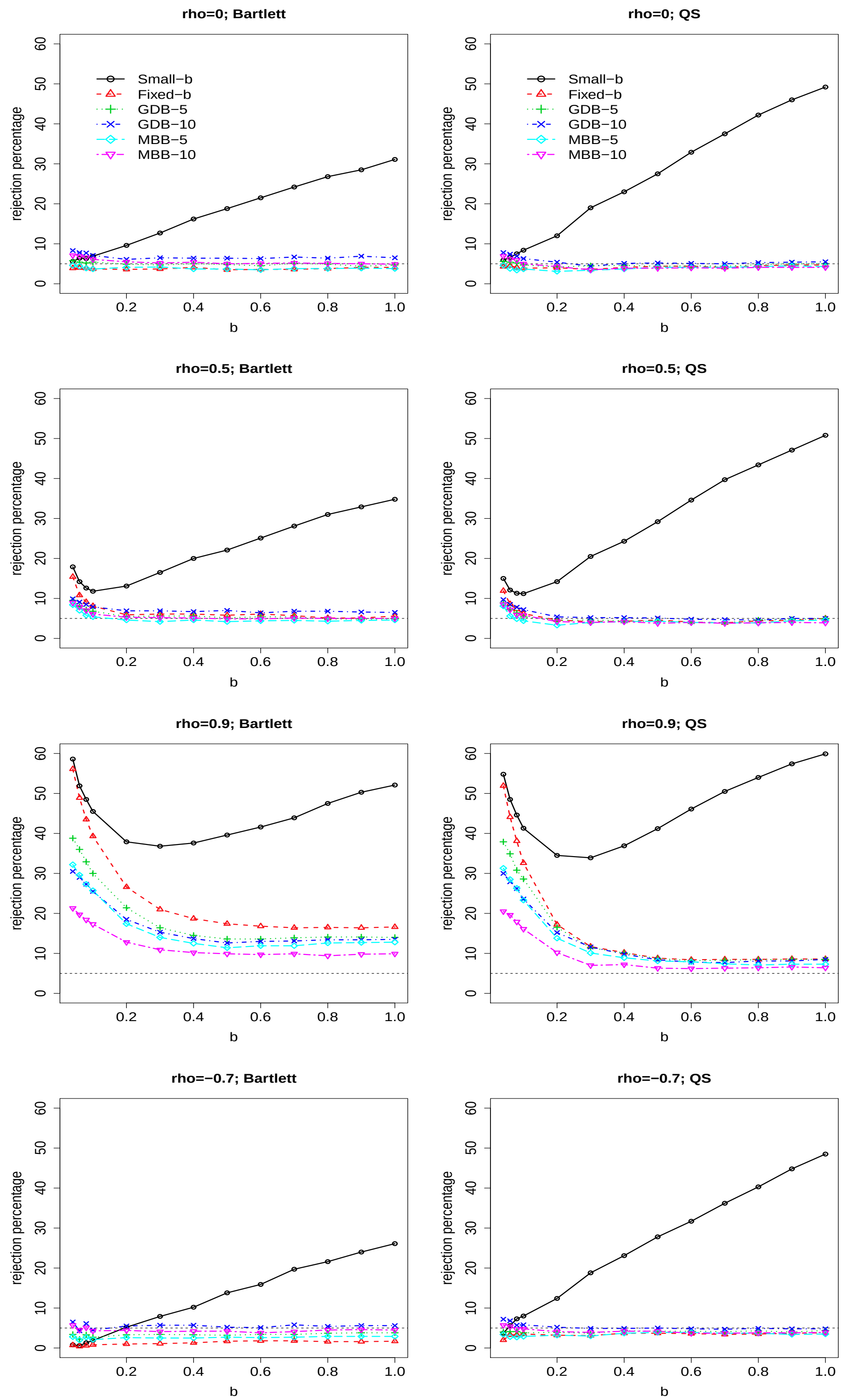

FIG 2. *Empirical rejection probabilities for the Wald statistic with the Bartlett kernel (left panel) and QS kernel (right panel) and for the AR(1) model with* $t(3)$ *innovations*

