Supplementary Materials: The Dependent Wild Bootstrap

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The following supplementary materials contain proofs of Theorems 4.1, 4.2 and 5.2, some useful lemmas and their proofs, as well as some additional tables that show the normalized MSEs and coverages for the case of randomly sampled time points with spherical covariance functions.

0.1 Proofs of Theorems 4.1, 4.2 & 5.2

Proof of Theorem 4.1: Let $b_n = p_n + l_n$, where $p_n = \lfloor \sqrt{n} \rfloor$. Define a block of observations as $\mathcal{B}_n(i) = i + b_n(0,1]$, $i \in \mathbb{Z}$. We first divide R_n into non-overlapped blocks of observations. Let $\mathcal{K}_n = \{k \in \mathbb{Z} : \ \mathcal{B}_n(b_n k) \subset R_n\}$ represents the index set of all complete blocks $\mathcal{B}_n(b_n k) = b_n(k + (0,1])$ lying inside R_n. For each $k \in \mathcal{K}_n$, we further divide each block into large and small blocks, i.e., $\mathcal{B}_n(k) = \mathcal{B}_n^{(L)}(k) \cup \mathcal{B}_n^{(S)}(k)$, where $\mathcal{B}_n^{(L)}(k) = k + p_n(0,1]$ and $\mathcal{B}_n^{(S)}(k) = \mathcal{B}_n(k) - \mathcal{B}_n^{(L)}(k)$. Denote by Σ_L (Σ_S) the sum over all time points in the big (small) blocks and Σ_{NB} the sum over all time points that are not in the complete blocks, i.e., $\{R_n - \cup_{k \in \mathcal{K}_n} \mathcal{B}_n(k)\} \cap \mathbb{Z}$. Write $\Sigma_B = \Sigma_L + \Sigma_S$. For $\|\mathbf{v}\|_1 = 1$, denote by $\hat{f}_{n,\mathbf{v}}(0)$ the lag window estimator of the univariate time series $\{\mathbf{X}_t^{\mathbf{v}}\}$. Denote by $c_{\mathbf{v}} = D^{\mathbf{v}} H(\boldsymbol{\mu})/\mathbf{v}!$ and $\hat{c}_{\mathbf{v}} = D^{\mathbf{v}} H(\bar{\mathbf{X}}_n)/\mathbf{v}!$.

We apply a third-order Taylor expansion to $\hat{\theta}_n^* = H(\bar{\mathbf{X}}_n^*)$ around $\bar{\mathbf{X}}_n$ and write $\hat{\theta}_n^* - H(\bar{\mathbf{X}}_n) = J_{1n} + J_{2n} + J_{3n}$, where

$$J_{1n} = \sum_{\|\mathbf{v}\|_1=1} \hat{c}_{\mathbf{v}} (\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}}, \quad J_{2n} = \sum_{\|\mathbf{v}\|_1=2} \hat{c}_{\mathbf{v}} (\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}},$$

$$J_{3n} = 3 \sum_{\|\mathbf{v}\|_1=3} (\mathbf{v}!)^{-1} (\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}} \int_0^1 (1-w)^2 D^{\mathbf{v}} H\{\bar{\mathbf{X}}_n + w(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)\} dw.$$

In the sequel, we shall show

$$\begin{aligned}\mathbb{E}\{n\mathrm{var}^*(J_{1n})\} &= \tau_\infty^2 + B_0/l_n^2 + o(l_n^{-2}), &(1)\\ \mathbb{E}\{n\mathrm{var}^*(J_{2n})\} &= O(l_n^2/n), &(2)\\ \mathbb{E}\{n\mathrm{var}^*(J_{3n})\} &= O(l_n^3/n^2), &(3)\\ \mathbb{E}\{n\mathrm{cov}^*(J_{1n}, J_{2n})\} &= O(l_n^2/n). &(4)\end{aligned}$$

If (1)-(4) hold, then by the Cauchy-Schwarz inequality, we get $n\mathbb{E}|\mathrm{cov}^*(J_{1n}, J_{3n})| = O(l_n^{3/2}/n) = o(l_n^{-2})$ and $n\mathbb{E}|\mathrm{cov}^*(J_{2n}, J_{3n})| = O(l_n^{5/2}/n^{3/2}) = o(l_n^{-2})$ under the assumption that $l_n = o(n^{1/4})$. Thus the conclusion follows.

