$L^\natural$-Convexity and Its Applications in Operations

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Abstract: $L^\natural$-convexity, one of the central concepts in discrete convex analysis, receives significant attentions in the operations literature in recent years as it provides a powerful tool to derive structures of optimal policies and allows for efficient computational procedures. In this paper, we present a survey of key properties of $L^\natural$-convexity and some closely related results in lattice programming, several of which were developed recently and motivated by operations applications. As a new contribution to the literature, we establish the relationship between a notion called μ-differential monotonicity and $L^\natural$-convexity. We then illustrate the techniques of applying $L^\natural$-convexity through a detailed analysis of a perishable inventory model and a joint inventory and transshipment control model with random capacities.

Key words: $L^\natural$-convexity, Lattice programming, Perishable inventory models, Random capacity
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1. Introduction

$L^\natural$-convexity is a key concept in discrete convex analysis introduced by Murota (1998) in an attempt to extend powerful convex analysis from continuous spaces to discrete spaces with lattice structures (here L stands for "lattice"). A $L^\natural$-convex function on an integer space has several salient properties: it can be extended to a convex function on a continuous space; local optimality guarantees global optimality; elegant duality results similar to what we see in convex programming duality hold. Though $L^\natural$-convexity was originally defined on integer spaces, the definition can be naturally extended to continuous spaces and imposes certain combinatorial structures on convex functions. Efficient algorithms have been developed for $L^\natural$-convex function minimization. We refer to Murota (2003) and Murota (2009) for a comprehensive treatment of discrete convex analysis including $L^\natural$-convexity.

The primary purpose of introducing $L^\natural$-convexity and some other related discrete convexity concepts is to provide a theoretical framework of tractable combinatorial optimization problems. Interestingly, the close connection of $L^\natural$-convexity with lattice programming, which will become clear in the next section, allows us to derive monotone comparative statics in many inventory models. Indeed, it was used by Zipkin (2008) to derive the optimal structural policy of lost-sales inventory models with positive lead times, which shed new lights on a classical result of Karlin and Scarf (1958) and Morton (1969). Since then, $L^\natural$-convexity was found to be powerful to establish the structures of optimal policies in various other operations models: serial inventory systems (Huh and Janakiraman 2010); inventory-pricing models with positive lead times (Pang et al. 2012); capacitated inventory systems with remanufacturing (Gong and Chao 2013); perishable inventory models (Chen et al. 2014); dual-sourcing models with random capacities, assemble-to-order systems with random capacities, and revenue management using booking limits (Chen et al. 2017); a joint inventory and transshipment control model with random capacities (Chen et al. 2015); an instantaneous control model of Brownian motion with positive lead time (Xu et al. 2016); etc.

Instead of providing a comprehensive survey of operations papers which use $L^\natural$-convexity, the purpose of this paper is to give a brief review of the concepts and properties of $L^\natural$-convexity which are useful in deriving structural optimal policies in operations models and illustrate how these properties can be used as a powerful tool to derive monotone comparative statics. A brief overview of lattice programming is also provided due to its close connection with $L^\natural$-convexity. We demonstrate the techniques of applying $L^\natural$-convexity through detailed analysis of a perishable inventory model and a two-location joint inventory and transshipment control model with random capacities and hope that through the analysis the readers can get a sense how these applications motivate some of the newly developed properties of $L^\natural$-convexity. Though this paper is intended to serve as a survey, it does provide some new results. For example, we show that the notion of μ-differential monotonicity is equivalent to $L^\natural$-convexity subject to a simple linear transformation.

The organization of this paper is as follows. In Section 2, we briefly review the basic concepts and properties in lattice programming and $L^\natural$-convexity. Though many results can be presented in more general terms, we refrain from doing it for the sake of readability. In Section 3, we review two inventory models in which $L^\natural$-convexity plays a critical role. Finally, we provide some concluding remarks in Section 4.

For easy reference, we list the main terminologies and notations here. Throughout this paper, we use decreasing, increasing and monotonicity in a weak sense. We use $\Re$ and $\Re_+$ to denote the real space and the set with nonnegative reals, $\mathcal{Z}$ and $\mathcal{Z}_+$ to denote the set of integers and the set of nonnegative integers, respectively. For convenience, let $\mathcal{F}$ be either $\Re$ or $\mathcal{Z}$. Define $\bar{\Re} = \Re \cup \{+\infty\}$, $e \in \mathcal{F}^n$ a vector whose components are all ones, e_j a unit vector whose jth component is one, and for $x, y \in \mathcal{F}^n$, $x \le y$ if and only if $x_i \le y_i$ for any $i = 1, ..., n$, $x^+ = \max(x, 0)$, $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$ (the component-wise minimum and maximum operations). The effective domain of a function $f : \Re^n \to \bar{\Re}$ is defined as $\mathrm{dom}(f) = \{x \in \Re^n | f(x) < +\infty\}$. The indicator function of any set $\mathcal{V} \subseteq \mathcal{F}^n$, denoted by $\delta_{\mathcal{V}}$, is defined as $\delta_{\mathcal{V}}(x) = 0$ for $x \in \mathcal{V}$ and $+\infty$ otherwise. We use the superscript T to denote the transpose of a vector or a matrix. We use uppercase letters (e.g. Ξ) to denote random vectors and lowercase letters (e.g., ξ) for their realizations. Given a random vector

$\Xi = (\Xi_1, ..., \Xi_n)^T$, we use $\mathcal{X} = Supp(\Xi)$ to denote the support of this random vector. In addition, we define $\bar{\xi}_j = \text{ess}\sup\{\xi_j | \xi_j \in \mathcal{X}_j\}$, $\underline{\xi}_j = \text{ess}\inf\{\xi_j | \xi_j \in \mathcal{X}_j\}$ for $j = 1, ..., n$, where $\mathcal{X}_j$ is $\mathcal{X}$'s projection into the j-th coordinate. Let $\bar{\xi} = (\bar{\xi}_1, ..., \bar{\xi}_n)^T$, $\underline{\xi} = (\underline{\xi}_1, ..., \underline{\xi}_n)^T$, and almost surely is abbreviated as a.s..

2. Lattice Programming and $L^\natural$-Convexity

In this section, we provide a brief review of lattice programming and $L^\natural$-convexity. The materials on lattice programming and $L^\natural$-convexity, unless specified, follow Topkis (1998) and Murota (2003) respectively. The readers may also refer to Simchi-Levi et al. (2014).

2.1. Lattice Programming

Since $L^\natural$-convexity is the combination of convex analysis with a lattice structure, we first introduce the concepts of lattice and supermodularity. Though the concepts can be defined on general partially ordered sets, for our purpose we focus on the Euclidean space $\Re^n$ with the standard partial order $\leq$, i.e., for any $x, x' \in \Re^n$, $x \leq x'$ if and only if $x_i \leq x'_i$ for $i = 1, 2, \ldots, n$.

To present the definitions of lattice and supermodularity, we first introduce two operations, *join* and *meet* operations, in $\Re^n$. For any two points $x = (x_1, x_2, \ldots, x_n)$ and $x' = (x'_1, x'_2, \ldots, x'_n)$ in $\Re^n$, define their *join* as

$$x \vee x' = (\max\{x_1, x'_1\}, \max\{x_2, x'_2\}, \ldots, \max\{x_n, x'_n\}),$$

and their *meet* as

$$x \wedge x' = (\min\{x_1, x'_1\}, \min\{x_2, x'_2\}, \ldots, \min\{x_n, x'_n\}).$$

Of course, if $x \leq x'$, i.e., $x_i \leq x'_i$ for $i = 1, 2, \ldots, n$, then $x \vee x' = x'$ and $x \wedge x' = x$. If none of $x \leq x'$ nor $x \geq x'$ is true, x and x' are called unordered. A set $X \subseteq \Re^n$ is called a *lattice* if for any $x, x' \in X$, $x \vee x', x \wedge x' \in X$. Note that X is called a *sublattice* of $\Re^n$ in some literature as it inherits the infimum and supremum from $\Re^n$.

Definition 1. A function $f : \Re^n \to \bar{\Re}$. The function f is supermodular, if for any $x, x' \in \Re^n$,

$$f(x) + f(x') \leq f(x \vee x') + f(x \wedge x'). \tag{1}$$

f is strictly supermodular, if the inequality (1) holds strictly for unordered pairs $x, x' \in \text{dom}(f)$. A function f is (strictly) submodular if $-f$ is (strictly) supermodular.

For a supermodular function f, its effective domain is a lattice. We say f is supermodular on a set $X \subseteq \Re^n$ if X is a lattice and inequality (1) holds for any $x, x' \in X$.

One can show that f is supermodular in $\Re^n$ if and only if f has increasing difference. That is, for any $x \in \Re^n$, any $i = 1, \ldots, n$ and a scalar $t > 0$, $f(x + te_i) - f(x)$ is increasing in x_j for any $j \neq i$. If a function $f : \Re^n \to \Re$ is differentiable, it is easy to verify that f is supermodular if and only if the partial derivative $\frac{\partial f(x)}{\partial x_i}$ is nondecreasing in x_j for all distinct indexes i and j and for any $x \in \Re^n$. Furthermore, if f is twice differentiable, then f is supermodular if and only if $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$ for any distinct indexes i and j and for any $x \in \Re^n$.

