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A tuning parameter free test for properties of space–time covariance functions

Xiaofeng Shao[a,*], Bo Li[b]

[a]Department of Statistics, University of Illinois at Urbana-Champaign, Champaign, IL 61820, USA
[b]Department of Statistics, Purdue University, West Lafayette, IN 47906, USA

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ABSTRACT

We propose a new nonparametric test to test for symmetry and separability of space–time covariance functions. Unlike the existing nonparametric tests, our test has the attractive convenience of being free of choosing any user-chosen number or smoothing parameter. The asymptotic null distributions of the test statistics are free of nuisance parameters and the critical values have been tabulated in the literature. From a practical point of view, our test is easy to implement and can be readily used by the practitioner. A Monte-Carlo experiment and real data analysis illustrate the finite sample performance of the new test.

1. Introduction

In the recent decade, there has been a growing literature on modeling space–time data using both parametric and nonparametric techniques; see Finkenstädt et al. (2007) for a review. The space–time covariance function plays an important role in describing spatio-temporal variability of the data in a parametric (or nonparametric) way and predicting the observations at unknown sites or future time points. Due to large size of space–time data, the implementation of the traditional technique, such as kriging, is computationally costly and sometimes impossible. One way to alleviate the computational burden is to impose certain structure on the covariance function, such as full symmetry and separability, which greatly reduces the dimension of the covariance matrix. Although symmetry and separability are desirable assumptions from a computational point of view, they may not be appropriate for real data, see, e.g. Irish wind data (Gneiting et al., 2007). This motivates two lines of research: the development of parametric classes of covariance functions that allow space–time asymmetry and/or nonseparability (Cressie and Huang, 1999; Gneiting, 2002; Stein, 2005; Jun and Stein, 2007; Park and Fuentes, 2009) and formal statistical assessment of symmetry and separability for space–time data (Mitchell et al., 2005; Fuentes, 2006; Li et al., 2007, 2008; Park and Fuentes, 2008).

In this article, we consider the latter problem, i.e. testing a given (possibly multivariate) spatio-temporal process for symmetry and separability. We restrict our discussion to testing of symmetry and separability, but an extension of our method to testing for other properties of space–time covariance functions is straightforward. In addition to the possible computational gain, we note that a proper assessment of properties of space–time covariance functions could provide a meaningful restriction on the class of models one needs to investigate when modeling the data.

Many approaches have been proposed to test for symmetry and separability of space–time covariance functions. Mitchell et al. (2005) extended the likelihood ratio test of Mitchell et al. (2006), which was proposed in the context of multivariate repeated measures, to test for separability of a space–time process. Fuentes (2006) developed a nonparametric test for separability

* Corresponding author.
E-mail addresses: xshao@uiuc.edu (X. Shao), boli@purdue.edu (B. Li).

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of a space–time process based on the properties of the spectral representation. Lately, Park and Fuentes (2008) extended the spectral method to detect lack of symmetry. In Li et al. (2007), a unified methodology was developed to assess the symmetry and separability based on the asymptotic joint normality of sample space–time covariance estimators. All the works mentioned above focus on the univariate space–time process, whereas the testing problem for multivariate space–time data has been tackled recently in Li et al. (2008) as a useful extension of Li et al. (2007). In addition, there is also a literature on testing symmetry, separability and other properties of covariance functions in the purely spatial context, see, e.g. Shitan and Brockwell (1995), Scaccia and Martin (2005), Lu and Zimmerman (2005) and Guan et al. (2004).

A common feature of nonparametric tests (e.g. Li et al., 2007, 2008; Fuentes, 2006; Park and Fuentes, 2008) is that a tuning parameter is involved in the calculation of test statistics. For example, in Li et al.'s test, one needs to determine the subsampling window size to estimate the covariance matrix of the selected contrasts, whereas the estimation of joint space–time spectral density involved in Fuentes (2006) and Park and Fuentes (2008) also requires the choice of a bandwidth parameter. Although the tuning parameter only has to satisfy very mild growth conditions in theory, the finite sample performance of these tests can be sensitive to these user-chosen numbers. Note that Li et al. (2007, 2008) employ automatic data-driven numbers, for which more comments on its drawback will be made in Section 2.

In this paper, we propose a nonparametric test that is free of any tuning parameters. Building on the work of Li et al. (2007, 2008), we modify their test statistics by employing an inconsistent estimator of the asymptotic covariance matrix of the selected contrasts. This inconsistent estimator is formed by recursively estimating the space–time covariances and is proportional to the asymptotic covariance matrix of the contrasts. The asymptotic null distributions of the resulting test statistics do not contain any unknown nuisance parameters and the critical values have been tabulated in previous work. The self-normalized test statistics proposed here are nontrivial extensions of the early work by Lobato (2001), who focused on testing for noncorrelation of a univariate time series, to testing problems in the space–time context.