To show (1), we write $J_{1n} = J_{11n} + J_{12n}$ where $J_{11n} = \boldsymbol{\nabla}'(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)$ and $J_{12n} = \sum_{\|\mathbf{v}\|_1=1}(\hat{c}_{\mathbf{v}} - c_{\mathbf{v}})(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}}$. Denote by $Y_{t_j} = \boldsymbol{\nabla}'\mathbf{X}_{t_j}$ and $\bar{Y}_n = \boldsymbol{\nabla}'\bar{\mathbf{X}}_n$. We can write

$$\begin{aligned}n\mathrm{var}^*(J_{11n}) &= n^{-1}\mathrm{var}^*\left[(\Sigma_B + \Sigma_{NB})\, W_{t_j}\{Y_{t_j} - \bar{Y}_n\}\right]\\ &= n^{-1}\mathrm{var}^*(\Sigma_{NB}) + n^{-1}\mathrm{cov}^*(\Sigma_B, \Sigma_{NB}) + n^{-1}\mathrm{var}^*(\Sigma_B).\end{aligned}$$

Further write $n^{-1}\mathrm{var}^*(\Sigma_B) = V_{1n} + V_{2n}$, where

$$\begin{aligned}V_{1n} &= n^{-1}\sum_{k\in\mathcal{K}_n}\sum_{t,t'=k+1}^{k+b_n}\{Y_t - \bar{Y}_n\}\{Y_{t'} - \bar{Y}_n\}a(|t-t'|/l_n),\\ V_{2n} &= n^{-1}\sum_{k\neq k'\in\mathcal{K}_n}\sum_{t=k+1}^{k+b_n}\sum_{t'=k'+1}^{k'+b_n}\{Y_t - \bar{Y}_n\}\{Y_{t'} - \bar{Y}_n\}a(|t-t'|/l_n).\end{aligned}$$

By Theorem 9.3.3 in Anderson (1971) and a similar argument used in Lemma 0.2, we can derive that $\mathbb{E}(V_{1n}) = |\mathcal{K}_n|b_n/n\{\tau_\infty^2 + l_n^{-2}B_0 + o(l_n^{-2})\} = \tau_\infty^2 + l_n^{-2}B_0 + o(l_n^{-2})$. Concerning V_{2n}, since $a(\cdot)$ has compact support on $[-1, 1]$, the summand in V_{2n} is non-vanishing only when $\mathcal{B}_n(k)$ and $\mathcal{B}_n(k')$ are two neighboring blocks. Moreover, for each such pair (k, k'), since $\sum_{k\in\mathbb{Z}}|kr_k| < \infty$, it is easy to see that

$$\mathbb{E}\left[\sum_{t=k+1}^{k+b_n}\sum_{t'=k'+1}^{k'+b_n}\{Y_{t'} - \bar{Y}_n\}\{Y_t - \bar{Y}_n\}a(|t-t'|/l_n)\right] = O(1).$$

Therefore, $\mathbb{E}(V_{2n}) = O(|\mathcal{K}_n|/n) = o(l_n^{-2})$. Regarding the other two terms in $n\mathrm{var}^*(J_{11n})$, since the number of blocks on the boundary is $O(1)$, i.e., $|R_n - \cup_{k\in\mathcal{K}_n}\mathcal{B}_n(b_nk)| \leq Cp_n$, we have $\mathbb{E}\{n^{-1}\text{var}^*(\Sigma_{NB})\} \le Cp_n/n = o(l_n^{-2})$. It follows from the same argument that $\mathbb{E}\{n^{-1}\text{cov}^*(\Sigma_B, \Sigma_{NB})\} = O(n^{-1})$. Hence, $\mathbb{E}\{n\text{var}^*(J_{11n})\} = \tau_\infty^2 + B_0 l_n^{-2} + o(l_n^{-2})$.

Next we consider $\mathbb{E}^*(J_{12n}^2)$. Denote by $c_{\mathbf{vu}}(\mathbf{x}) = D^{\mathbf{v}}\{D^{\mathbf{u}}H(\mathbf{x})\}$ for $\|\mathbf{v}\|_1 = 1, \|\mathbf{u}\|_1 = 1$. Note that $J_{12n} = \sum_{\|\mathbf{u}\|_1=1}\sum_{\|\mathbf{v}\|_1=1} c_{\mathbf{vu}}(\tilde{\mathbf{X}}_n)(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}}(\bar{\mathbf{X}}_n - \boldsymbol{\mu})^{\mathbf{u}}$, where $\tilde{\mathbf{X}}_n = \boldsymbol{\mu} + w(\bar{\mathbf{X}}_n - \boldsymbol{\mu})$ for some $w \in (0,1)$. For each $\|\mathbf{v}\|_1 = 1$, $\hat{f}_{n,\mathbf{v}}(0) = n\mathbb{E}^*\{(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}}\}^2/(2\pi)$. Then we derive

$$\begin{aligned}
\mathbb{E}\{n\mathbb{E}^*(J_{12n}^2)\} &\le Cn \sum_{\|\mathbf{u}\|_1=1}\sum_{\|\mathbf{v}\|_1=1} \mathbb{E}[\mathbb{E}^*\{c_{\mathbf{vu}}(\tilde{\mathbf{X}}_n)(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}}(\bar{\mathbf{X}}_n - \boldsymbol{\mu})^{\mathbf{u}}\}^2] \\
&\le C \sum_{\|\mathbf{u}\|_1=1}\sum_{\|\mathbf{v}\|_1=1} \mathbb{E}[c_{\mathbf{vu}}^2(\tilde{\mathbf{X}}_n)\{(\bar{\mathbf{X}}_n - \boldsymbol{\mu})^{\mathbf{u}}\}^2 \hat{f}_{n,\mathbf{v}}(0)] \\
&\le C \sum_{\|\mathbf{u}\|_1=1}\sum_{\|\mathbf{v}\|_1=1} \mathbb{E}^{1/4}\{\hat{f}_{n,\mathbf{v}}^4(0)\}\mathbb{E}^{1/2}\{c_{\mathbf{vu}}^4(\tilde{\mathbf{X}}_n)\}\mathbb{E}^{1/4}\{(\bar{\mathbf{X}}_n - \boldsymbol{\mu})^{\mathbf{u}}\}^8.
\end{aligned}$$