We now list below some useful properties about supermodular functions.

Proposition 1. *(a) Any nonnegative linear combination of supermodular functions is supermodular. That is, if $f_i : \Re^n \to \bar{\Re}$ $(i = 1, 2, \ldots, m)$ are supermodular, then for any scalar $\alpha_i \geq 0$, $\sum_{i=1}^m \alpha_i f_i$ is still supermodular.*

(b) If f_k is supermodular for $k = 1, 2, \ldots$ and $\lim_{k\to\infty} f_k(x) = f(x)$ for any $x \in \Re^n$, then $f(x)$ is supermodular.

(c) A composition of an increasing (decreasing) convex function and an increasing supermodular (submodular) function is still supermodular. That is, if $f : \Re \to \Re$ is convex and nondecreasing (nonincreasing) and $g : \Re^n \to \Re$ is increasing and supermodular (submodular), then $f(g(x))$ is supermodular.

(d) Assume that a function $f(\cdot, \cdot)$ is defined in the product space $\Re^n \times \Re^m$. If $f(\cdot, y)$ is supermodular for any given $y \in \Re^m$, then for a random vector ζ in $\Re^m$, $E_\zeta[f(x, \zeta)]$ is supermodular, provided it is well defined.

The following result establishes some connections between convexity and supermodularity, which are quite commonly used in operations models.

Theorem 1. *Let X be a lattice in $\Re^n$, $a_i \in \Re$ $(i = 1, 2, \ldots, n)$, and Y be a convex subset of $\Re$. Assume $\{\sum_{i=1}^n a_i x_i | x \in X\} \subseteq Y$. For a function $f : Y \to \Re$, define $g : X \to \Re$ with $g(x) := f(\sum_{i=1}^n a_i x_i)$ for any $x = (x_1, x_2, \ldots, x_n) \in X$. We have the following.*

(a) If $a_i \geq 0$ for $i = 1, 2, \ldots, n$, and f is convex on Y, then g is supermodular on X.

(b) If $n = 2$, $a_1 > 0$ and $a_2 < 0$, and f is concave on Y, then g is supermodular on X.

Suppose, in addition, that for any x, x' with $x \leq x'$, $x \in X$ implies $x' \in X$, $Y = \{\sum_{i=1}^n a_i x_i \mid x \in X\}$ and f is continuous on the interior of Y.

(c) If $n \geq 2$, $a_1 > 0$ and $a_2 > 0$, and g is supermodular on X, then f is convex on Y.

d) If $n \geq 2$, $a_1 > 0$ and $a_2 < 0$, and g is supermodular on X, then f is concave on Y.

(e) If $n \geq 3$, $a_1 > 0$, $a_2 > 0$ and $a_3 < 0$, and g is supermodular on X, then f is linear on Y.

One of the most widely used properties associated with supermodularity is on the monotonicity of the sets of optimal solutions of a class of parametric optimization problems. To present this property, we define the *induced set ordering* $\sqsubseteq$ which defines $X \sqsubseteq X'$ for two nonempty sets X and

X' if $x \in X$ and $x' \in X'$ imply that $x \wedge x' \in X$ and $x \vee x' \in X'$. Roughly speaking, $X \sqsubseteq X'$ implies that X contains smaller elements and X' contains larger elements.

Let $S(t)$ be a set function in $\Re^n$ parameterized by $t \in T \subseteq \Re^m$, i.e., for a parameter $t \in T$, $S(t)$ is a subset of $\Re^n$. Throughout this paper, the set function $S(t)$ is called increasing (or increasing set function) in t, if for any $t, t' \in T$ with $t \leq t'$, $S(t) \sqsubseteq S(t')$.

The concept of increasing set functions is different from set inclusion. To see this, notice that $S(t) = [t, +\infty)$ is an increasing set function for $t \in \Re$ but $S(t') \subset S(t)$ for $t \leq t'$. However, for an increasing set function S, it is straightforward to show that given $t, t' \in T$ with $t \leq t'$, for any $x \in S(t)$, there exists a point $x' \in S(t')$ and for any $x' \in S(t')$, there exists a point $x \in S(t)$ such that $x \leq x'$.

Proposition 2. *Let $S(t)$ be an increasing set function in $\Re^n$ parameterized by $t \in T \subset \Re^m$. We have that $S(t)$ is a lattice of $\Re^n$ for any $t \in T$. If in addition $S(t)$ is nonempty and compact for any $t \in T$, we can show that $S(t)$ has a largest and a smallest elements, which are increasing in t respectively.*

Under some supermodularity assumptions, the sets of optimal solutions for a collection of optimization problems parameterized by a parameter are increasing in the parameter. In addition, for a given parameter, there exist a largest and a smallest optimal solutions, which are increasing in the parameter as well. Consider the parametric optimization problem

$$f(t) := \sup_{x \in S(t)} g(x, t). \tag{2}$$

Let $\mathcal{A} := \{(x, t) \mid t \in T, x \in S(t)\} \subseteq \Re^n \times \Re^m$, the graph of the parametric constraint set $S(\cdot)$, and $S^*(t) := \text{argmax}_{x \in S(t)} g(x, t)$, the set of maximizers of problem (2).

Theorem 2. *Assume that $S(t)$ is increasing in $t \in T$, and $g(x, t) : \mathcal{A} \to \Re$ is supermodular in x for any fixed $t \in T$ and has increasing differences in (x, t).*

(a) $S^*(t)$ is increasing in t on $\{t \in T \mid S^*(t) \neq \emptyset\}$.

(b) Assume, in addition, that $S(t)$ is a nonempty and compact set of $\Re^n$ for any $t \in T$, and $g(x, t)$ is continuous in x on $S(t)$ for any $t \in T$. Then $S^*(t)$ is a nonempty and compact lattice and thus there exist $\bar{x}(t), \underline{x}(t) \in S^*(t)$ such that for any $x \in S^*(t)$, $\underline{x}(t) \leq x \leq \bar{x}(t)$. Furthermore, $\bar{x}(t)$ and $\underline{x}(t)$ are increasing.

Supermodularity can be preserved under the optimization operation (2), which is particularly useful to prove that supermodularity can be carried out in dynamic programming recursions.

Theorem 3. *Consider the optimization problem (2). Assume that the graph of the parametric constraint set, $\mathcal{A}$, is a lattice in $\Re^n \times \Re^m$ and $g(\cdot, \cdot) : \mathcal{A} \to \Re$ is supermodular. Let $\Pi_t(\mathcal{A}) = \{t \in T \mid S(t) \neq \emptyset\}$. Then $\Pi_t(\mathcal{A})$ is a lattice. In addition, f is supermodular over $\Pi_t(\mathcal{A})$.*

The critical assumption in the above theorem is that the graph of the parametric constraint set is a lattice. Yet in many inventory models this assumption is not valid. The following theorem relaxes the lattice condition yet imposes some other conditions. Consider the following optimization problem parameterized by a *two dimensional* vector $t \in T = \{Ax : x \in S\}$:

$$f(t) = \sup_x \{g(x) : Ax = t, x \in S\}, \tag{3}$$

where A is a $2 \times n$ matrix, S is a subset of $\Re^n$ and g is an n-dimensional function defined on S.

THEOREM 4. *(Chen et al. 2013) Given any $2 \times n$ matrix $A \geq 0$, if S is a nonempty closed convex sublattice, then so is the set T; moreover, if g is concave and supermodular on S, then so is f on T.*

2.2. $L^\natural$-Convexity

$L^\natural$-convexity was first proposed by Murota (1998) for functions defined on integer spaces. Specifically, a function $f : \mathcal{Z}^n \to \bar{\Re}$ is (discrete) $L^\natural$-convex if for any $x, y \in \mathcal{Z}^n$,

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right),$$

where $\lceil z \rceil$ is the smallest integer vector no less than z and $\lfloor z \rfloor$ is the largest integer vector no more than z. The above definition is an extension of the midpoint convexity defined in continuous spaces. Interestingly, if the effective domain of f is $\{0,1\}^n$, it is straightforward to verify that f is supermodular over $\{0,1\}^n$, which illustrates a strong connection between $L^\natural$-convexity and supermodularity. In fact, for our purpose, an equivalent but more convenient definition of $L^\natural$-convexity on $\mathcal{Z}^n$ is given as follows. In addition, the definition can be extended to functions defined on $\Re^n$. For some applications, we need the extension of $L^\natural$-convexity to more general spaces (see for example Chen et al. 2017 and Xu et al. 2016). For simplicity, we restrict to finite dimensional spaces.