The rest of the article is organized as follows. Section 2 presents the framework and the new test statistics along with asymptotic theory. Some simulation results are reported in Section 3 to compare the finite sample performance of our test with that of Li et al.'s. Section 4 illustrates the new test using real data sets. Section 5 concludes.

2. Statistical framework and methodology

We follow the notation used in Li et al. (2007) and let $\{Z(\mathbf{s},t):\mathbf{s}\in\mathbb{R}^d, t\in\mathbb{R}\}$ denote a strictly stationary space–time random field with covariance function $C(\mathbf{h},u)=\mathrm{cov}\{Z(\mathbf{s},t),Z(\mathbf{s}+\mathbf{h},t+u)\}$, where $\mathbf{h}$ and u denote a spatial and temporal lag. The covariance function is fully symmetric if $C(\mathbf{h},u)=C(\mathbf{h},-u)$ or if $C(\mathbf{h},u)=C(-\mathbf{h},u)$. The separable covariance function is a subclass of symmetric ones, which satisfy that $C(\mathbf{h},u)/C(\mathbf{h},0)=C(\mathbf{0},u)/C(\mathbf{0},0)$. Gneiting et al. (2007) give a very clear illustration of the relationship among stationarity, symmetry, separability and other assumptions for space–time processes. To perform Li et al.'s test, we need to define a set of space–time lags, denoted by $\boldsymbol{\Lambda}$. For instance, $\boldsymbol{\Lambda}=\{(\mathbf{1},1),(\mathbf{1},2)\}$. We formulate the null hypothesis of full symmetry or separability as

$$H_0:\mathbf{Af}(\mathbf{G})=0,$$

where $\mathbf{A}$ is a contrast matrix of row rank q, $\mathbf{G}=\{C(\mathbf{h},u),(\mathbf{h},u)\in\boldsymbol{\Lambda}\}$ and $\mathbf{f}=(f_1,\dots,f_r)'$ are real-valued functions that are differentiable at $\mathbf{G}$. See Section 2 of Li et al. (2007) for detailed examples of $\mathbf{A}$, $\mathbf{f}$ and $\mathbf{G}$ for symmetry and separability. Following Li et al. (2007), we assume the observations are taken from a fixed space $S\subset\mathbb{R}^d$ at regularly spaced times $T_n=\{1,\dots,n\}$. This framework allows irregular nonlattice spatial configuration and seems natural for many real space–time data sets.

Let $S(\mathbf{h})=\{\mathbf{s}:\mathbf{s}\in S,\mathbf{s}+\mathbf{h}\in S\}$ and $|S(\mathbf{h})|$ be the number of elements in $S(\mathbf{h})$. Define

$$\widehat{C}_n(\mathbf{h},u)=\frac{1}{|S(\mathbf{h})||T_n|}\sum_{\mathbf{s}\in S(\mathbf{h})}\sum_{t=1}^{n-u}Z(\mathbf{s},t)Z(\mathbf{s}+\mathbf{h},t+u),\quad u\geq 0$$

to be a sample estimate of $C(\mathbf{h},u)$. Denote by $\widehat{\mathbf{G}}_n=\{\widehat{C}_n(\mathbf{h},u):(\mathbf{h},u)\in\boldsymbol{\Lambda}\}$ the estimate of $\mathbf{G}$ computed over the domain $S\times T_n$. Here we implicitly assume the mean of $Z(\mathbf{s},t)$ to be zero for convenience of presentation. If we remove this assumption, then we can use mean-corrected estimator of $C(\mathbf{h},u)$, which would not affect the theory presented below. Under appropriate conditions, Theorem 1 of Li et al. (2007) asserts that $\sqrt{|T_n|}(\widehat{\mathbf{G}}_n-\mathbf{G})\rightarrow_D \mathrm{N}_p(0,\boldsymbol{\Sigma})$, where $p=|\boldsymbol{\Lambda}|$, "$\rightarrow_D$" denotes convergence in distribution and $\mathrm{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ denotes p-th dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. By the multivariate delta theorem (Mardia et al., 1979, p. 52), we have

$$\sqrt{|T_n|}\{\mathbf{f}(\widehat{\mathbf{G}}_n)-\mathbf{f}(\mathbf{G})\}\rightarrow_D \mathrm{N}_r(0,\mathbf{B}'\boldsymbol{\Sigma}\mathbf{B}),$$

where $B_{ij}=\partial f_j/\partial G_i$, $i=1,\dots,p$, $j=1,\dots,r$. The test statistic defined by Li et al. (2007) takes the form

