Duality Approaches to Economic Lot-Sizing Games

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Abstract

Sharing common production, resources and services to reduce cost are important for not for profit operations due to limited and mission-oriented budget and effective cost allocation mechanisms are essential for encouraging effective collaborations. In this paper, we illustrate how rigorous methodologies can be developed to derive effective cost allocations to facilitate sustainable collaborations in not for profit operations by modeling the cost allocation problem arising from an economic lot-sizing (ELS) setting as a cooperative game.

Specifically, we consider the economic lot-sizing (ELS) game with general concave ordering cost. In this cooperative game, multiple retailers form a coalition by placing joint orders to a single supplier in order to reduce ordering cost. When both the inventory holding cost and backlogging cost are linear functions, it can be shown that the core of this game is non-empty. The main contribution of this paper is to show that a core allocation can be computed in polynomial time under the assumption that all retailers have the same cost parameters.

Our approach is based on linear programming (LP) duality. More specifically, we study an integer programming formulation for the ELS problem and show that its LP relaxation admits zero integrality gap, which makes it possible to analyze the ELS game by using LP duality. We show that there exists an optimal dual solution that defines an allocation in the core.

An interesting feature of our approach is that it is not necessarily true that every optimal dual solution defines a core allocation. This is in contrast to the duality approach for other known cooperative games in the literature.

Keywords: Economic Lot-Sizing Problems; Cooperative Games; Cost Allocation; Core; Duality.

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1 Introduction

To employ economics of scale and risk pooling, business entities build and share common facilities, resources or service capacities. For example, nearby towns build and share a common water supply system (Young et al. 1982); different market agents in power industry use common transmission and distribution networks (Zolezzi and Rudnick 2002); recipients of multicasted contents make shared use of network transmission services (Feigenbaum et al. 2000); in states under Extended Producer Responsibility regulations, producers share the cost for the post-use collection, recycling and disposal of their products when a state-run agency is responsible for processing all E-waste collected (Gui et al. 2014); humanitarian organizations cooperate in fundraising, procurement, transportation, and stock storage to improve efficiency and speed of disaster response (Arikan et al. 2015); small or isolated countries use pooled procurement mechanisms to "access a sustainable supply of quality vaccines, achieve greater demand predictability, reduce transaction costs, and (sometimes) reduce the total price paid for vaccines and related products" (WHO website on vaccine procurement).

A key issue is how the resulting cost should be allocated to various stake holders. Poorly designed and implemented allocation policies may lead to distorted incentives and result in inefficient operations (Shapley 1962) or even lead to lawsuit (Armed Services Board of Contract Appeals No. 58587). Sharing common production, resources and services to reduce cost are especially important for not for profit operations due to limited and mission-oriented budget and effective cost allocation mechanisms are essential for encouraging effective collaborations. In this paper, we model the cost allocation problem arising from an economic lot-sizing (ELS) setting as a cooperative game and illustrate how linear programming duality can be used to derive cost allocations leading to stable cooperations. In our presentation, the agents involved are referred to as retailers to reflect the tradition of the ELS model. However, the same model can be used to capture the essence of pooled procurements in which economics of scale are critical. Our hope is to provide rigorous methodologies useful to derive effective cost allocations to facilitate sustainable collaborations in not for profit operations.

We consider a situation where multiple retailers sell the same product, which is ordered from a single manufacturer. In a decentralized system, each retailer would solve an ELS problem, i.e., each retailer needs to decide the order quantity at each time period of a finite planning horizon so that the demand is satisfied at a minimum total cost, including ordering cost and inventory holding cost. Throughout the paper, we assume that demand of a period can be backlogged and be fulfilled by orders at later periods while incurring a backlogging cost (this is referred to as *ELS with backlogging*).

By exploiting economies of scale, captured by concave ordering cost here, the retailers may find it beneficial to form coalitions and place joint orders. One important issue is how to allocate the cost or profit in such a way that is considered advantageous by all the retailers, i.e., no retailer(s) gain more by deviating from the cooperation. This naturally gives rise to a cooperative game, referred to as the ELS game, and its cost allocation can be studied by using concepts from the cooperative game theory. In this paper, we mainly focus on the core of the ELS game with concave ordering cost.