DEFINITION 2 ($L^\natural$-CONVEXITY). A function $f : \mathcal{F}^n \to \bar{\Re}$ is $L^\natural$-convex if for any $x, x' \in \mathcal{F}^n$, $\alpha \in \mathcal{F}_+$,

$$f(x) + f(x') \geq f((x + \alpha e) \wedge x') + f(x \vee (x' - \alpha e)), \tag{4}$$

where e is the n-dimensional all-ones vector. A function f is $L^\natural$-concave if $-f$ is $L^\natural$-convex. A set $\mathcal{V}$ is $L^\natural$-convex if its indicator function, $\delta_\mathcal{V} : \mathcal{F}^n \to \bar{\Re}$, defined as $\delta_\mathcal{V}(x) = 0$ if $x \in \mathcal{V}$ and $+\infty$ otherwise, is $L^\natural$-convex.

We sometimes say a function f is $L^\natural$-convex on a set $\mathcal{V} \subseteq \mathcal{F}^n$ with the understanding that $\mathcal{V}$ is an $L^\natural$-convex set and the extension of f to the whole space $\mathcal{F}^n$ by defining $f(x) = +\infty$ for $x \notin \mathcal{V}$ is $L^\natural$-convex.

We have the following equivalent definition of $L^\natural$-convexity.

Proposition 3. *A function $f:\mathcal{F}^n \to \bar{\Re}$ is $L^\natural$-convex if and only if $g(x,\xi):=f(x-\xi e)$ is submodular on $(x,\xi)\in\mathcal{F}^n\times\mathcal{S}$, where $\mathcal{S}$ is the intersection of $\mathcal{F}$ and any unbounded interval in $\Re$.*

We can show that if $f:\Re^n \to \bar{\Re}$ is $L^\natural$-convex and continuous, then f is convex and submodular.

In the following we present some examples of $L^\natural$-convex functions and $L^\natural$-convex sets.

Proposition 4. *(a) Given any univariate convex (or discrete convex if $\mathcal{F}=\mathcal{Z}$) functions $g_i:\mathcal{F}\to\bar{\Re}$ $(i=1,\cdots,n)$ and $h_{ij}:\Re\to\bar{\Re}$ $(i\neq j)$, the function $f:\mathcal{F}^n\to\bar{\Re}$ defined by*

$$f(x):=\sum_{i=1}^{n} g_i(x_i)+\sum_{i\neq j} h_{ij}(x_i-x_j)$$

is $L^\natural$-convex. As a special case, any linear function is $L^\natural$-convex.

(b) A quadratic function $f:\Re^n\to\Re$ defined by $f(x)=\sum_{i,j=1}^n a_{ij}x_ix_j$ with $a_{ij}=a_{ji}\in\Re$ is $L^\natural$-convex if and only if the matrix A with its ij-th component being a_{ij} is a diagonally dominant M-matrix, i.e.,

$$a_{ij}\leq 0, \forall\ i\neq j, a_{ii}\geq 0, \text{ and } \sum_{j=1}^{n} a_{ij}\geq 0, \forall\ i.$$

(c) A twice continuously differentiable function $f:\Re^n\to\Re$ is $L^\natural$-convex if and only if its Hessian is a diagonally dominant M-matrix.

(d) For a given vector $a\in\Re^n$ and a nondecreasing univariate function $f:\Re\to\Re$, the function $g:\Re^n\to\Re$ defined by $g(x)=f(\max_{i=1:n}\{a_i+x_i\})$ is $L^\natural$-convex.

(e) A set with a representation $\{x\in\mathcal{F}^n: l\leq x\leq u, x_i-x_j\leq v_{ij}, \forall\ i\neq j\}$ is $L^\natural$-convex in the space $\mathcal{F}^n$, where $l,u\in\mathcal{F}^n$ and $v_{ij}\in\mathcal{F}$ $(i\neq j)$. In fact, any closed $L^\natural$-convex set in the space $\mathcal{F}^n$ can have such a representation.

We now list below some useful preservation properties about $L^\natural$-convex functions. Some of these properties are similar to but more restrictive compared with those in Proposition 1 which deals with supermodular functions.

Proposition 5. *(a) Any nonnegative linear combination of $L^\natural$-convex functions is $L^\natural$-convex. That is, if $f_i:\mathcal{F}^n\to\bar{\Re}$ $(i=1,2,\ldots,m)$ are $L^\natural$-convex, then for any scalar $\alpha_i\geq 0$, $\sum_{i=1}^m \alpha_i f_i$ is also $L^\natural$-convex.*

(b) If f_k is $L^\natural$-convex for $k=1,2,\ldots$ and $\lim_{k\to\infty} f_k(x)=f(x)$ for any $x\in\mathcal{F}^n$, then $f(x)$ is $L^\natural$-convex.

(c) Assume that a function $f(\cdot,\cdot)$ is defined on the product space $\mathcal{F}^n\times\mathcal{F}^m$. If $f(\cdot,y)$ is $L^\natural$-convex for any given $y\in\mathcal{F}^m$, then for a random vector ζ in $\mathcal{F}^m$, $E_\zeta[f(x,\zeta)]$ is $L^\natural$-convex, provided it is well defined.

(d) If $f:\mathcal{F}^n\to\bar{\Re}$ is an $L^\natural$-convex function, then $g:\mathcal{F}^n\times\mathcal{F}\to\bar{\Re}$ defined by $g(x,\xi)=f(x-\xi e)$ is also $L^\natural$-convex.

$L^\natural$-convexity is closely related to a notion called μ-differential monotone, introduced by Chen (2004) to analyze the optimality of hedging point policies for a stochastic two-product flexible manufacturing systems.

DEFINITION 3. Given a given positive vector $\mu \in \Re^2$, a convex function $f : \Re^2 \to \Re$ is said to be μ-differential monotone if for any $t > 0$,

(1) $f(x_1 + t, x_2) - f(x_1, x_2)$ is increasing in x_1 and x_2;

(2) $f(x_1, x_2 + t) - f(x_1, x_2)$ is increasing in x_1 and x_2;

(3) $f(x_1 + \mu_1 t, x_2) - f(x_1, x_2 + \mu_2 t)$ is increasing in x_1 and decreasing in x_2.

REMARK 1. Chen (2004) introduces the definition of μ-differential monotone based on directional derivatives, but shows that it is equivalent to the above one in function difference forms.

THEOREM 5. *Given a positive vector $\mu \in \Re^2$, if $f : \Re^2 \to \Re$ is convex, then f is μ-differential monotone if and only if $f_\mu : \Re^2 \to \Re$ defined by $f_\mu(x) = f(\mu_1 x_1, -\mu_2 x_2)$ is $L^\natural$-convex.*

Proof. We first assume f is twice continuously differentiable. From Proposition 4 part (c), we have that f_μ is $L^\natural$-convex if and only if its Hessian is a diagonally dominant M-matrix. That is, for any $x \in \Re^2$,

$$\frac{\partial^2 f_\mu(x_1,x_2)}{\partial x_1 \partial x_2} \le 0, \qquad \frac{\partial^2 f_\mu(x_1,x_2)}{\partial x_1^2} \ge -\frac{\partial^2 f_\mu(x_1,x_2)}{\partial x_1 \partial x_2}, \qquad \frac{\partial^2 f_\mu(x_1,x_2)}{\partial x_2^2} \ge -\frac{\partial^2 f_\mu(x_1,x_2)}{\partial x_1 \partial x_2}.$$

Since for $i, j = 1, 2$,

$$\frac{\partial^2 f_\mu(x_1,x_2)}{\partial x_i \partial x_j} = -\mu_i \mu_j \frac{\partial^2 f(\mu_1 x_1, -\mu_2 x_2)}{\partial x_i \partial x_j},$$

the above inequalities are equivalent to the following ones respectively: for any $x \in \Re^2$,

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} \ge 0, \qquad \mu_1 \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \ge \mu_2 \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \qquad \mu_2 \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \ge \mu_1 \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}. \tag{5}$$

On the other hand, the conditions in the definition of μ-differential monotone is equivalent to the inequalities (5) as well. Thus, our proposition holds for twice continuously differentiable functions.

If f is not twice continuously differentiable, we can smoothen f by $f^\epsilon(x) = E[f(x - \epsilon \tilde{z})]$, where $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$, and $\tilde{z}_1$ and $\tilde{z}_2$ are independent and follow the standard normal distribution. It is easy to see that if f is μ-differential monotone, then so does f^ϵ for any $\epsilon > 0$, which implies that f^ϵ_μ is $L^\natural$-convex. Letting $\epsilon \to 0$, from Proposition 5 part (b), we have f_μ is $L^\natural$-convex. Conversely, if f_μ is $L^\natural$-convex, then so does f^ϵ_μ by Proposition 5 part (c), which implies that f^ϵ is μ-differential monotone. Letting $\epsilon \to 0$, we have that f is μ-differential monotone. Q.E.D.

$L^\natural$-convexity is also closely related to the notion of multimodularity in discrete-event control (Hajek 1985). A function $f : \mathcal{F}^n \to \bar{\Re}$ with $\mathrm{dom}(f) \neq \emptyset$ is said to be multimodular if the function $\tilde{f} : \mathcal{F}^{n+1} \to \bar{\Re}$ defined by

$$\tilde{f}(x_0, x) = f(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1})$$

is submodular in $n+1$ variables. Murota (2005) establishes the following relationship.