By Lemmas 0.1 and 0.2 and Condition D_2, $\mathbb{E}(n\mathbb{E}^* J_{12n}^2) = O(n^{-1})$. By the Cauchy-Schwarz inequality, $|\mathbb{E}\{n\text{cov}^*(J_{11n}, J_{12n})\}| \le \sqrt{\mathbb{E}(n\mathbb{E}^* J_{12n}^2)}\sqrt{\mathbb{E}\{n\text{var}^*(J_{11n})\}} = o(l_n^{-2})$, which implies (1).

As for (2), applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\mathbb{E}\{n\text{var}^*(J_{2n})\} \le Cn\mathbb{E}\left[\sum_{\|\mathbf{v}\|_1=2}\sum_{\|\mathbf{v}'\|_1=2} \hat{c}_{\mathbf{v}}\hat{c}_{\mathbf{v}'}\mathbb{E}^*\{(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}}(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}'}\}\right] \\
&\quad \le Cn \sum_{\|\mathbf{v}\|_1=2}\sum_{\|\mathbf{v}'\|_1=2} \mathbb{E}^{1/4}\hat{c}_{\mathbf{v}}^4\mathbb{E}^{1/4}\hat{c}_{\mathbf{v}'}\mathbb{E}^{1/4}\{\mathbb{E}^*|(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}}|^4\}\mathbb{E}^{1/4}\{\mathbb{E}^*|(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}'}|^4\} \\
&\quad \le Cn \sum_{\|\mathbf{v}\|_1=2}\sum_{\|\mathbf{v}'\|_1=2} \mathbb{E}^{1/2}(1 + \|\bar{\mathbf{X}}_n\|^{4\kappa_2})\mathbb{E}^{1/2}\{\mathbb{E}^*\|\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n\|^8\} = O(l_n^2/n)
\end{aligned}$$

in view of Lemma 0.3. Regarding (3), we apply the condition D_3 and get

$$\begin{aligned}
\mathbb{E}^*(J_{3n}^2) &\le C\mathbb{E}^*\{\|\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n\|^6(1 + \|\bar{\mathbf{X}}_n\|^{2\kappa_3} + \|\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n\|^{2\kappa_3})\} \\
&\le C\{(1 + \|\bar{\mathbf{X}}_n\|^{2\kappa_3})\mathbb{E}^*\|\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n\|^6 + \mathbb{E}^*\|\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n\|^{6+2\kappa_3}\}.
\end{aligned}$$

By Lemma 0.3 and Hölder's inequality, (3) follows. Finally, we have

$$\begin{aligned}\mathrm{cov}^*(J_{1n}, J_{2n}) &= \sum_{\|\mathbf{v}\|_1=1} \sum_{\|\mathbf{u}\|_1=2} \hat{c}_{\mathbf{v}} \hat{c}_{\mathbf{u}} \mathbb{E}^* \{ (\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{v}} (\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)^{\mathbf{u}} \} \\ &= \sum_{\|\mathbf{v}\|_1=1} \sum_{\|\mathbf{u}_1\|_1=1} \sum_{\|\mathbf{u}_2\|_1=1} \hat{c}_{\mathbf{v}} \hat{c}_{\mathbf{u}_1+\mathbf{u}_2} n^{-3} \sum_{j_1,j_2,j_3=1} \{\mathbf{X}_{t_{j_1}} - \bar{\mathbf{X}}_n\}^{\mathbf{v}} \\ &\qquad \{\mathbf{X}_{t_{j_2}} - \bar{\mathbf{X}}_n\}^{\mathbf{u}_1} \{\mathbf{X}_{t_{j_3}} - \bar{\mathbf{X}}_n\}^{\mathbf{u}_2} \mathbb{E}^* \{ W_{t_{j_1}} W_{t_{j_2}} W_{t_{j_3}} \}.\end{aligned}$$

Due to the l_n-dependence of W_{t_j}, the number of non-vanishing terms involved in the above sum $\sum_{j_1,j_2,j_3=1}^n$ is $O(l_n^2 n)$. Further, for each $(\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2)$, it follows from the condition D_3 that $\left|\mathbb{E}\left[\hat{c}_{\mathbf{v}} \hat{c}_{\mathbf{u}_1+\mathbf{u}_2} \{\mathbf{X}_{t_{j_1}} - \bar{\mathbf{X}}_n\}^{\mathbf{v}} \{\mathbf{X}_{t_{j_2}} - \bar{\mathbf{X}}_n\}^{\mathbf{u}_1} \{\mathbf{X}_{t_{j_3}} - \bar{\mathbf{X}}_n\}^{\mathbf{u}_2}\right]\right|$ is bounded by $C\mathbb{E}^{1/4}(1+ \|\bar{\mathbf{X}}_n\|^{\kappa_1})^4 \mathbb{E}^{1/4}(1 + \|\bar{\mathbf{X}}_n\|^{\kappa_2})^4 \mathbb{E}^{1/2} \|\mathbf{X}_{t_j}\|^6 < \infty$. This implies (4). So the proof is complete.