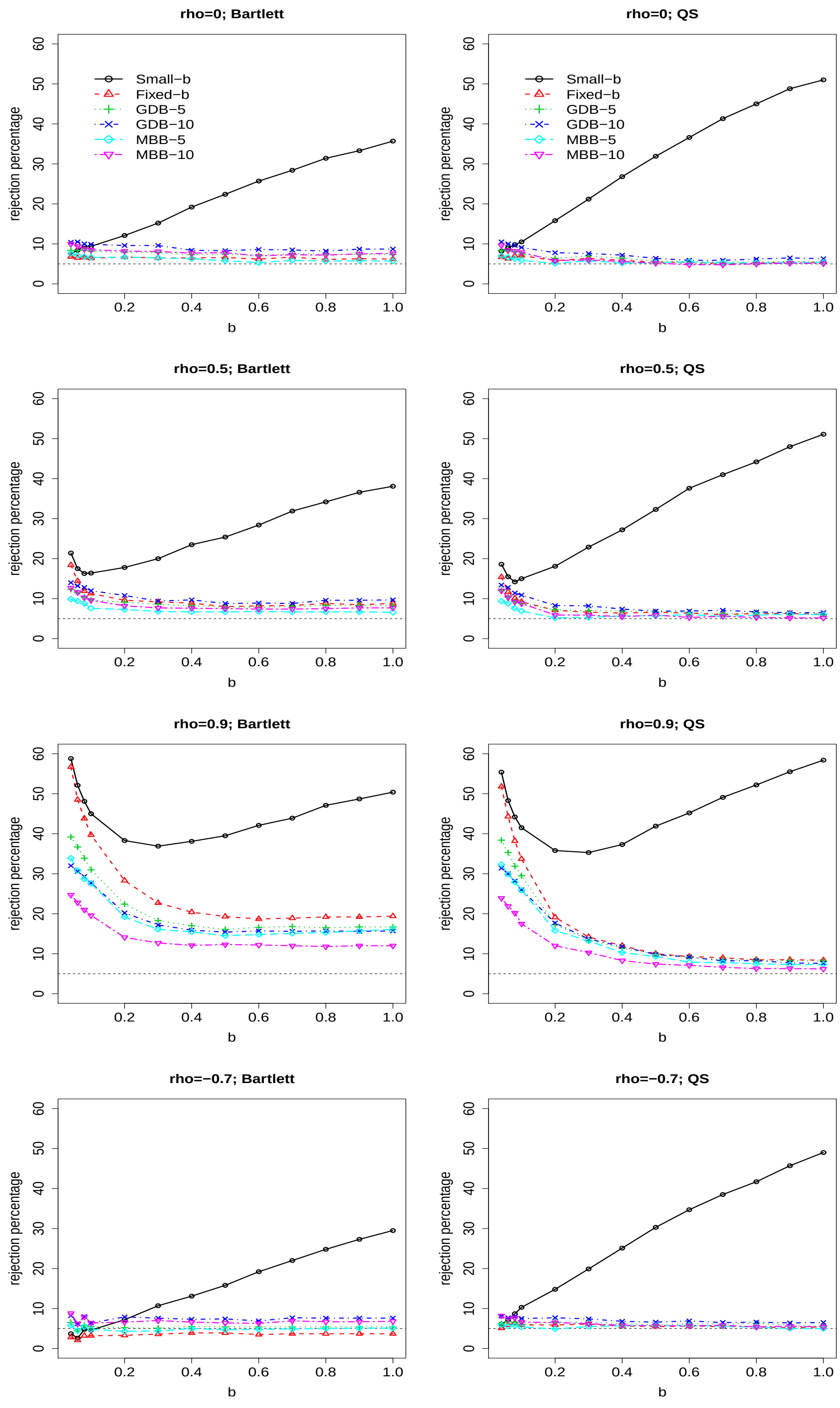

FIG 3. *Empirical rejection probabilities for the Wald statistic with the Bartlett kernel (left panel) and QS kernel (right panel) and for the AR(1) model with* $\exp(1) - 1$ *innovations*