PROPOSITION 6. *A function $f:\mathcal{F}^n \to \bar{\Re}$ is multimodular if and only if it can be represented as*

$$f(x) = g(x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots, x_1 + \dots, +x_n)$$

for some $L^\natural$-convex function g. Conversely, a function $g:\mathcal{F}^n \to \bar{\Re}$ is $L^\natural$-convex if and only if it can be represented as

$$g(p) = f(p_1, p_2 - p_1, p_3 - p_2, \dots, p_n - p_{n-1})$$

for some multimodular function f.

REMARK 2. Murota (2005) proves the above result for $\mathcal{F} = \mathcal{Z}$. It is straightforward to show that it is still true if $\mathcal{F} = \Re$.

Let

$$\mathcal{V}_{n,k} = \{(s_1, \dots, s_n) \in \mathcal{F}^n : s_1 \le \dots \le s_k\}$$

and

$$\mathcal{V}^+_{n,k} = \{(s_1, \dots, s_n) \in \mathcal{F}^n : 0 \le s_1 \le \dots \le s_k\}.$$

Note that both sets are $L^\natural$-convex.

The following result is developed in Chen et al. (2014) to analyze perishable inventory models.

PROPOSITION 7. *(Chen et al. 2014) Assume that $f:\mathcal{F}^n \to \bar{\Re}$ is an $L^\natural$-convex function. If f is nondecreasing in its first k $(1 \le k \le n)$ variables for $\boldsymbol{s} \in \mathcal{V}^+_{n,k}$, then the function*

$$\hat{f}(s_1, ..., s_n, s_{n+1}) := f((s_1 - s_{n+1})^+, ..., (s_k - s_{n+1})^+, s_{k+1} - s_{n+1}, ..., s_n - s_{n+1})$$

is $L^\natural$-convex for $(s_1, ..., s_n, s_{n+1}) \in \mathcal{V}_{n+1,k}$.If $f(\boldsymbol{s})$ is nondecreasing in all variables for $\boldsymbol{s} \in \mathcal{V}^+_{n,n}$, then the function

$$g(s_1, ..., s_n, s_{n+1}) := f((s_1 - s_{n+1})^+, ..., (s_k - s_{n+1})^+, s_{k+1} - s_k \wedge s_{n+1}, ..., s_n - s_k \wedge s_{n+1})$$

is $L^\natural$-convex on the $L^\natural$-convex set $\mathcal{V}_{n+1,n}$.

We now come back to the parametric optimization problem (2) and impose $L^\natural$-convexity. Similar to Theorem 2, we can show that the optimal solution set $S^*(t)$ is increasing in t. In addition, it has a bounded sensitivity.

THEOREM 6. *In the parametric optimization problem (2), assume that the graph of the parametric constraint set, $\mathcal{A}$, is an $L^\natural$-convex set of $\mathcal{F}^n \times \mathcal{F}^m$ and $g(\cdot,\cdot):\mathcal{F}^n \times \mathcal{F}^m \to \bar{\Re}$ is an $L^\natural$-convex function. Then the optimal solution set $S^*(t)$ is increasing in $t \in T$. In addition, for any $t \in T$ and a scaler $\omega \ge 0$ with $t + \omega e \in T$,*

$$S^*(t + \omega e) \sqsubseteq \omega\tilde{e} + S^*(t),$$

where $T = \{t \in \mathcal{F}^m | S^*(t) \neq \emptyset\}$, and e and $\tilde{e}$ are the all-ones vectors with dimensions consistent with t and x respectively.

REMARK 3. The above result first appears in Chen et al. (2017) and is a slight generalization of a corresponding one in Zipkin (2008) which deals with $m = 1$.

We also have the following preservation property of $L^\natural$-convexity, which again is very useful in dynamic programming recursions.

THEOREM 7. *In the parametric optimization problem (2), assume that the graph of the parametric constraint set,* $\mathcal{A}$, *is an* $L^\natural$*-convex set of* $\mathcal{F}^n \times \mathcal{F}^m$ *and* $g(\cdot,\cdot) : \mathcal{F}^n \times \mathcal{F}^m \to \bar{\Re}$ *is an* $L^\natural$*-convex function. Then* f *is* $L^\natural$*-convex over* $\mathcal{F}^m$ *if* $f(t) \neq -\infty$ *for any* $t \in \mathcal{F}^m$.

Similar to Theorem 4, we can relax the $L^\natural$-convexity assumption on the graph of the parametric constraint set when the parameters are restricted to a two-dimensional space.

THEOREM 8. *(Chen et al. 2013) Consider the parametric optimization problem:*

$$f(t) = \max_x \left\{ \sum_{i=1}^n g_i(x_i) : \sum_{i=1}^n x_i = t, x_i \in S_i \ \forall i = 1, \ldots, n \right\},$$

where $t, x_i \in \mathcal{F}^2$ *and* $S_i \subseteq \mathcal{F}^2$, $i = 1, \ldots, n$. *If all* $g_i : \mathcal{F}^2 \to \bar{\Re}$ *are* $L^\natural$*-convex functions and all* S_i *are* $L^\natural$*-convex sets, then* f *is* $L^\natural$*-convex on* $T = \{\sum_{i=1}^n x_i : x_i \in S_i \ \forall i = 1, \ldots, n\}$.

REMARK 4. Chen et al. (2013) presents the above theorem for $\mathcal{F} = \Re$. It is easy to see that it extends to $\mathcal{F} = \mathcal{Z}$ as well.

In an attempt to deal with inventory models with random capacity or revenue management problems using booking limit control in which decisions are truncated by random variables, Chen et al. (2017) derive useful properties of convexity, submodularity and $L^\natural$-convexity for the following problem.

$$f(x,z) = \inf_{u:(x,z,u)\in\mathcal{A}} E_\Xi[g(x, u\Diamond_k(z+\Xi))], \tag{6}$$

where $g(\cdot,\cdot) : \mathcal{F}^m \times \mathcal{F}^n \to \bar{\Re}$, $x \in \mathcal{F}^m, z \in \mathcal{F}^n$, the set $\mathcal{A} \subseteq \mathcal{F}^m \times \mathcal{F}^n \times \mathcal{F}^n$ is non-empty, and $\Diamond_k$ is defined as $u\Diamond_k\xi \triangleq (u_1 \wedge \xi_1, ..., u_k \wedge \xi_k, u_{k+1} \vee \xi_{k+1}, ..., u_n \vee \xi_n)$.

ASSUMPTION 1. *For any given* x, *(a)* $g(x,\cdot)$ *is lower semi-continuous with* $g(x,u) \to +\infty$ *for* $\|u\|_2 \to \infty$; *(b)* $g(x,\cdot)$ *is component-wise convex (component-wise discrete convex if* $\mathcal{F} = \mathcal{Z}$*).*

ASSUMPTION 2. *The random vector* Ξ *has independent components. Its support is denoted by* $\mathcal{X}$ *and the support of the* j*-th component is denoted by* $\mathcal{X}_j$.

ASSUMPTION 3. *The set* $\mathcal{A} = \{(x,z,u) | Au \le b, u_1 \ge \underline{u}_1, ..., u_k \ge \underline{u}_k, u_{k+1} \le \bar{u}_{k+1}, ..., u_n \le \bar{u}_n\}$, *where* $b, \underline{u}_1, ..., \underline{u}_k, \bar{u}_{k+1}, ..., \bar{u}_n$ *are parameters that may depend on* x *and* z, $A = (a_{ij})$ *with entries* $a_{ij} \ge 0$ *for any* i *and* $j = 1, ..., k$, *and* $a_{ij} \le 0$ *for any* i *and* $j = k+1, ..., n$. *In addition,* $\mathcal{X}_j$ *is contained in* $[\underline{u}_j - z_j, +\infty)$ *for* $j = 1, ..., k$, *and* $\mathcal{X}_j$ *is contained in* $(-\infty, \overline{u}_j - z_j]$ *for* $j = k+1, ..., n$.

Theorem 9. *(Chen et al. 2017) Consider the optimization problem* (6)*. Under Assumptions 1-3, problem* (6) *and the following optimization problem have the same optimal objective value.*

$$\begin{aligned} \inf\ & E[g(x, v_1(\Xi_1), \ldots, v_n(\Xi_n))] \\ s.t.\ & v_j(\xi_j) \le z_j + \xi_j \quad \forall \xi_j \in \mathcal{X}_j,\ \forall\ j = 1, \ldots, k \\ & v_j(\xi_j) \ge z_j + \xi_j \quad \forall \xi_j \in \mathcal{X}_j,\ \forall\ j = k+1, \ldots, n \\ & (x, z, v_1(\xi_1), \ldots, v_n(\xi_n)) \in \mathcal{A}^\Xi \quad \forall \xi \in \mathcal{X}, \end{aligned} \tag{7}$$

where $\mathcal{A}^\Xi = \{(x,z,w) | w = u \Diamond_k (z+\xi), (x,z,u) \in \mathcal{A}, \xi \in \mathcal{X}\}$*. Furthermore,*

(a) If g *and* $\mathcal{A}^\Xi$ *are convex, then* f *is also convex.*

(b) If g *is submodular and* $\mathcal{A}^\Xi$ *is a lattice, then* f *is also submodular.*

(c) If g *and* $\mathcal{A}^\Xi$ *are* $L^\natural$*-convex, then* f *is also* $L^\natural$*-convex.*

We refer to Gao (2017) for several generalizations of the above theorem: models with dependent random variables; models with a cost term $c(u)$; models taking risk preference into account.