$\diamondsuit$

Proof of Theorem 4.2: The proof follows the same argument as presented in the proof of Theorem 4.1. Since there is no additional technical difficulties, we omit the details.

$\diamondsuit$

Proof of Theorem 5.2: We shall only deal with case (i) using the large-block-small-block argument as case (ii) can be handled in a similar fashion. Following the notation introduced in the proof of Theorem 4.1, we let p_n (l_n) be the size of a large (small) block, where p_n satisfies $l_n/p_n + p_n/\lambda_n = o(1)$ (e.g., $p_n = \lfloor \sqrt{l_n \lambda_n} \rfloor$). Let $b_n = p_n + l_n$. Denote by $\mathcal{K}_n = \{k \in \mathbb{Z} : \mathcal{B}_n(b_n k) \subset R_n\}$ the index set of all complete blocks $\mathcal{B}_n(b_n k) = b_n(k + (0, 1])$ lying inside R_n and by $\bar{\mathcal{K}}_n = \{k \in \mathbb{Z} : \mathcal{B}_n(b_n k) \cap R_n \neq \emptyset,\ k \notin \mathcal{K}_n\}$ the index set of all incomplete blocks. Denote by $\mathcal{B}_n^{(L)}(k) = k + p_n(0, 1]$ and $\mathcal{B}_n^{(S)}(k) = \mathcal{B}_n(b_n k) - \mathcal{B}_n^{(L)}(k)$, $k \in \mathcal{K}_n$ a large block and a small block respectively. Write

$$\begin{aligned}\sqrt{n}(\bar{X}_n^* - \bar{X}_n) &= n^{-1/2} \sum_{j=1}^n W(t_j)\{X(t_j) - \bar{X}_n\} \\ &= n^{-1/2} \sum_{j=1}^n W(t_j)\{X(t_j) - \mu\} + n^{-1/2} \sum_{j=1}^n W(t_j)(\mu - \bar{X}_n) =: D_{1n} + D_{2n},\end{aligned}$$

where D_{1n} and D_{2n} are implicitly defined. It is easy to show that $\mathbb{E}^*\{\sum_{j=1}^n W(t_j)\}^2 = \sum_{j,j'=1}^n a\{(t_j - t_{j'})/l_n\}$, whose expectation is $O(l_n n^2 \lambda_n^{-1})$. Since $\bar{X}_n - \mu = O_p(n^{-1/2})$ under case (i), $\mathbb{E}^*(D_{2n}^2) = O_p(l_n/\lambda_n) = o_p(1)$, which implies that D_{2n} is asymptotically negligible. Define $s_n^{(L)}(k) = n^{-1/2}\sum_{j:t_j\in\mathcal{B}_n^{(L)}(k)} W(t_j)\{X(t_j)-\mu\}$ for $k \in \mathcal{K}_n$. By the summation convention, the above summation $s_n^{(L)}(k)$ is zero if the set $\{j : t_j \in \mathcal{B}_n^{(L)}(k)\}$ is empty. Similarly, we define

$$\begin{aligned} s_n^{(S)}(k) &= n^{-1/2} \sum_{j:t_j\in\mathcal{B}_n^{(S)}(k)} W(t_j)\{X(t_j)-\mu\},\ k \in \mathcal{K}_n, \\ s_n^{(NB)}(k) &= n^{-1/2} \sum_{j:t_j\in R_n\cap\mathcal{B}_n(b_n k)} W(t_j)\{X(t_j)-\mu\} \text{ for } k \in \bar{\mathcal{K}}_n. \end{aligned}$$

Now we can write

$$D_{1n} = \sum_{k\in\mathcal{K}_n} s_n^{(L)}(k) + \sum_{k\in\mathcal{K}_n} s_n^{(S)}(k) + \sum_{k\in\bar{\mathcal{K}}_n} s_n^{(NB)}(k) =: S_n^{(L)} + S_n^{(S)} + S_n^{(NB)},$$

which represent the contributions from large blocks, small blocks and boundary blocks, respectively. In the sequel, we shall show that

$$S_n^{(L)} \to_D N\left(0, \gamma(0) + \kappa\iota\int_{\mathbb{R}}\gamma(s)ds\right) \text{ in probability,} \tag{5}$$

$$\mathbb{E}^*\{S_n^{(S)}\}^2 = o_p(1),\quad \mathbb{E}^*\{S_n^{(NB)}\}^2 = o_p(1). \tag{6}$$