$$\mathrm{TS}_L=|T_n|\{\mathbf{Af}(\widehat{\mathbf{G}}_n)\}'(\mathbf{A}\widehat{\mathbf{B}}'\widehat{\boldsymbol{\Sigma}}\widehat{\mathbf{B}}\mathbf{A}')^{-1}\{\mathbf{Af}(\widehat{\mathbf{G}}_n)\},$$

where $\widehat{\mathbf{B}}$ is a consistent estimator of $\mathbf{B}=\mathbf{B}(\mathbf{G})$, which can be formed by replacing $\mathbf{G}$ with $\widehat{\mathbf{G}}_n$. The asymptotic covariance matrix $\boldsymbol{\Sigma}$ is consistently estimated by $\widehat{\boldsymbol{\Sigma}}$ via a subsampling approach (Carlstein, 1986; Politis et al., 1999). In Li et al. (2007), a formula for the subsampling window size is given, which is derived under the assumption that the underlying data structure follows a univariate AR(1) model and the statistic of interest is the sample mean. Specifically, the window size

$$l(n) = \{2\gamma/(1-\gamma^2)\}^{2/3}(3n/2)^{1/3}, \tag{1}$$

where γ is estimated by $\hat{\gamma}_n = \widehat{C}_n(\mathbf{0},1)/\widehat{C}_n(\mathbf{0},0)$.

The simulation studies in Li et al. (2007) show that the procedure works reasonably well for VAR(1) (i.e. first-order vector autoregressive model). However, for more general class of models, the empirical performance is yet to be investigated. In fact, we find through our unreported simulation studies that the selection rule cannot be applied without modifications to some commonly used models. For example, when the space–time process follows a VAR(2) model with AR(1) coefficients being zero, the block size can take value 1, which makes the calculation of $\widehat{\mathbf{\Sigma}}$ infeasible since it involves computing the empirical space–time covariances at non-zero temporal lags based on blocks of length 1. The main reason is that in this particular case, the estimate $\hat{\gamma}_n$ is near zero thus yields very small block size with a chance of getting $l(n) = 1$. This artifact, which is due to the fact that the optimal block size is tailed to the AR(1) model, can be circumvented by using the kernel smoothed estimate of covariance matrix (Hall et al., 1994). However, the practical issue of bandwidth selection poses difficulty to the user, and no satisfactory rule seems existing for the testing problem addressed here. There is also a closely related literature on the block size selection in the context of spectrum estimation and resampling dependent data (Hall et al., 1995; Politis et al., 1999; Lahiri, 2003). The proposed methodologies often require heavy computation or a choice of another tuning parameter, for which no good guidance is available.

To this end, we shall develop a test that is free of the choice of any tuning parameters. Extending the random normalization idea presented in Lobato (2001), we replace $\mathbf{\Sigma}$ by an inconsistent estimator $\widetilde{\mathbf{\Sigma}}$ so that the resulting test statistic has an asymptotic pivotal distribution under the null hypothesis. Let $m = \max\{|u| : (\mathbf{h}, u) \in \mathbf{\Lambda}\}$ and $N = |T_n| - m$. Define the recursive estimate of $C(\mathbf{h}, u)$ based on the (temporal) subsample $\{Z(\mathbf{s}, t), \mathbf{s} \in S, t = 1, \ldots, J + m\}$, i.e.

$$\widetilde{C}_J(\mathbf{h}, u) = \frac{1}{|S(\mathbf{h})||J|} \sum_{\mathbf{s}\in S(\mathbf{h})} \sum_{k=1}^{J} Z(\mathbf{s}, k) Z(\mathbf{s}+\mathbf{h}, k+u), \quad u \geq 0$$

and $\widetilde{\mathbf{G}}_J = \{\widetilde{C}_J(\mathbf{h}, u), (\mathbf{h}, u) \in \mathbf{\Lambda}\}$ for $J = 1, \ldots, N$. Note that $\widetilde{C}_N(\mathbf{h}, u)$ is slightly different from $\widehat{C}_n(\mathbf{h}, u)$ unless $m = 0$. The first test statistic is defined as

$$\mathrm{TS}_1 = |T_n| \{\mathbf{Af}(\widetilde{\mathbf{G}}_N)\}' (\mathbf{A}\widetilde{\mathbf{B}}'\widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{B}}\mathbf{A}')^{-1} \{\mathbf{Af}(\widetilde{\mathbf{G}}_N)\},$$

where $\widetilde{\mathbf{B}} = \mathbf{B}(\widetilde{\mathbf{G}}_N)$ and $\widetilde{\mathbf{\Sigma}} = N^{-2}\sum_{J=1}^{N} J^2 (\widetilde{\mathbf{G}}_J - \widetilde{\mathbf{G}}_N)(\widetilde{\mathbf{G}}_J - \widetilde{\mathbf{G}}_N)'$. Theorem 2.1 asserts that the asymptotic null distribution of TS_1 is U_q (see (2) for the definition), for which the critical values have been tabulated in Lobato (2001) for $q = 1, \ldots, 20$. So at the $100\alpha\%$ significance level, we reject the null hypothesis if $\mathrm{TS}_1 > U_{q,1-\alpha}$, where $U_{q,1-\alpha}$ is the $100(1-\alpha)\%$ upper percentile of U_q.