The following theorem characterizes the monotonicity properties of the solution set to the optimization problem (6). Let $\mathcal{U}^*(x,z)$ denote the the optimal solution set of (6).

Theorem 10. *(Chen et al. 2017) Consider the optimization problem* (6)*. Under Assumptions 1-3, if* $\mathcal{A}^\Xi$ *is closed, and* $u_j \le z_j + \bar{\xi}_j, j = 1, ..., k, u_j \ge z_j + \underline{\xi}_j, j = k+1, ..., n$*, then we have the following results:*

(a) If g *is a submodular function, and* $\mathcal{A}, \mathcal{A}^\Xi$ *are lattices, then* $\mathcal{U}^*(x,z)$ *is increasing in* (x,z)*. There exist a greatest element and a least element in* $\mathcal{U}^*(x,z)$*, which are increasing in* (x,z)*.*

(b) If g *is an* $L^\natural$*-convex function, and* $\mathcal{A}, \mathcal{A}^\Xi$ *are* $L^\natural$*-convex sets, then* $\mathcal{U}^*(x,z)$ *is increasing in* (x,z) *and* $\mathcal{U}^*((x,z)+\omega e) \sqsubseteq \mathcal{U}^*(x,z) + \omega e$ *for any* $\omega > 0$*. Within* $\mathcal{U}^*(x,z)$*, there exist a greatest element and a least element, which have the above monotonicity properties with limited sensitivity.*

3. Inventory Applications

In this section, we illustrate how to apply $L^\natural$-convexity to a perishable inventory model and a joint inventory and transshipment control model.

3.1. Perishable Inventory Models

Dynamic inventory control for perishable products with fixed-lifetime was studied by Nahmias and Pierskalla (1973) in a two-period lifetime setting with zero lead time and demand uncertainty. Nahmias (1975) and Fries (1975) analyze the case with multi-period lifetime and zero lead time. Yet, their analysis is lengthy and difficult to be generalized. To highlight the complexity, note that "The main theorem requires 17 steps and is proven via a complex induction argument" (Nahmias 2011, page 10) and for models with discrete demand, a separate argument of using a sequence of continuous demand distributions to approximate the discrete demand distribution is needed (Nahmias and Schmidt 1986).

Employing $L^\natural$-convexity, Chen et al. (2014) give a significantly simpler proof of the structural result in Nahmias (1975) and Fries (1975)). In addition,they provide a unified approach to deal with both backlogging and lost sales models, and their approach allows for both continuous and discrete demand and many other extensions. Our presentation follows Chen et al. (2014) but focuses on a simplified version with lost sales and zero lead time.

Consider a periodic-review single-product inventory system over a finite horizon of T periods. The product is perishable and has a finite lifetime of exactly l periods. The replenishment lead time is assumed to be zero. In each period, a single price is charged for inventories of different ages, which are equally useful to fill consumer's price-sensitive demand. Demand is always met to the maximum extent with the on-hand inventory and we assume that unmet demand is lost. We assume that the retailer has the power or mechanism to determine how inventory is issued, and can also decide how much inventory to be carried over to the next period and how much inventory in addition to that at the end of its lifetime to be intentionally disposed of. The objective is to dynamically determine ordering, disposal and pricing decisions in all periods so as to maximize the total expected discounted profit over the planning horizon.

For convenience, we assume that the age of the inventory is counted from the period when the replenishment order is placed. If $l=1$, then the model reduces to a newsvendor model in the lost-sales case. If $l=\infty$, then it becomes a standard non-perishable inventory model. We assume that the costs and demand distributions are stationary.

The demand takes an additive form as is commonly used in the literature (see, e.g., Petruzzi and Dada 1999, Chen and Simchi-Levi 2004a,b). That is, the demand in period t is given as follows:

$$d_t := D(p) + \epsilon_t, \tag{8}$$

where $D(p)$ is the expected demand in period t and is strictly decreasing in the selling price p in this period, and ϵ_t is a random variable with zero mean. We assume that $\{\epsilon_t, t \geq 1\}$ are independently and identically distributed over time with a bounded support $[A, B]$, $(A \leq 0 \leq B)$. Let $F(\cdot)$ be the probability distribution function of ϵ_t. The selling price p is restricted to an interval $[\underline{p}, \overline{p}]$. To ensure non-negativity, we assume that $D(\bar{p}) + A \geq 0$.

Note that the monotonicity of the expected demand function implies a one-to-one correspondence between the selling price p and the expected demand $d \in \mathcal{D} \equiv [\underline{d}, \overline{d}]$, where $\underline{d} = D(\overline{p})$ and $\overline{d} = D(\underline{p})$. For convenience, we use the expected demand instead of the price as the decision variable in our analysis.

Let $R(d, y) = P(d)E[\min(d + \epsilon_t, y)]$ be the expected revenue for any given expected demand level d and on-hand inventory level y. Here $P(d)$ is the inverse function of $D(p)$.

Assumption 4. *$R(d,y)$ is continuous and $L^\natural$-concave in $(d,y)\in\mathcal{D}\times\Re_+$.*

In general, the concavity of the unconstrained revenue $P(d)d$ cannot guarantee the joint concavity of the inventory-truncated revenue $R(d,y)$. We refer to Chen et al. (2014) for conditions under which Assumption 4 holds. Under this assumption, 4 implies that the marginal value of the expected revenue is decreasing not only in the demand level but also in the on-hand inventory level. In addition, the higher the inventory level, the higher the marginal revenue of increasing demand level. In other words, the demand and on-hand inventory are complementary to each other.

The sequence of events in period t is as follows.

1. At the beginning of the period, the inventory levels of different residual useful lifetimes are observed.
2. Based on the inventory levels of different residual useful lifetimes, an order is placed and will be delivered immediately. At the same time, the selling price p_t of period t is determined.
3. During period t, demand d_t arrives, which is stochastic and depends on the selling price p_t, and is satisfied by on-hand inventory.
4. Unsatisfied demand is lost and the remaining inventory with zero useful lifetime has to be discarded. Meanwhile, unused inventory with positive useful lifetimes can be either intentionally discarded or carried over to the next period. In the latter case, their lifetimes decrease by one.

Each order incurs a variable cost c. Inventory carried over from one period to the next incurs a holding cost of h^+ per unit, and demand that is not satisfied from on-hand inventory incurs a cost of h^- per unit which represents the backlogging cost or the lost-sales penalty cost. Inventory that is disposed of incurs a disposal cost of θ per unit. Let $\gamma\in[0,1]$ be the discount factor.

The system state after receiving the order placed k periods ago but before placing an order can be represented by an $(l-1)$-dimensional vector $\boldsymbol{s}=[s_1,...,s_{l-1}]$. Here, s_i is the total amount of inventory with residual lifetimes no more than i periods, $i=1,...,l-1$. In particular, s_{l-1} is the system inventory position and s_{l-k} is the net on-hand inventory level. The state variables satisfy the condition that $0\le s_1\le s_2\le ...\le s_{l-1}$ and the set of feasible states is given by

$$\mathcal{F}_l=\mathcal{V}^+_{l-1,l-1}.$$

In the literature, it is common to denote the system state by an $(l-1)$-dimensional vector $\boldsymbol{x}=[x_1,...,x_{l-1}]$ where x_i is the amount of inventory on hand of age $l-i$. Clearly,

$$s_1=x_1, s_2=s_1+x_2,...,s_{l-1}=s_{l-1-1}+x_{l-1}.$$

Let s_l be the order-up-to level and a be the amount of on-hand inventory to be depleted or the realized demand of period t, whichever is greater. We have that

$$s_1\vee d_t\le a\le s_{l-k}\vee d_t.$$

When demand is greater than the total on-hand inventory level s_{l-k}, the above inequalities imply that $a = d_t$. Otherwise, they imply that $s_1 \vee d_t \leq a \leq s_{l-k}$, which in turn implies that (1) the firm satisfies the demand to the maximum extent, and (2) in addition to the inventory at the end of life, the firm could also intentionally dispose of some more inventory that will expire in later periods. We also note that since on-hand inventory can be intentionally disposed of, it is not difficult to show that it is optimal to deplete on-hand inventory sequentially with increasing useful lifetimes, i.e., FIFO depletion policy is optimal.