To the best of our knowledge, Tamir (1992) is the first to analyze the ELS game with fixed ordering cost. To study the ELS game with general concave ordering cost, one may formulate the ELS problem with concave ordering cost as an uncapacitated facility location problem with concave facility cost, which can be further formulated as an uncapacitated facility location problem with fixed facility cost (see, for example, Mahdian et al. 2006). The linear programming (LP) relaxation of such a formulation always has an integral solution, i.e., there is no integrality gap between the integer program and its LP relaxation. This fact allows one to show the existence of a core allocation of the ELS game with concave ordering cost; see Tamir (1992). However, the size of the LP relaxation is not necessarily polynomial in the input of the game. Thus, this approach does not provide a polynomial time algorithm for finding a core allocation of the ELS game with general concave ordering cost. Instead, we consider an *alternative* integer programming formulation for the ELS problem whose LP relaxation always has an integral solution as well. Our main contribution is to show that, for the ELS game with general concave ordering cost, an allocation in the core can be computed in polynomial time under the assumption that all retailers have the same cost parameters.

Our approach is based on LP duality, and is inspired by the work of Owen (1975), who used LP duality to show the non-emptiness of the core for a class of linear production games. Owen's approach has been applied and/or extended to other cooperative games; see, for instance, Granot (1986), Tamir (1991,1992), and Goemans and Skutella (2004). One scheme commonly seen in all these approaches is that one first formulates the underlying optimization problem of the cooperative game as a linear program, and then use the dual variables to define an allocation that can be proven in the core. In fact, for all these games, the set of allocations defined by optimal dual variables, often referred to as Owen set, is always a subset of the core (see Samet and Zemel (1984) for the relationship between Owen set and the core).

Our approach has an interesting feature that is in contrast to the duality approach for other known cooperative games in the literature. On the one hand, allocations defined by some optimal solutions to the dual of the LP relaxation may not be in the core of the ELS game. On the other hand, there always exists an optimal dual solution that defines an allocation in the core, which can be found in polynomial time.

Our analysis is crucially based on the fact that there always exists an optimal dual solution that has certain monotonicity. Such a property is quite intuitive and might be interesting on its own.

The topic of this paper falls into a stream of recent research on applying cooperative game theory in the area of inventory management; see, for instance, Anily and Haviv (2007), Chen (2009), Chen and Zhang (2009), Dror and Hartman (2007), Muller et al. (2002), and Zhang (2009) for detailed discussions on inventory games. After the first appearance of our paper, Gopaladesikan et al. (2012) and Toriello and Uhan (2014), building upon some structural properties established here, propose primal-dual algorithms for computing a core allocation, and develop cost allocations in the strong sequential core for economic lot-sizing games without backlogging, respectively.

The organization of this paper is as follows. In Section 2 we review the basic concepts in cooperative game theory and present the ELS game model. In Section 3 we analyze the ELS game with setup cost. In Section 4, we consider an LP formulation for the ELS problem with general concave ordering cost, which allows us to compute a core allocation for the ELS game in polynomial time. Some concluding remarks and open problems are presented in Section 5.

2 Preliminaries and Economic Lot Sizing Game Model

2.1 Cooperative Games

Here we briefly introduce some basic concepts of cooperative game theory that will be used in this paper; see Peleg and Sudhölter (2003) for more details. Let $N = \{1, 2 \cdots, n\}$ be the set of players. A collection of players $S \subseteq N$ is called a coalition. The set N is sometimes referred to as the grand coalition. A characteristic cost function $F(S)$ is defined for each coalition $S \subseteq N$, which could be the minimum total cost that coalition S should pay if the members of S decide to secede from the grand coalition and cooperate only among themselves. A cooperative game is determined by the pair (N, F). For each subset $S \subset N$, the cooperative game (S, F) is called a subgame.