Alternatively, we can use the following test statistic that admits a simpler form than TS_1. Define

$$\mathrm{TS}_2 = |T_n| \{\mathbf{Af}(\widetilde{\mathbf{G}}_N)\}' \widetilde{\mathbf{V}}_N^{-1} \{\mathbf{Af}(\widetilde{\mathbf{G}}_N)\},$$

where $\widetilde{\mathbf{V}}_N = N^{-2}\sum_{J=1}^{N} J^2 \{\mathbf{Af}(\widetilde{\mathbf{G}}_J) - \mathbf{Af}(\widetilde{\mathbf{G}}_N)\}\{\mathbf{Af}(\widetilde{\mathbf{G}}_J) - \mathbf{Af}(\widetilde{\mathbf{G}}_N)\}'$. The null limiting distribution of TS_2 is also U_q, as stated in Theorem 2.1. An implementational advantage of TS_2 over TS_1 is that there is no need to calculate the derivative of $\mathbf{f}$, i.e. $\mathbf{B}$ in the calculation of TS_2. Note that for testing symmetry, $\mathrm{TS}_1 = \mathrm{TS}_2$ due to $\mathbf{f}(\mathbf{G}) = \mathbf{G}$. An important practical issue is how to choose the testing lags and contrast matrices. In general, the choice of testing lags and contrast matrices has little effect on the sizes, yet the powers do depend on the specific choices. The bottom line is to choose testing lags associated with strong correlation. This usually points to a combination of small spatial and small time lags, but there is no rule of thumb for this choice. For example, when the observations are strongly influenced by a physical process such as wind or current, the choice of testing lags should take the direction of the physical process into consideration, other than simply seeking small lags, in order to catch the lags with strong correlation. For more details, we refer the reader to Section 4.2 of Li et al. (2007) for some discussions and guidelines.

To develop suitable asymptotic theory, we follow Li et al. (2007) and restrict our attention to the space–time random field with a fixed spatial domain and an increasing temporal domain. Equivalently, we can view the space–time data as a multivariate time series with the number of time series fixed and the length of each time series going to infinity. This asymptotic framework is basically the one adopted in asymptotic problems of multivariate time series. Note that Matsuda and Yajima (2004) have developed spectral tests for separable correlations in multivariate time series. Their tests also involve the choice of a bandwidth parameter and they found the performance heavily depends on the bandwidth selection. Although they use a frequency domain cross-validation procedure that seems to work reasonably well, their procedure is computationally expensive and the implementation seems rather complex.

Let $\mathbf{Y}(k) = \{|S(\mathbf{h})|^{-1}\sum_{\mathbf{s}\in S(\mathbf{h})} Z(\mathbf{s}, k) Z(\mathbf{s}+\mathbf{h}, k+u), (\mathbf{h}, u) \in \mathbf{\Lambda}\}, k = 1, \ldots, N$. Denote by "$\Rightarrow$" weak convergence in the space of functions on [0, 1] which are right continuous and have left limits, endowed with the Skorokhod topology; by $\lfloor a \rfloor$ the integer part of a; by $o_p(1)$ convergence to zero in probability. Let $\mathbf{V}_p = \int_0^1 \{\mathbf{B}_p(r) - r\mathbf{B}_p(1)\}\{\mathbf{B}_p(r) - r\mathbf{B}_p(1)\}'\, dr$, where $\mathbf{B}_p(r)$ is a p-dimensional vector of independent Brownian motions. Define

$$U_p = \mathbf{B}_p(1)' \mathbf{V}_p^{-1} \mathbf{B}_p(1). \tag{2}$$

Theorem 2.1. *Assume that*

$$\frac{1}{\sqrt{N}}\sum_{k=1}^{\lfloor Nr\rfloor}[\mathbf{Y}(k)-\mathbb{E}\{\mathbf{Y}(k)\}]\Rightarrow \Phi\mathbf{B}_p(r), \tag{3}$$

where Φ is a $p\times p$ lower triangular matrix with nonnegative diagonal entries. Suppose that $\mathbf{D}=\mathbf{A}\mathbf{B}'\in R_{q\times p}$ has rank q. Then under H_0, $\mathrm{TS}_1\to_D U_q$. Further assume that

$$\sqrt{J}\{\mathbf{f}(\widetilde{\mathbf{G}}_J)-\mathbf{f}(\mathbf{G})-\mathbf{B}'(\widetilde{\mathbf{G}}_J-\mathbf{G})\}=o_p(1) \text{ uniformly in } J=1,\ldots,N. \tag{4}$$

Then under H_0, $\mathrm{TS}_2\to_D U_q$.