The system state at the beginning of the next period before ordering, denoted by $\widetilde{\boldsymbol{s}}$, evolves as follows:

$$\widetilde{\boldsymbol{s}} = [(s_2 - a)^+, ..., (s_l - a)^+]. \tag{9}$$

We are now ready to present the model formulation. Let $\hat{f}_t(\boldsymbol{s})$ be the profit-to-go function when the system state is specified by $\boldsymbol{s} \in \mathcal{F}_l$ at the beginning of period t before ordering. We can write the optimality equation as follows:

$$\hat{f}_t(\boldsymbol{s}) = \max_{s_l \geq s_{l-1}, d \in \mathcal{D}} \{R(d, s_l) + \mathbb{E}[\hat{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t)]\},$$

where

$$\hat{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \max_{s_1 \vee d_t \leq a \leq s_l \vee d_t} \{-c(s_l - s_{l-1}) - \theta(a - d_t) - h^+(s_l - a)^+ - h^-(a - s_l)^+ + \gamma \hat{f}_{t+1}(\widetilde{\boldsymbol{s}})\}.$$

Here the four terms in the maximization problem defining $\hat{g}_t$ represent the ordering cost, disposal cost, inventory holding cost, and lost-sales penalty cost, respectively. Also recall that $d_t = d + \epsilon_t$. For simplicity, we assume that $\hat{f}_{T+1}(\boldsymbol{s}) = cs_{l-1}$, i.e., inventory (or unfilled orders) at the end of the planning horizon is salvaged (or filled) with a unit price (or cost) equal to the unit ordering cost.

It is more convenient to work with a slightly modified profit-to-go function. Define for $\boldsymbol{s} \notin \mathcal{F}_l$, $f_t(\boldsymbol{s}) = +\infty$ and for $\boldsymbol{s} \in \mathcal{F}_l$, $f_t(\boldsymbol{s}) = \hat{f}_t(\boldsymbol{s}) - cs_{l-1}$ for all t. Then for $\boldsymbol{s} \in \mathcal{F}_l$, $f_{T+1}(\boldsymbol{s}) = 0$ and the optimality equation can be rewritten as follows:

$$f_t(\boldsymbol{s}) = \max_{s_l \geq s_{l-1}, d \in \mathcal{D}_t} G_t(\boldsymbol{s}, s_l, d) \tag{10}$$

with

$$G_t(\boldsymbol{s}, s_l, d) = R(d, s_l) + \mathbb{E}\left[g_t(\boldsymbol{s}, s_l, d|\epsilon_t)\right],$$

where

$$g_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \max_{s_1 \vee d_t \leq a \leq s_l \vee d_t} \{\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)\}, \tag{11}$$

$$\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t) = -cs_l + h^-(s_l - a) - (h^+ + h^- - \gamma c)(s_l - a)^+ - \theta(a - d_t) + \gamma f_{t+1}(\widetilde{\boldsymbol{s}}).. \tag{12}$$

Denote by $s_{lt}(\boldsymbol{s}), d_t(\boldsymbol{s})$ the optimal order-up-to inventory position and demand level decisions and $a_t(\boldsymbol{s}, s_l, d)$ the optimal inventory depletion solution for any given $(\boldsymbol{s}, s_l, d)$.

We assume that $h^+ + h^- \geq \gamma c$ which implies that the cost of carrying a unit of inventory to the next period while facing lost sales is larger than the potential salvage value of the inventory.

We have the following monotonicity property of the value-to-go functions, which allows us to use the preservation property of $L^\natural$-convexity in Proposition 7.

LEMMA 1. *For $t = 1, ..., T+1$ and any $\boldsymbol{s} \in \mathcal{F}_l$, $f_t(\mathbf{s})$ is nonincreasing in $s_1, ..., s_{l-1}$, respectively.*

We are ready to present our main result in this subsection.

THEOREM 11 (MONOTONICITY PROPERTIES OF LOST-SALES MODEL). *Under Assumption 4, for $t = 1, ..., T$, the functions $f_t(\boldsymbol{s})$, $g_t(\boldsymbol{s}, s_l, d)$ and $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ are $L^\natural$-concave in $\boldsymbol{s}$, $(\boldsymbol{s}, s_l, d)$ and $(\boldsymbol{s}, s_l, d, a)$, respectively. Thus, the optimal order-up-to level $s_{lt}(\boldsymbol{s})$ and the optimal demand level $d_t(\boldsymbol{s})$ are nondecreasing in $\boldsymbol{s}$ (i.e., the optimal price $p_t(\boldsymbol{s})$ is nonincreasing in $\boldsymbol{s}$), and for any $\omega \geq 0$*

$$s_{lt}(\boldsymbol{s} + \omega \boldsymbol{e}) \leq s_{lt}(\boldsymbol{s}) + \omega, \text{ and } d_t(\boldsymbol{s} + \omega \boldsymbol{e}) \leq d_t(\boldsymbol{s}) + \omega. \tag{13}$$

Given the realized demand d_t, the optimal depletion decision $a_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ is nondecreasing in $(\boldsymbol{s}, s_l, d)$ and for any $\omega \geq 0$

$$a_t(\boldsymbol{s} + \omega \boldsymbol{e}, s_l + \omega, d + \omega|\epsilon_t) \leq a_t(\boldsymbol{s}, s_l, d|\epsilon_t) + \omega. \tag{14}$$

Proof. First, applying Lemma 1, we know that $f_t(\boldsymbol{s})$ is nonincreasing in $\boldsymbol{s}$ for all t. Then, by Proposition 7, the $L^\natural$-concavity of f_{t+1} implies that $f_{t+1}(\tilde{\mathbf{s}})$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, d, a) \in \mathcal{V}^+_{l+1,l}$. The other terms in $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ are all $L^\natural$-concave in their variables. Thus, $\phi_t(\boldsymbol{s}, s_l, d, a|\epsilon_t)$ is $L^\natural$-concave for$(\boldsymbol{s}, s_l, d, a) \in \mathcal{V}^+_{l+2,l}$.

We next show that $g_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ is $L^\natural$-concave in $(\boldsymbol{s}, s_l, d) \in \mathcal{V}_{l+1,l}$. The idea is to use Theorem 7 to show that $L^\natural$-concavity can be preserved under the optimization problem (11). Unfortunately, the constraint in (11), $s_1 \vee d_t \leq a \leq s_{l-k} \vee d_t$, is not $L^\natural$-convex. Interestingly, we can prove that this constraint $a \leq s_l \vee d_t$ can be removed, i.e., $g_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \tilde{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t)$, where

$$\tilde{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t) = \max_{a \geq s_1 \vee d_t} \{\phi_t(\boldsymbol{s}, s_l, a, d|\epsilon_t)\}.$$

Although $a > s_l \vee d_t$ does not have any physical meaning, it is well defined mathematically. It suffices to show $\phi_t(\boldsymbol{s}, s_l, a, d)$ is decreasing in a for $a \geq s_l \vee d_t$, which is quite straightforward to verify from (12).

It is easy to check that the set $\{(s_1, d, a) : a \geq s_1 \vee d_t\} = \{(s_1, d, a) : a \geq s_1, a \geq d_t\}$ is $L^\natural$-convex. From Theorem 7, we have $\tilde{g}_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ and hence $g_t(\boldsymbol{s}, s_l, d|\epsilon_t)$ are $L^\natural$-concave in $(\boldsymbol{s}, s_l, d)$. Since $L^\natural$-concavity is preserved under expectation and $R(d, s_l)$ is $L^\natural$-concave by Assumption 4, the objective function in the optimization problem (10) is $L^\natural$-concave. This, together with $L^\natural$-convexity of the set associated with the constraints in the optimization problem (10) and Theorem 7, implies that $f_t(\boldsymbol{s})$ is $L^\natural$-concave in $\boldsymbol{s}$.

By Theorem 6, we know that the optimal order-up-to level $s_{lt}(\boldsymbol{s})$ and the optimal demand level $d_t(\boldsymbol{s})$ are nondecreasing in $\boldsymbol{s}$ and the inequalities in (13) hold (the existence of optimal solutions is straightforward to check and is thus omitted). The monotonicity of $P(d)$ implies that the optimal price $p_t(\boldsymbol{s})$ is nonincreasing in $\boldsymbol{s}$. The desired results hold. Q.E.D.

The inequalities in (13) imply that the optimal pricing and ordering decisions have bounded sensitivity. That is, a unit increase in some or all of the state variables will increase the order-up-to inventory position level $s_{lt}(\boldsymbol{s})$ and the optimal demand level $d_t(\boldsymbol{s})$ by at most one unit. The rates of the increase in decision variables are slower than that of the increase in state variables. These inequalities also provide insight into how the freshness of the inventory affects the inventory and pricing decisions. Comparing the states $\boldsymbol{s}$ and $\boldsymbol{s}+\boldsymbol{e}_i$, the latter has one more unit of inventory with residual lifetime of i periods but one less unit of inventory with residual lifetime of $i+1$ periods, $i=1,...,l-2$. These monotonicity properties imply that the fresher the inventory in the system, the less inventory to order and the higher price to charge.