In cooperative games with transferable cost, the cost of a coalition S can be transferred between the players of S. In such games a coalition S can be completely characterized by $F(S)$. The coalition is allowed to split the cost $F(S)$ among its members in any possible way.

A vector $l = (l_1, l_2, \cdots, l_N)$ is called an allocation for the game (N, F) if $\sum_{r\in N} l_r = F(N)$. The core of a cooperative game is a solution concept which requires that no subset of players has an incentive to secede.

Definition 1. *An allocation l is in the* core *of the game (N, F), if $\sum_{r\in N} l_r = F(N)$ and for any subset $S \subseteq N$, $\sum_{r\in S} l_r \leq F(S)$.*

There are several interesting questions related to the core of a cooperative game (N, F). In particular, we would like to know whether the core of (N, F) is non-empty or not, and if yes, how to design an algorithm to find a core allocation efficiently.

2.2 The Economic Lot-Sizing Problem

The basic economic lot-sizing model was proposed by Manne (1958) and Wagner and Whitin (1958). In this model, demand for a single product occurs during each of T consecutive time periods numbered through 1 to T. The demand of a given time period can be satisfied by orders at that period or at pervious periods, i.e., backlogging is not allowed. The model includes ordering cost and inventory holding cost. The objective is to decide the order quantity at each time period so that the demand is satisfied at a minimum total cost. Without loss of generality, we assume that the initial inventory is zero and lead time is also zero.

The ELS model considered in this paper, which was proposed by Zangwill (1969), allows backlogging. At any period, the unfulfilled demand incurs penalty cost referred to as backlogging cost.

In order to present a mathematical formulation for the ELS problem, we introduce the following notations. In particular, for each $t : 1 \leq t \leq T$, define: $d_t \geq 0$ – the amount of demand in period t, which is assumed to be an integer; z_t – the order quantity at period t (t is called an ordering period if $z_t > 0$); I_t^+ – the amount of (non-negative) inventory at the end of period t; I_t^- – the amount of backlogged demand at period t; $c_t(z_t)$ – the cost of ordering z_t units at period t; $h_t^+ \geq 0$ – the unit cost of holding inventory at the end of period t; $h_t^- \geq 0$ – the unit cost of having backlogged demand at period t.

The following mathematical formulation is sometimes called the flow formulation for the ELS problem:

$$\begin{aligned} C(d) := \quad \min \quad & \sum_{1 \leq t \leq T} \left\{ c_t(z_t) + h_t^+ I_t^+ + h_t^- I_t^- \right\} \\ \text{s.t.} \quad & z_t + I_{t-1}^+ - I_{t-1}^- = d_t + I_t^+ - I_t^-, \forall\ t = 1, \ldots, T, \\ & I_0^+ = I_0^- = 0, \\ & z_t \geq 0, I_t^+ \geq 0, I_t^- \geq 0\ \forall t = 1, \ldots, T, \end{aligned} \tag{1}$$

where $d = (d_1, d_2, \ldots, d_T)^T$ is a vector in Z_+^T, and the first constraint is the inventory balance equation.

We assume that the ordering cost function $c_t(\cdot)$ is non-decreasing (however all our results in this paper continue to hold without the non-decreasing assumption) and concave with $c_t(0) = 0$. Under the concavity assumption, problem (1) is a concave minimization problem over a polyhedron.

Therefore, there exists an optimal solution that is an extreme point of the polyhedron. It follows that, as proved by Zangwill (1969), there exists an optimal solution to (1) with the following properties: a) $I_{t-1}^{+} > 0$ implies $z_t = 0$; b) $z_t > 0$ implies $I_t^{-} = 0$; and c) $I_{t-1}^{+} > 0$ implies $I_t^{-} = 0$.

Define any period for which $I_t^{+} = I_t^{-} = 0$ as a regeneration period. Then the above properties imply that there exists an optimal solution to (1) such that there is always a regeneration period between two ordering periods. An optimal solution with such structure can be found in polynomial time Zangwill (1969).