Proof of Theorem 2.1. Note that under H_0, it follows from the multivariate delta theorem and the continuous mapping theorem that

$$\begin{aligned}\mathrm{TS}_1 &= |T_n|\{\mathbf{A}\mathbf{f}(\widetilde{\mathbf{G}}_N)\}'(\mathbf{A}\widetilde{\mathbf{B}}'\widetilde{\boldsymbol{\Sigma}}\widetilde{\mathbf{B}}\mathbf{A}')^{-1}\{\mathbf{A}\mathbf{f}(\widetilde{\mathbf{G}}_N)\}\\ &= |T_n|\{\mathbf{A}\mathbf{B}'(\widetilde{\mathbf{G}}_N-\mathbf{G})\}'(\mathbf{A}\mathbf{B}'\widetilde{\boldsymbol{\Sigma}}\mathbf{B}\mathbf{A}')^{-1}\{\mathbf{A}\mathbf{B}'(\widetilde{\mathbf{G}}_N-\mathbf{G})\}+o_p(1)\\ &= \{\mathbf{D}\sqrt{|T_n|}(\widetilde{\mathbf{G}}_N-\mathbf{G})\}'(\mathbf{D}'\widetilde{\boldsymbol{\Sigma}}\mathbf{D}')^{-1}\{\mathbf{D}\sqrt{|T_n|}(\widetilde{\mathbf{G}}_N-\mathbf{G})\}'+o_p(1)\\ &\to_D \{\mathbf{D}\mathbf{B}_p(1)\}'(\mathbf{D}'\mathbf{V}_p\mathbf{D}')^{-1}\{\mathbf{D}\mathbf{B}_p(1)\}=_D U_q,\end{aligned}$$

where the symbol "$=_D$" stands for equality in distribution. As for TS_2, it is easy to see that under (4), we have $\widetilde{\mathbf{V}}_N=\mathbf{D}\widetilde{\boldsymbol{\Sigma}}\mathbf{D}'+o_p(1)$, which implies that $\mathrm{TS}_2\to_D U_q$ in view of the above derivation for TS_1. The proof is complete. □

The self-normalization idea has been used in Lobato (2001) and Shao (2009) in the context of robust testing and confidence interval construction for univariate time series. The key ingredient is to avoid the estimation of asymptotic covariance matrix by inefficient studentization. The functional central limit theorem (3) allows us to combine the asymptotic behavior of two statistics (e.g. $\mathbf{A}\mathbf{f}(\widetilde{\mathbf{G}}_N)$ and $\widetilde{\mathbf{V}}_N$ in the case of TS_2) so that the asymptotic covariance matrix is canceled out in the asymptotic distribution of our test statistics. The two test statistics we propose here can be regarded as an extension of the self-normalization idea to testing problems for space–time processes. A distinctive feature of the self-normalization proposed in this paper is that we use recursive estimate of autocovariances and its functionals as the normalization factor, which seems new and has not been considered before. Note that Kuan and Lee (2006) have used recursive estimate in the normalization, but their discussions are restricted to the robust testing context, where the statistic of interest is a function of residuals and the use of recursive residuals is able to remove the estimation effect.

The main assumption (3) is not primitive. For primitive conditions, we can impose mixing assumptions or near epoch dependence (Davidson, 1994) on the process $Z(\mathbf{s},t)$, which guarantees the validity of (3) along with appropriate moment conditions on $Z(\mathbf{s},t)$. See Lobato (2001) and Shao (2009) for related discussions. To verify (4), one can impose suitable smoothness conditions on $\mathbf{f}$ (including growth and boundedness conditions on suitable derivatives of $\mathbf{f}$) as well as moment and mixing assumptions on $Z(\mathbf{s},t)$. Note that the $\mathbf{f}$ described in Section 2 of Li et al. (2007) for separability is infinitely differentiable at the true values of the selected contrasts. Since the primitive conditions and technical proofs are quite standard without any methodological difficulty, we prefer to leave out the details.

Our testing methodology was developed for stationary space–time random fields, yet it is worth noting that the stationarity in space is not needed in our asymptotic theory. Our test statistics can be readily extended to multivariate space–time data to test for symmetry and separability. Since the derivation of our test statistics and related asymptotic theory for the multivariate case differs only notationally from the displayed univariate case, we omit the details. We examine the finite sample performance of our test statistics for the multivariate case in the simulation studies.