The inequalities of (14) imply that the optimal depletion decisions also have bounded sensitivity. Furthermore, since $s_{lt}(\boldsymbol{s})$ and $d_t(\boldsymbol{s})$ are increasing in $\boldsymbol{s}$, the optimal depletion decision under the optimal policy, $a_t(\boldsymbol{s}, s_{lt}(\boldsymbol{s}), d_t(\boldsymbol{s})|\epsilon_t)$, is increasing in $\boldsymbol{s}$ and satisfies $a_t(\boldsymbol{s}+\omega\boldsymbol{e}, s_{lt}(\boldsymbol{s}+\omega\boldsymbol{e}), d_t(\boldsymbol{s}+\omega\boldsymbol{e})|\epsilon_t) \leq a_t(\boldsymbol{s}, s_{lt}(\boldsymbol{s}+\omega\boldsymbol{e}), d_t(\boldsymbol{s}+\omega\boldsymbol{e})|\epsilon_t)+\omega$ for any ϵ_t and $\omega > 0$. That is, the higher the total on-hand inventory level or the more aged the inventory in the system, the more inventory to deplete and to dispose of.

We now translate the structural properties of the optimal decisions with respect to $\boldsymbol{s}$ back to that with respect to $\boldsymbol{x}$. Let $\hat{x}_{lt}(\boldsymbol{x})$ and $\hat{d}_t(\boldsymbol{x})$ be the optimal order quantity and demand level with respect to $\boldsymbol{x}$, respectively.

Corollary 1 (Monotone Sensitivity). *For $t=1,...,T+1$ and for any $\omega \geq 0$, the following inequalities hold:*

$$-\omega \leq \hat{x}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_{l-1}) - \hat{x}_{lt}(\boldsymbol{x}) \leq ... \leq \hat{x}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_1) - \hat{x}_{lt}(\boldsymbol{x}) \leq 0, \tag{15}$$

$$0 \leq \hat{d}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_{l-1}) - \hat{d}_{lt}(\boldsymbol{x}) \leq ... \leq \hat{d}_{lt}(\boldsymbol{x}+\omega\boldsymbol{e}_1) - \hat{d}_{lt}(\boldsymbol{x}) \leq \omega. \tag{16}$$

Corollary 1 reveals that the optimal order quantity and demand level have bounded and monotone sensitivity. In particular, the inequalities in (15) imply that the optimal order quantity decreases in the inventory level of each age and the sensitivity decreases in age. That is, it is more sensitive to the younger inventory or outstanding order and least sensitive to the oldest order. The inequalities in (16) show that the optimal demand level increases in the level of inventory of each age and the sensitivity increases in age as well. In particular, it is most sensitive to the inventory close to expiration. That is, the more inventory to expire the more discount the seller should offer to induce more sales and avoid the disposals.

3.2. A Joint Inventory and Transshipment Control Model with Random Capacities

The joint inventory and transshipment control model is introduced in Hu et al. (2008) with a very lengthy and complicated analysis. Here we follow Chen et al. (2015), which, based on $L^\natural$-convexity, provide a significantly simplified proof of the main technical result in Hu et al. (2008).

Consider a firm operating two manufacturing facilities in separate markets through multiple time periods. Each facility faces uncertain capacities that are independent in time and of each other. Facilities also face uncertain demands which are independent in time but can be correlated across the two facilities. In each period, the firm's decisions can be divided into two stages. The first stage is the production stage where the firm decides how much it will produce in each of the facilities. After the production stage, the capacities and demands are realized. The firm's actual production quantity, which is the minimum of the planned production quantity and the realized capacity, incurs a unit production cost. The firm then enters the transshipment stage where it decides how much inventory to be transshipped from one facility to another. Finally, the demands are met and unsatisfied demands are lost. The firm receives linear revenue on satisfied demands and pays linear holding and transshipment costs. The problem is then to find the optimal production and transshipment quantities in each period so that the firm maximizes the total discounted profit over the planning horizon.

We now introduce the dynamic programming formulation of the optimization problem in Hu et al. (2008) as follows. Let $G_*^k(x_1^k, x_2^k)$ be the profit-to-go function when the current inventory levels at the two facilities are x_1^k and x_2^k respectively and there are k periods left in the planning horizon.
Production Stage:

$$\begin{aligned} G_*^k(x_1^k, x_2^k) = \max_{y_1^k \geq x_1^k, y_2^k \geq x_2^k} & E_{T_1^k, T_2^k, D_1^k, D_2^k}\{-c_1(y_1^k \wedge (x_1^k + T_1^k) - x_1^k) \\ & - c_2(y_2^k \wedge (x_2^k + T_2^k) - x_2^k) + r_1 D_1^k + r_2 D_2^k \\ & + J_*^k(y_1^k \wedge (x_1^k + T_1^k) - D_1^k, y_2^k \wedge (x_2^k + T_2^k) - D_2^k)\} \end{aligned} \tag{17}$$

Transshipment Stage:

$$J_*^k(z_1^k, z_2^k) = \max_{\hat{z}_1^k + \hat{z}_2^k = z_1^k + z_2^k} J^k(z_1^k, z_2^k, \hat{z}_1^k, \hat{z}_2^k), \tag{18}$$

where

$$\begin{aligned} J^k(z_1^k, z_2^k, \hat{z}_1^k, \hat{z}_2^k) = & -r_1(-\hat{z}_1^k)^+ - r_2(-\hat{z}_2^k)^+ - h_1(\hat{z}_1^k)^+ - h_2(\hat{z}_2^k)^+ \\ & -s_1(z_1^k - \hat{z}_1^k)^+ - s_2(z_2^k - \hat{z}_2^k)^+ + \alpha G_*^{k-1}((\hat{z}_1^k)^+, (\hat{z}_2^k)^+), \end{aligned} \tag{19}$$

and $G_*^0(x_1^0, x_2^0) \equiv 0$.

In the production stage, in period k, the target inventory levels at the two facilities y_1^k and y_2^k are decided. They are constrained to be no smaller than the current inventory levels at the two facilities x_1^k and x_2^k. The first two terms on the right hand side of (17) are the production costs with c_1, c_2 and T_1^k, T_2^k representing the marginal production costs and random capacities at the two facilities respectively. The next two terms are the full revenue collected over the realized demands, where r_1, r_2 and D_1^k, D_2^k are marginal revenue and random demand respectively. The revenue for the lost sales is deducted in the transshipment stage.

In the transshipment stage, in period k, the transshipment quantities or equivalently, the inventory levels after transshipment $\hat{z}_1^k$ and $\hat{z}_2^k$ are decided, whose sum is constrained to be equal to the inventory levels before transshipment (but after demands realization) z_1^k and z_2^k. The first two terms on the right hand side of (19) are the deducted revenue for the lost sales. The next two terms are the holding costs, where h_1 and h_2 are unit holding costs at the two facilities respectively. The two terms following are transshipment costs with s_1 (s_2) being the unit transshipment cost from facility 1 (2) to 2 (1). We assume that the marginal profit is always higher when the demand at a facility is satisfied by inventory at this particular facility than using transshipped inventory from the other facility. Note that, though the constraints in (18) seem to allow the transshipped quantity from a facility to exceed its available inventory, this assumption implies that it is never optimal to do so. Finally, α in (19) is the discount factor.

Under the assumption of continuous demands and capacities, Hu et al. (2008) prove the following properties on the profit-to-go function $G_*^k(x_1, x_2)$.

$\mathbb{A}_1$*:* $G_*^{k-1}(x_1, x_2)$ is jointly concave in x_1 and x_2, and

$$\frac{\partial^2}{\partial x_1^2} G_*^{k-1}(x_1, x_2) \le \frac{\partial^2}{\partial x_1 \partial x_2} G_*^{k-1}(x_1, x_2),$$
$$\frac{\partial^2}{\partial x_2^2} G_*^{k-1}(x_1, x_2) \le \frac{\partial^2}{\partial x_2 \partial x_1} G_*^{k-1}(x_1, x_2);$$

$\mathbb{A}_2$*:* $G_*^{k-1}(x_1, x_2)$ is submodular and

$$\frac{\partial^2}{\partial x_1 \partial x_2} G_*^{k-1}(x_1, x_2) = \frac{\partial^2}{\partial x_2 \partial x_1} G_*^{k-1}(x_1, x_2).$$

The above properties are essential for Hu et al. (2008) to derive the optimal transshipment and production policies. Through an inductive argument, their proof relies on a full characterization of the optimal transshipment policy and a rather involved analysis of the derivatives which spans several pages. In the following, we illustrate how the concept and properties of $L^\natural$-convexity in Section 2 can be used to simplify the proof of their key result. Interestingly, our approach does not rely on the characterization of the optimal policy and it applies to discrete demands as well as capacities without any further efforts.