To prove (5), we note that conditional on the data, $S_n^{(L)}$ is a sum of mean zero independent random variables. Thus according to the argument in the proof of Theorem 3.2 of Lahiri (2003a), it suffices to verify the following two conditions:

$$\sum_{k\in\mathcal{K}_n} \mathbb{E}^*\{s_n^{(L)}(k)\}^2 = \left\{\gamma(0) + \kappa\iota\int_{\mathbb{R}}\gamma(z)dz\right\}\{1 + o_p(1)\}, \tag{7}$$

$$\sum_{k\in\mathcal{K}_n} \mathbb{E}^*\{s_n^{(L)}(k)\}^4 = o_p(1). \tag{8}$$

The assertion (7) follows from Theorem 5.1 and (6), the latter of which will be shown below. As to (8), we have for $k \in \mathcal{K}_n$,

$$\begin{aligned} \mathbb{E}_{X|Z}[\mathbb{E}^*\{s_n^{(L)}(k)\}^4] = n^{-2} &\sum_{j_1,j_2,j_3,j_4=1}^{n} \mathbf{1}\{t_{j_1} \in \mathcal{B}_n^{(L)}(k),\ t_{j_2} \in \mathcal{B}_n^{(L)}(k),\ t_{j_3} \in \mathcal{B}_n^{(L)}(k), \\ &t_{j_4} \in \mathcal{B}_n^{(L)}(k)\}\mathbb{E}\{W(t_{j_1})W(t_{j_2})W(t_{j_3})W(t_{j_4})\}[\gamma(t_{j_1}-t_{j_2})\gamma(t_{j_3}-t_{j_4}) + \gamma(t_{j_1}-t_{j_3}) \\ &\times\gamma(t_{j_2}-t_{j_4}) + \gamma(t_{j_1}-t_{j_4})\gamma(t_{j_2}-t_{j_3}) + \mathrm{cum}\{X(t_{j_1}), X(t_{j_2}), X(t_{j_3}), X(t_{j_4})\}]. \end{aligned}$$

Since $W(t) \in \mathcal{L}^4$, we can derive that

$$
\begin{aligned}
|\mathbb{E}[\mathbb{E}^*\{s_n^{(L)}(k)\}^4]| \le n^{-2}\mathbb{E}_Z & \sum_{j_1,j_2,j_3,j_4=1}^{n} \mathbf{1}\{t_{j_1} \in \mathcal{B}_n^{(L)}(k),\ t_{j_2} \in \mathcal{B}_n^{(L)}(k),\ t_{j_3} \in \mathcal{B}_n^{(L)}(k), \\
& t_{j_4} \in \mathcal{B}_n^{(L)}(k)\}[|\gamma(t_{j_1}-t_{j_2})||\gamma(t_{j_3}-t_{j_4})| + |\gamma(t_{j_1}-t_{j_3})||\gamma(t_{j_2}-t_{j_4})| + |\gamma(t_{j_1}-t_{j_4})| \\
& \times|\gamma(t_{j_2}-t_{j_3})| + |\mathrm{cum}\{X(t_{j_1}), X(t_{j_2}), X(t_{j_3}), X(t_{j_4})\}|] =: \sum_{k=1}^{4} J_{kn}.
\end{aligned}
$$

For J_{kn}, $k = 1, \cdots, 4$, we can discuss their magnitudes by treating $N(j_1, j_2, j_3, j_4)$ and its complement separately. For example, for J_{1n} and $(j_1, j_2, j_3, j_4) \in N(j_1, j_2, j_3, j_4)$,

$$
\begin{aligned}
& \mathbb{E}_Z[\mathbf{1}\{t_{j_1} \in \mathcal{B}_n^{(L)}(k),\ t_{j_2} \in \mathcal{B}_n^{(L)}(k),\ t_{j_3} \in \mathcal{B}_n^{(L)}(k),\ t_{j_4} \in \mathcal{B}_n^{(L)}(k)\}|\gamma(t_{j_1}-t_{j_2})||\gamma(t_{j_3}-t_{j_4})|] \\
&= \int_{R_0^4} \mathbf{1}\{\lambda_n z_1 \in \mathcal{B}_n^{(L)}(k),\ \lambda_n z_2 \in \mathcal{B}_n^{(L)}(k),\ \lambda_n z_3 \in \mathcal{B}_n^{(L)}(k),\ \lambda_n z_4 \in \mathcal{B}_n^{(L)}(k)\} \\
&\qquad |\gamma\{\lambda_n(z_1-z_2)\}\gamma\{\lambda_n(z_3-z_4)\}|\eta(z_1)\eta(z_2)\eta(z_3)\eta(z_4)dz_1dz_2dz_3dz_4 \\
&= \lambda_n^{-4}\int_{R_n^4} \mathbf{1}\{z_1' \in \mathcal{B}_n^{(L)}(k),\ z_2' \in \mathcal{B}_n^{(L)}(k),\ z_3' \in \mathcal{B}_n^{(L)}(k),\ z_4' \in \mathcal{B}_n^{(L)}(k)\} \\
&\qquad |\gamma(z_1'-z_2')||\gamma(z_3'-z_4')|dz_1'dz_2'dz_3'dz_4' = O(\lambda_n^{-4}b_n^2)
\end{aligned}
$$