3. Simulation studies

In this section, we use the same simulation settings in Li et al. (2007, 2008) and make a thorough comparison with their tests in terms of finite-sample size and power. For the sake of readership, we present the details of the simulation setup. Throughout simulations, the significance level is 5% and the number of replications is 3000.

3.1. Separability for univariate space–time process

In the univariate case, we use the same model as presented in Section 4.2 of Li et al. (2007). Let $\mathbf{Z}_t=\mathbf{R}\mathbf{Z}_{t-1}+\epsilon_t$, where $\mathbf{Z}_t=\{Z(\mathbf{s}_1,t),\ldots,Z(\mathbf{s}_K,t)\}'$, K is number of spatial locations on a 3×3 (or 5×5) grid, $\mathbf{R}$ is a matrix of coefficients, ϵ_t is a Gaussian multivariate white noise process with a spatially stationary and isotropic exponential correlation function. We use the exponential covariance function for ϵ, i.e. $C(\mathbf{h},0)=\exp(-\|\mathbf{h}\|/\phi)$, $\phi=3.476$. The testing lags are ($\|\mathbf{h}\|=1,u=1$) and ($\|\mathbf{h}\|=1,u=2$), so $q=2$. To examine the size, let $\mathbf{R}=\rho\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix and ρ indicates the strength of temporal correlation. For the power assessment, we define $\mathbf{R}$ as follows. For each $(\mathbf{s}_i,t)$, the coefficient ρ is 0.05 for $\{(\mathbf{s}_j,t-1):\|\mathbf{s}_j-\mathbf{s}_i\|=1\}$, whereas 0 for the remaining $(\mathbf{s}_j,t-1)$'s.

Table 1
Empirical sizes (in percentage) of the tests for separability in the univariate case.

Grid size	ρ	$\lvert T_n\rvert = 200$			$\lvert T_n\rvert = 500$		
		TS_L	TS_1	TS_2	TS_L	TS_1	TS_2
3 × 3	0.3	2.5	6.1	5.9	2.4	6.0	5.5
	0.6	5.1	6.3	6.2	4.1	5.5	5.2
	0.8	7.1	5.3	5.5	5.3	5.2	4.8
	0.9	9.5	4.3	4.5	6.2	4.8	4.3
5 × 5	0.3	2.7	6.0	5.9	2.9	5.4	5.4
	0.6	5.5	6.7	6.4	4.1	5.1	5.1
	0.8	7.2	6.0	5.4	5.2	5.0	5.0
	0.9	10.3	5.0	5.0	6.3	4.2	4.3

Nominal level is 0.05. The largest standard error is 0.55%.

Table 2
Empirical size-adjusted powers (in percentage) of the tests for separability in the univariate case.

Grid size	ρ	$\lvert T_n\rvert = 200$			$\lvert T_n\rvert = 500$		
		TS_L	TS_1	TS_2	TS_L	TS_1	TS_2
3 × 3	0.3	41.6	26.4	23.9	80.1	53.5	53.8
	0.6	49.8	35.2	31.7	90.2	70.5	68.6
	0.8	60.4	39.7	31.3	97.6	81.9	76.8
5 × 5	0.3	69.8	46.8	45.5	97.2	81.9	81.3
	0.6	78.9	60.9	56.3	99.4	92.9	91.2
	0.8	81.2	54.7	47.7	99.9	90.7	87.8

Nominal level is 0.05. The largest standard error is 0.91%.

Table 1 shows the empirical sizes of our test statistics TS_1 and TS_2, and Li et al.'s TS_L. When $n = 200$, Li et al.'s test tends to be undersized for $\rho = 0.3$ and oversized for $\rho = 0.8, 0.9$, whereas the size of our tests are fairly close to the nominal level across different ρ's and is noticeably good when the temporal correlation is strong, i.e. $\rho = 0.9$. At a larger sample size $n = 500$, the size distortion of TS_L improves, but is still apparent when $\rho = 0.3$. A comparison of TS_1 and TS_2 shows that they offer similar sizes. To make the power comparison informative, we compute the size-adjusted power of the three test statistics, as displayed in Table 2. It appears that TS_1 and TS_2 have less power than Li et al.'s test, which holds almost uniformly for all grid sizes, AR(1) coefficients and both sample sizes. The loss of power can be attributed to the random normalization (or inefficient studentization) adopted in our test statistics. A similar phenomenon has also been observed by Lobato (2001), who found that the power loss due to self-normalization is modest compared to the alternative tests. Further we note that TS_1 outperforms TS_2 slightly in power.