Denote d_i^k as the realization of demand for facility i in period k, and define $q_i^k = z_i^k + d_i^k$, $w_i^k = \hat{z}_i^k + d_i^k$, i.e., q_i^k and w_i^k are the before and after transshipment inventory levels at facility i respectively after production is done but before demand is fulfilled. Clearly, $q_i^k, w_i^k \ge 0$. To employ $L^\natural$-convexity, it is more convenient to change variables by letting $\tilde{y}_2^k = -y_2^k$, $\tilde{x}_2^k = -x_2^k$, $\tilde{T}_2^k = -T_2^k$, $\tilde{q}_2^k = -q_2^k$, $\tilde{w}_2^k = -w_2^k$. Then the original problem can be equivalently reformulated as

$$\tilde{G}_*^k(x_1^k,\tilde{x}_2^k)=\max_{y_1^k\ge x_1^k,\tilde{y}_2^k\le\tilde{x}_2^k} E_{T_1^k,\tilde{T}_2^k,D_1^k,D_2^k}\{-c_1(y_1^k\wedge(x_1^k+T_1^k)-x_1^k) \\ +c_2(\tilde{y}_2^k\vee(\tilde{x}_2^k+\tilde{T}_2^k)-\tilde{x}_2^k)+\tilde{J}_*^k(y_1^k\wedge(x_1^k+T_1^k),\tilde{y}_2^k\vee(\tilde{x}_2^k+\tilde{T}_2^k)|D_1^k,D_2^k)\}, \tag{20}$$

where $\tilde{G}_*^k(x_1^k,\tilde{x}_2^k)=G_*^k(x_1^k,-\tilde{x}_2^k)$ and by introducing a new variable v

$$\begin{aligned}\tilde{J}_*^k(q_1^k,\tilde{q}_2^k|d_1^k,d_2^k)=&\max_{w_1^k,\tilde{w}_2^k,v}\tilde{J}(w_1^k,\tilde{w}_2^k,v|d_1^k,d_2^k)\\ s.t.\quad & w_1^k+v=q_1^k\\ & \tilde{w}_2^k+v=\tilde{q}_2^k\\ & w_1^k\ge 0,\tilde{w}_2^k\le 0,\end{aligned} \tag{21}$$

where

$$\begin{aligned}&\tilde{J}(w_1^k,\tilde{w}_2^k,v|d_1^k,d_2^k)=r_1(w_1^k\wedge d_1^k)+r_2((-\tilde{w}_2^k)\wedge d_2^k)\\ &-h_1(w_1^k-d_1^k)^+-h_2(-\tilde{w}_2^k-d_2^k)^+-s_1v^+-s_2(-v)^+\\ &+\alpha\tilde{G}_*^{k-1}((w_1^k-d_1^k)^+,-(-\tilde{w}_2^k-d_2^k)^+).\end{aligned} \tag{22}$$

Note that in (21), we impose $w_1^k\ge 0$ and $\tilde{w}_2^k\le 0$, which guarantee that the transshipped quantities won't exceed the inventory levels at the facilities. These constraints can be safely removed as done in Chen et al. (2015) under the assumption on the marginal profit made earlier.

Now we are ready to state and prove our main result, which offers a new approach that proves the key properties $\mathbb{A}_1$ and $\mathbb{A}_2$ when demands and capacities are continuous .

THEOREM 12. *Suppose that $\tilde{G}_*^{k-1}(\cdot,\cdot)$ is $L^\natural$-concave, then $\tilde{G}_*^k(\cdot,\cdot)$ is also $L^\natural$-concave.*

Proof. For notational brevity, we omit the superscript k in the proof when there is no ambiguity. Define u_1, u_2 as the inventory level after the sales assuming that *the firm can hold inventory with some demand unsatisfied.* Then the realized sales are given by w_1-u_1 and w_2-u_2 at the two facilities respectively. By letting $\tilde{u}_2=-u_2$, we claim that $\tilde{J}(w_1,\tilde{w}_2,v|d_1,d_2)$ equals the optimal objective value of the following problem:

$$\begin{aligned}&\max_{u_1,\tilde{u}_2} r_1(w_1-u_1)-r_2(\tilde{w}_2-\tilde{u}_2)-h_1u_1+h_2\tilde{u}_2\\ &-s_1v^+-s_2(-v)^++\alpha\tilde{G}_*^{k-1}(u_1,\tilde{u}_2)\\ s.t.\quad & 0\le u_1,u_1-w_1\le 0,\\ &\tilde{w}_2-\tilde{u}_2\le 0,\tilde{u}_2\le 0,\\ &w_1-u_1\le d_1,u_2-\tilde{w}_2\le d_2.\end{aligned} \tag{23}$$

Note that facing a stationary system, the firm should never hold inventory and reject demand at the same time since it is always more profitable to satisfy the current demand than holding the inventory to fulfill future demands. Therefore, the optimal solution is $u_1=(w_1-d_1)^+$, $u_2=-(-\tilde{w}_2-d_2)^+$ and our claim is correct.

We further claim that the objective function of the problem (23) is $L^\natural$-concave in $(w_1, \tilde{w}_2, v, u_1, \tilde{u}_2)$. To see this, note that $\tilde{G}_*^{k-1}(u_1, \tilde{u}_2)$ is $L^\natural$-concave by our induction hypothesis. The $L^\natural$-concavity of the rest of terms in the objective function is straightforward to verify. The constraint set is $L^\natural$-convex according to Proposition 4 part (e). Then the $L^\natural$-concavity of $\tilde{J}(w_1, \tilde{w}_2, v|d_1, d_2)$ follows from Theorem 7.

Note that the objective function in (21) is separable in variables $(w_1, \tilde{w}_2)$ and (v, v). Thus, the $L^\natural$-concavity of $\tilde{J}_*^k(q_1, \tilde{q}_2|d_1, d_2)$ follows from Theorem 8.

By defining $\tilde{G}(y_1, y_2) = E_{D_1, D_2}\{-c_1 y_1 + c_2 y_2 + \tilde{J}_*^k(y_1, y_2|D_1, D_2)\}$, (20) can be expressed as

$$\tilde{G}_*(x_1, \tilde{x}_2) = \max_{y_1 \geq x_1, \tilde{y}_2 \leq \tilde{x}_2} E_{T_1, \tilde{T}_2}\{\tilde{G}(y_1 \wedge (x_1 + T_1), \tilde{y}_2 \vee (\tilde{x}_2 + \tilde{T}_2))\} + c_1 x_1 - c_2 \tilde{x}_2$$

Clearly $\tilde{G}(y_1, y_2)$ is $L^\natural$-concave in (y_1, y_2). Moreover, $y_1 \wedge (x_1 + T_1) = (y_1 - x_1) \wedge T_1 + x_1$ and $\tilde{y}_2 \vee (\tilde{x}_2 + \tilde{T}_2) = (\tilde{y}_2 - \tilde{x}_2) \vee \tilde{T}_2 + \tilde{x}_2$. It is easy to see that by transforming the variables $\hat{y}_1 = y_1 - x_1$ and $\hat{y}_2 = \tilde{y}_2 - \tilde{x}_2$, the above problem can be expressed in the form of (6). Then Theorem 9 implies that the profit-to-go function $\tilde{G}_*(x_1, \tilde{x}_2)$ is $L^\natural$-concave. Q.E.D.

Using Proposition 4 part (b), it is straightforward to check that Theorem 12 implies the properties $\mathbb{A}_1$ and $\mathbb{A}_2$ of $G_*^k(\cdot, \cdot)$ when demands and capacities are continuous, which can then be used to establish the structure of the optimal policies. Interestingly, as $L^\natural$-convexity is defined for both continuous space and discrete space, with some minor modifications of the analysis in Hu et al. (2008), the structure of the optimal policies can be derived from Theorem 12 for both continuous and discrete demands as well as capacities.

4. Conclusion

In this paper, we provide a brief survey of the properties of $L^\natural$-convexity and closely related results in lattice programming. These properties are then used to analyze a perishable inventory model and a joint inventory and transshipment control model with random capacities. When applying $L^\natural$-convexity to derive monotone comparative statics of optimal policies, one has to choose the representations of states appropriately. Once this is done, the analysis is more or less standard and the structures of optimal policies have similar flavors. Our choice of the perishable inventory model and the joint inventory and transshipment control model highlights how operations applications can motivate the development of some new properties of $L^\natural$-convexity.

$L^\natural$-convexity has important computational implications. Indeed, when minimizing a $L^\natural$-convex function, local optimality leads to global optimality and the steepest descent algorithm can be applied to find an optimal solution efficiently with proven computational complexity bounds polynomial in the number of variables (see Murota 2003 for $L^\natural$-convex function minimization in integer spaces). Based on discrete $L^\natural$-convexity, Lu and Song (2005) show that in an order-based multi-product assemble-to-order system, the optimal base-stock levels can be obtained in a greedy fashion.

For stochastic dynamic programs with cost-to-go functions being discrete $L^\natural$-convex, Chen et al. (2014) propose a pseudo-polynomial time approximation scheme and demonstrate its effectiveness on a stochastic inventory model with lost sales and a positive lead time. Sun et al. (2014) develop quadratic approximation of cost-to-go functions for the lost sales and perishable inventory control problems. Finally, we refer to Begen and Queyranne (2011) and Ge et al. (2014) for applications of $L^\natural$-convexity to appointment scheduling for a given sequence of jobs on a single processor with random durations.

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