under the assumption that $\int_{\mathbb{R}}|\gamma(z)|dz < \infty$. So uniformly in $k \in \mathcal{K}_n$, the overall contribution from $N(j_1, j_2, j_3, j_4)$ to J_{1n} is $O(n^2\lambda_n^{-4}b_n^2)$. When $(j_1, j_2, j_3, j_4) \notin N(j_1, j_2, j_3, j_4)$, we have the following several cases. If $j_1 = j_2 = j_3 = j_4$, then the corresponding sum can be bounded by Cn^{-1}. If there are two equalities among (j_1, j_2, j_3, j_4) (e.g., $j_1 = j_2 \neq j_3 = j_4$), then we can bound the corresponding sum by $C\lambda_n^{-2}b_n$. If there is only one equality among (j_1, j_2, j_3, j_4) (e.g., $j_1 = j_2 \neq j_3 \neq j_4$, $j_2 \neq j_4$), then the bound for the corresponding sum is $Cn\lambda_n^{-3}b_n^2 = o(1)$. Therefore, $|\mathbb{E}[\mathbb{E}^*\{s_n^{(L)}(k)\}^4]| = o(1)$, which yields (8).

Next we consider (6). For $k \in \mathcal{K}_n$,

$$
\begin{aligned}
\mathbb{E}_{X|Z}\mathbb{E}^*\{s_n^{(S)}(k)\}^2 &= n^{-1}\mathbb{E}_{X|Z}\sum_{j,j'=1}^{n}\{X(t_j)-\mu\}\{X(t_{j'})-\mu\}\mathbf{1}\{t_j \in \mathcal{B}_n^{(S)}(k),\ t_{j'} \in \mathcal{B}_n^{(S)}(k)\} \\
&= n^{-1}\sum_{j,j'=1}^{n}\gamma(t_j-t_{j'})\mathbf{1}\{t_j \in \mathcal{B}_n^{(S)}(k),\ t_{j'} \in \mathcal{B}_n^{(S)}(k)\}.
\end{aligned}
$$

So uniformly in $k \in \mathcal{K}_n$, $\mathbb{E}[\mathbb{E}^*\{s_n^{(S)}(k)\}^2]$ equals to

$$
\gamma(0)P_Z[\lambda_n Z_1 \in \mathcal{B}_n^{(S)}(k)] + (n-1)\int_{R_0^2}\gamma\{\lambda_n(z_1-z_2)\}\mathbf{1}\{\lambda_n z_1 \in \mathcal{B}_n^{(S)}(k),\ \lambda_n z_2 \in \mathcal{B}_n^{(S)}(k)\}dz_1dz_2,
$$

which is bounded by $C\lambda_n^{-1}l_n + Cnl_n\lambda_n^{-2}$. Therefore, $\mathbb{E}[\mathbb{E}^*\{S_n^{(S)}\}^2] = \sum_{k\in\mathcal{K}_n}\mathbb{E}[\mathbb{E}^*\{s_n^{(S)}(k)\}^2] \le C|\mathcal{K}_n|(\lambda_n^{-1}l_n + nl_n\lambda_n^{-2}) = o(1)$, which implies the first equation in (6). As to $S_n^{(NB)}$, we can derive under the assumptions on R_0 that $|\mathbb{E}[\mathbb{E}^*\{S_n^{(NB)}\}^2]| \le C\lambda_n^{-1}b_n + Cnb_n\lambda_n^{-2} = o(1)$. This completes the proof of (6). The conclusion follows.

$\diamondsuit$

0.2 Some lemmas

The following lemma, which states a moment bound for partial sums of a stationary time series, can be derived from Doukhan (1994, p.9, 26).

LEMMA **0.1**. *Under Condition M_r, $r \in \mathbb{Z}_+$, it holds that for any $1 \le q \le 2r$ and any $A \subset \mathbb{Z}$, $\mathbb{E}\|\sum_{t\in A}\{\mathbf{X}_t - \boldsymbol{\mu}\}\|^q \le C|A|^{q/2}$.*

LEMMA **0.2**. *For $k = 2, 4$, assume $\mathbf{X}_t \in \mathcal{L}^{2k}$ and the condition C_{2k} holds. Suppose that $l_n^{-1} + l_n/n = o(1)$. Then for $v = 1, \cdots, p$, $\mathbb{E}\{\hat{f}_{n,v}^k(0)\} = O(1)$, $k = 2, 4$.*

Proof of Lemma 0.2: For the convenience of presentation, let Z_{t_j} be the v-th element of $\mathbf{X}_{t_j}$, $\bar{Z}_n = n^{-1}\sum_{j=1}^n Z_{t_j}$ and $\mu_Z = \mathbb{E}\{Z_{t_1}\}$. Note that $2\pi\hat{f}_{n,v}(0) = K_{1n} + K_{2n} + K_{3n}$, where

$$\begin{aligned}
K_{1n} &= n^{-1}\sum_{j,j'=1}^{n}\{Z_{t_j} - \mu_Z\}\{Z_{t_{j'}} - \mu_Z\}a\{(t_j - t_{j'})/l_n\}, \\
K_{2n} &= -2n^{-1}(\bar{Z}_n - \mu_Z)\sum_{j,j'=1}^{n}\{Z_{t_{j'}} - \mu_Z\}a\{(t_j - t_{j'})/l_n\}, \\
K_{3n} &= n^{-1}(\bar{Z}_n - \mu_Z)^2\sum_{j,j'=1}^{n}a\{(t_j - t_{j'})/l_n\}.
\end{aligned}$$