3.2. Separability for bivariate space–time processes

The simulation studies are conducted according to the setup in Section 5.2 of Li et al. (2008). Specifically, we generate a bivariate space–time field $Z(\mathbf{s}, t) = \{Z_1(\mathbf{s}, t), Z_2(\mathbf{s}, t)\}' = FW(\mathbf{s}, t)$, where F is a 2 × 2 lower triangular matrix such that

$$FF' = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

Let $W(\mathbf{s}, t) = \{W_1(\mathbf{s}, t), W_2(\mathbf{s}, t)\}'$, where $W_1(\mathbf{s}, t)$ and $W_2(\mathbf{s}, t)$ are two independent vector autoregressive models of order 1 with Gaussian random noise and a spatial exponential covariance function. The testing lags are $k_1 = (\|\mathbf{h}\| = 1, u = 0)$, $k_2 = (\|\mathbf{h}\| = 2^{1/2}, u = 0)$, $k_3 = (\|\mathbf{h}\| = 2, u = 0)$ and $k_4 = (\|\mathbf{h}\| = 5^{1/2}, u = 0)$. Using the two contrasts defined in Li et al. (2008), we examine the size of the test for separability by specifying the spatial range parameters of W_1 and W_2 as $\phi_1 = \phi_2 = 3$, whereas we set $\phi_1 = 2$ and $\phi_2 = 4$ for the power assessment. Following Li et al. (2008), the subsampling window width is obtained by taking the average of the estimated temporal correlation for each variable and then plugging into formula (1).

From Tables 3 and 4, we see that for Li et al.'s test, the size distortion increases as the temporal correlation becomes stronger. However, both TS_1 and TS_2 exhibit accurate sizes, although they are a bit less powerful than Li et al.'s test. Nevertheless, considering the fact that Li et al.'s test overrejects under the null, the loss of the power of our test statistics is modest. Under this particular simulation setting, the performance of TS_1 and TS_2 is almost the same in terms of both sizes and powers. We also tried the test using four contrasts with four additional spatio-temporal lags $k_1' = (\|\mathbf{h}\| = 1, u = 1)$, $k_2' = (\|\mathbf{h}\| = 2^{1/2}, u = 1)$, $k_3' = (\|\mathbf{h}\| = 2, u = 1)$ and $k_4' = (\|\mathbf{h}\| = 5^{1/2}, u = 1)$ as done in Li et al. (2008). No substantial power improvement for all the tests was found, which is consistent with the findings reported in Table 2(b) of Li et al. (2008). We further compared the three test statistics in the symmetry test for a bivariate space–time field following the setup in Section 5.1 of Li et al. (2008). Since the findings are qualitatively similar to those reported above, we omit the results. Another comment to the above comparisons is that, to some

Table 3
Empirical sizes (in percent) of the tests for separability in the bivariate case.

Grid size	ρ	$\lvert T_n\rvert = 200$			$\lvert T_n\rvert = 500$		
		(a)	(b)	(c)	(a)	(b)	(c)
3 × 3	0.4	6.2	4.5	4.7	5.5	4.7	4.5
	0.6	8	5.1	5	6.6	4.4	4.5
	0.8	10.2	5.1	5	7.2	4.8	4.9
5 × 5	0.4	6.8	4.3	4.2	5.3	4.2	4.3
	0.6	7.7	4.8	4.5	6.9	4.7	4.6
	0.8	11.5	5.2	5.3	7.9	5.2	5.4

Nominal level is 0.05. The largest standard error is 0.58%. The columns (a), (b) and (c) present the rejection rates in percentage for the multivariate versions of TS_L, TS_1 and TS_2, respectively.

Table 4
Empirical raw powers (in percent) of the tests for separability in the bivariate case.

Grid size	ρ	$\lvert T_n\rvert = 200$			$\lvert T_n\rvert = 500$		
		(a)	(b)	(c)	(a)	(b)	(c)
3 × 3	0.4	27.9	15.8	15.6	57.7	35.9	35.4
	0.6	21.6	11.4	11	42	24.1	24.2
	0.8	16.6	7.4	7.4	23.9	12.9	12.9
5 × 5	0.4	63.6	37.8	37.7	95.5	72.7	73
	0.6	48	26.4	26.6	85	55.9	56
	0.8	30.6	14.1	14.9	54.8	30.2	30.4

Nominal level is 0.05. The largest standard error is 0.91%. The format is same as Table 3.

Table 5
Pacific Ocean wind data: testing full symmetry.

$\lVert\mathbf{h}\rVert$	u	TS_L	p-Value	$TS_1(TS_2)$	p-Value
1,2,3	1,3,5,7,9	152.7	$< 1.0e-16$	6467.7	< 0.005
1,5,10	1,3,5,7,9	85.6	6.5e−12	3350.7	< 0.005
10,11,12	1,3,5,7,9	27.6	0.025	830.5	> 0.1
1,2,3	1,2,3,4,5	96.8	5.1e−14	3433.3	< 0.005
1,5,10	1,2,3,4,5	87.5	2.9e−12	2961.5	< 0.005
10,11,12	1,2,3,4,5	24.7	0.054	762.4	> 0.1

extent, the AR(1) model used in the simulation favors Li et al.'s test because the optimal subsample block length in those tests are specifically configured for AR(1) process. Yet in reality the data structure is often not known, so the performance of Li et al.'s test using data-dependent bandwidth selection rule (1) becomes not very clear given a complex data structure.