Then it follows from the condition C_4 (C_8) that $\mathbb{E}(K_{1n}^k) = O(1)$ for $k = 2$ (4). Further, since $a(\cdot)$ has support on $[-1, 1]$, it follows from the Cauchy-Schwarz inequality that $\mathbb{E}(K_{2n}^k) = O(l_n^k/n^k)$ and $\mathbb{E}(K_{3n}^k) = O(l_n^k/n^k)$ for $k = 2, 4$. This completes the proof.

$\diamondsuit$

Lemma **0.3**. *Suppose Assumptions 4.1 and 4.2 hold and* $\mathbf{X}_t \in \mathcal{L}^m$, *where* $m > 2$. *Then*

$$\mathbb{E}(\mathbb{E}^* \|\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n\|^m) = O(l^{m/2}/n^{m/2}).$$

Proof of Lemma 0.3: Let $\mathbf{e}_v$, $v = 1, \cdots, m$ be a p-dimensional unit vector with its v-th element being 1 and 0 otherwise. It suffices to show that $\mathbb{E}\{\mathbb{E}^* |\mathbf{e}_v'(\bar{\mathbf{X}}_n^* - \bar{\mathbf{X}}_n)|^m\} = O(l^{m/2}/n^{m/2})$. Denote by $Z_{nv}(h) = \mathbf{e}_v'\{\mathbf{X}_h - \bar{\mathbf{X}}_n\}$. Write $\sum_{j=1}^n W_{t_j} Z_{nv}(t_j) = (\Sigma_L + \Sigma_S + \Sigma_{NB}) W_{t_j} Z_{nv}(t_j)$. Let $M_k = M_k(v) = \sum_{h \in \mathcal{B}_n^{(L)}(k)} W_h Z_{nv}(h)$, $k \in \mathcal{K}_n$. In the sequel, we show that

$$\mathbb{E}\left\{\mathbb{E}^* \left|\sum_{k \in \mathcal{K}_n} M_k\right|^m\right\} = O(l_n^{m/2}/n^{m/2}). \tag{9}$$

Since M_k, $k \in \mathcal{K}_n$ are independent conditional on the data, we apply Rosenthal's inequality and get

$$\mathbb{E}^* \left|\sum_{k \in \mathcal{K}_n} M_k\right|^m \leq C\mathbb{E}^* \left|\sum_{k \in \mathcal{K}_n} M_k^2\right|^{m/2} \leq C\left\{\sum_{k \in \mathcal{K}_n} (\mathbb{E}^*|M_k|^m)^{2/m}\right\}^{m/2},$$

where $(\mathbb{E}^*|M_k|^m)^{2/m} \leq C l_n \sum_{t \in \mathcal{B}_n^{(L)}(k)} Z_{nv}^2(t)$ using the same argument as presented in the derivation of $\|U_1\|_{2+\delta}^*$; see the proof of Theorem 3.1. So (9) follows by an application of triangle inequality. Similarly, we can see that the contribution from the small blocks are negligible. Since the number of summands in Σ_{NB} is $O(p_n)$, $\mathbb{E}[\mathbb{E}^*|\Sigma_{NB}\{W_{t_j} Z_{nv}(t_j)\}|^m] \leq C p_n^m$. This completes the proof.

$\diamondsuit$

0.3 Additional tables

Table 1: The normalized MSEs for the bootstrap variance estimators of $n\text{var}(\bar{X}_n)$ using (a) the grid-based block bootstrap; (b) the DWB with $a(x) = w_{0.43}^{\text{trap}} * w_{0.43}^{\text{trap}}(x)/w_{0.43}^{\text{trap}} * w_{0.43}^{\text{trap}}(0)$; (c) the DWB with $a(x) = (1 - |x|)\mathbf{1}(|x| \leq 1)$. The data is mean zero Gaussian with a spherical covariance function. The box for each row indicates the smallest normalized MSE among $l = 1, \cdots, 10$. Part (A) corresponds to the truncated $N(0, 1)$ density function for the sampling design, whereas part (B) is for the truncated $N(0, 1/4)$ density function. The largest standard error is .019 for part (A) and .016 for part (B).