4. Real data illustration

We revisit the two real data sets analyzed in Li et al. (2007). For the Pacific Ocean wind data, it contains the east–west wind velocity from a region over the tropical western Pacific Ocean for the period from November 1992 to February 1993. The winds are given every 6 h and at a 17 × 17 grid with grid interval of about 210 km. Following Li et al. (2007), we remove the time-averaged mean for each grid location and choose the same spatio-temporal testing lags as used in their paper, which results in $q = 15$. Tables 5 and 6 show the test results for full symmetry and separability, respectively. Note that the 90%, 95%, 97.5%, 99%, 99.5% upper critical values of U_{15} are 1662, 1957, 2261, 2658 and 2956, respectively; see Table 1 in Lobato (2001). In our table, we only give a range for the p-values of our tests, but the exact p-values can be obtained through simulations when necessary. Compared to Li et al.'s test, the p-value corresponding to our test seems larger, which may lead to different conclusions. For example, in the case of symmetry test, when $\lVert\mathbf{h}\rVert = (10, 11, 12)$ and $u = (1, 3, 5, 7, 9)$, our test would fail to reject the null hypothesis at 5% level, while Li et al.'s test rejects. Nevertheless, since the truth is unknown (i.e. whether the symmetry holds in this case), it is difficult to make judgement on the validity of the different conclusions drawn from the two tests. Either our test fails to reject due to its relatively lower power or Li et al.'s test falsely rejects due to its relatively larger size. Note that for all testing lags, the p-values delivered by TS_1 and TS_2 are fairly close as seen from their ranges. We also applied our test to the Irish wind data following the choice of the testing lags in Li et al. (2007). In accordance with the conclusion in Li et al. (2007), our new test thoroughly rejects the assumptions of fully symmetry and separability.

Table 6
Pacific Ocean wind data: testing separability.

$\|\mathbf{h}\|$	u	TS_L	p-Value	TS_1	p-Value	TS_2	p-Value
1,2,3	1,3,5,7,9	459.7	< 1.0e − 16	18 030	< 0.005	35 395	< 0.005
1,5,10	1,3,5,7,9	205.5	< 1.0e − 16	7565.1	< 0.005	11 735	< 0.005
10,11,12	1,3,5,7,9	40.4	0.0004	2038.0	(0.025,0.05)	3742.1	< 0.005
1,2,3	1,2,3,4,5	458.1	< 1.0e − 16	16 013	< 0.005	25 914	< 0.005
1,5,10	1,2,3,4,5	130.3	< 1.0e − 16	3046.0	< 0.005	4803.2	< 0.005
10,11,12	1,2,3,4,5	49.7	1.4e−5	2544.2	(0.01,0.025)	5026.3	< 0.005

5. Conclusions

In summary, we present two simple test statistics to test for symmetry and separability of space–time processes. Our test is built on the work by Li et al. (2007, 2008) and is also based on the asymptotic joint normality of sample space–time covariances. Our test statistics are asymptotically pivotal and free of any user-chosen numbers. The major difference between Li et al.'s tests and ours is that the former used a consistent subsampling-based estimator of the asymptotic covariance matrix, whereas our new test uses an inconsistent estimator to normalize out the asymptotic covariance matrix. As a result, Li et al.'s test involved user-chosen numbers in order to implement the subsampling method, while ours are free of such subjective choices.

Our test is very easy to implement, since the calculation of recursive estimates only require additional computation without the need to design any new algorithms. It can be routinely used by the applied researchers who do not have sophisticated knowledge on the bandwidth selection in consistent estimate of asymptotic covariance matrix. In general, our test holds sizes better than Li et al.'s test, although this leads to relatively low powers of our test. The moderate power loss of our test, which is consistent with the early findings by Lobato (2001) in a univariate time series context, is on the other hand a price we pay for simplicity and convenience.

In this paper, we focus on the tests for symmetry and separability. However, under the same asymptotic framework, extending the self-normalization technique to tests for other properties of univariate and multivariate space–time covariance functions, such as identifying the order of linear model of coregionalization (Wackernagel, 2003), seems straightforward. The extension of the self-normalization idea to testing procedures in a purely spatial domain, for instance, testing isotropy for a pure spatial covariance function, requires more effort due to the possible irregularly spaced locations. How to adapt this technique to lattice/nonlattice spatial data or regular/irregular configuration is certainly an interesting topic and is currently under investigation.

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