(A)			l									
λ_n	R		1	2	3	4	5	6	7	8	9	10
18	4	(a)	0.54	0.4	**0.38**	0.4	0.44	0.51	0.51	0.55	0.62	0.61
		(b)	0.6	0.44	0.37	**0.35**	0.36	0.38	0.4	0.42	0.44	0.46
		(c)	0.55	0.4	**0.37**	0.38	0.41	0.43	0.46	0.48	0.49	0.51
	8	(a)	0.76	0.63	0.57	0.56	**0.55**	0.57	0.59	0.63	0.68	0.68
		(b)	0.8	0.69	0.61	0.56	0.53	**0.51**	0.51	0.51	0.52	0.53
		(c)	0.76	0.64	0.57	0.54	**0.53**	0.53	0.54	0.55	0.57	0.58
36	4	(a)	0.47	0.31	0.26	**0.25**	0.27	0.29	0.32	0.33	0.37	0.38
		(b)	0.53	0.37	0.28	0.24	**0.23**	0.24	0.25	0.27	0.28	0.3
		(c)	0.47	0.32	0.26	**0.25**	0.26	0.28	0.3	0.32	0.34	0.37
	8	(a)	0.69	0.55	0.46	0.41	0.4	**0.39**	0.4	0.41	0.44	0.45
		(b)	0.74	0.61	0.52	0.46	0.41	0.38	0.37	**0.36**	0.36	0.37
		(c)	0.7	0.56	0.47	0.41	0.39	**0.38**	0.38	0.39	0.41	0.42

(B)			l									
λ_n	R		1	2	3	4	5	6	7	8	9	10
18	4	(a)	0.6	0.48	**0.45**	0.47	0.5	0.56	0.58	0.66	0.75	0.76
		(b)	0.66	0.51	0.45	**0.43**	0.43	0.44	0.46	0.48	0.5	0.53
		(c)	0.6	0.48	**0.44**	0.46	0.48	0.51	0.54	0.57	0.59	0.62
	8	(a)	0.81	0.71	0.66	**0.65**	0.66	0.68	0.7	0.76	0.82	0.82
		(b)	0.84	0.75	0.69	0.65	0.63	**0.61**	0.61	0.61	0.62	0.63
		(c)	0.81	0.72	0.66	0.64	**0.63**	0.64	0.65	0.67	0.69	0.71
36	4	(a)	0.5	0.36	**0.32**	0.32	0.34	0.36	0.38	0.4	0.43	0.44
		(b)	0.56	0.4	0.33	**0.3**	0.3	0.31	0.33	0.35	0.37	0.39
		(c)	0.5	0.36	**0.31**	0.32	0.34	0.36	0.38	0.4	0.42	0.43
	8	(a)	0.73	0.61	0.53	0.49	0.47	**0.46**	0.47	0.48	0.5	0.51
		(b)	0.77	0.66	0.57	0.51	0.47	0.45	0.44	**0.43**	0.43	0.44
		(c)	0.73	0.6	0.52	0.48	0.46	**0.45**	0.45	0.46	0.47	0.48

Table 2: The empirical coverages (in percent) for the bootstrap-based confidence intervals of μ using (a) the grid-based block bootstrap; (b) the DWB with $a(x) = w_{0.43}^{\text{trap}} * w_{0.43}^{\text{trap}}(x)/w_{0.43}^{\text{trap}} * w_{0.43}^{\text{trap}}(0)$; (c) the DWB with $a(x) = (1 - |x|)\mathbf{1}(|x| \leq 1)$. The data is mean zero Gaussian with a spherical covariance function. The box for each row indicates the best coverage among $l = 1, \cdots, 10$ (i.e., closest to the nominal level 95%). The largest standard error is 1.6%. Part (A) corresponds to the truncated $N(0, 1)$ density function for the sampling design, whereas part (B) is for the truncated $N(0, 1/4)$ density function.

(A)			l									
λ_n	R		1	2	3	4	5	6	7	8	9	10
18	4	(a)	68	74	77	76	73	69	68	64	56	56
		(b)	64	73	76	77	77	78	77	76	75	74
		(c)	68	75	77	76	76	75	73	72	70	69
	8	(a)	50	61	64	64	65	62	60	57	51	51
		(b)	47	57	62	65	66	68	68	67	67	66
		(c)	50	60	64	65	66	66	65	64	63	61
36	4	(a)	70	79	82	84	84	82	80	80	77	76
		(b)	66	76	81	84	85	85	84	84	84	84
		(c)	70	79	83	83	84	83	82	82	80	80
	8	(a)	53	64	70	73	74	76	75	74	74	72
		(b)	50	60	66	71	73	75	77	78	79	78
		(c)	54	64	69	72	75	76	76	76	76	75

(B)			l									
λ_n	R		1	2	3	4	5	6	7	8	9	10
18	4	(a)	60	68	71	70	68	65	64	57	46	46
		(b)	58	67	72	74	75	74	74	72	70	68
		(c)	62	69	73	73	71	69	66	64	61	59
	8	(a)	41	49	52	52	52	52	50	44	38	38
		(b)	38	47	52	56	57	58	58	58	57	56
		(c)	41	51	55	56	56	56	55	52	50	48
36	4	(a)	68	74	77	78	77	76	76	74	71	72
		(b)	65	74	77	80	80	80	80	80	79	78
		(c)	68	75	78	79	79	78	76	76	75	73
	8	(a)	49	58	63	66	67	68	67	66	66	66
		(b)	47	56	62	64	67	70	70	71	71	72
		(c)	50	59	65	67	68	69	69	69	68	67