2.3 The Economic Lot-Sizing Game

We consider a set of retailers $N = \{1, 2, \cdots, n\}$, all of which sell the same product and face an ELS problem. For each $r \in N$, let $d^r = (d_1^r, d_2^r, \ldots, d_T^r)$, where d_t^r is the known demand of retailer r in time period $t : 1 \leq t \leq T$. We assume that the ordering cost, inventory holding cost, and backlogging cost are independent of retailers.

The retailers can place orders individually, i.e., each of them solves an ELS problem separately to satisfy its own demand. They can also cooperate by placing joint orders and keeping inventory at a central warehouse. For a collection of retailers $S \subseteq N$, the goal is to minimize the total cost for the coalition, while the aggregated demand being satisfied.

Since the ordering cost is a concave function of the ordering quantity, it is not hard to see that it leads to cost reduction if the retailers place joint orders. Therefore, a cooperative game, called the ELS game (N, F), can be defined in this setting with the tuple of demand vectors $(d^1, d^2, \ldots, d^n)$ (the tuple is referred to as the *demand profile* of the game). In the ELS game, for each subset $S \subseteq N$, the characteristic cost function $F(S)$ is defined by $F(S) = C(d^S)$, where $C(\cdot)$ is defined by optimization problem (1), $d^S = (d_1^S, d_2^S, \ldots, d_T^S)$, and $d_t^S = \sum_{r \in S} d_t^r$ for each $1 \leq t \leq T$.

3 ELS Game with Setup Cost

In this section, we focus on the ELS game with setup cost, that is, the ordering cost function is defined as $c_t(z_t) = K_t \delta(z_t) + c_t z_t$ where $\delta(z_t) = 1$ if $z_t > 0$, 0 otherwise. The results and analysis may shed some lights to the ELS game with general concave ordering cost analyzed in the next section.

It is well-known that the ELS problem with setup cost can be formulated as a facility location problem (see, for instance, Krarup and Bilde (1977) for the case without backlogging, and Pochet and Wolsey (1988) for the case with backlogging). Thus, the cost function $C(d)$ is equal to

$$\begin{aligned} \min \quad & \sum_{1\le t\le T}\sum_{1\le \tau\le T} d_\tau p_{t\tau}\lambda_{t\tau} + \sum_{1\le t\le T} K_t y_t \\ \text{s.t.} \quad & d_\tau \sum_{1\le t\le T} \lambda_{t\tau} = d_\tau, \quad \forall \tau = 1,\ldots,T, \\ & \lambda_{t\tau} \le y_t, \quad \forall 1 \le t,\tau \le T, \\ & \lambda_{t\tau}, y_t \in \{0,1\}, \quad \forall 1 \le t,\tau \le T, \end{aligned} \tag{2}$$

where the coefficient $p_{t\tau}$ is the cost of satisfying one unit demand at period τ by ordering at period t, i.e., $p_{t\tau}$ is equal to c_t if $t = \tau$, $c_t + \sum_{i=t}^{\tau-1} h_t^+$ if $t < \tau$, and $c_t + \sum_{i=\tau}^{t-1} h_t^-$ if $t > \tau$, the binary indicator variable $\lambda_{t\tau} = 1$ if and only if the demand at period τ is satisfied by the inventory ordered at period t, and $y_t = 1$ if and only if an order is placed at period t.

In the LP relaxation of problem (2), variables $\lambda_{t\tau}$ and y_t are required to be non-negative, rather than integral. It is well-known that this LP relaxation has an 1988). Therefore, the optimal objective value of its dual problem is also equal to $C(d)$, i.e.,

$$\begin{aligned} C(d) = \quad \max \quad & \sum_{1\le\tau\le T} d_\tau b_\tau \\ \text{s.t.} \quad & \sum_{1\le\tau\le T} d_\tau \beta_{t\tau} \le K_t, \quad \forall t = 1,\ldots,T, \\ & b_\tau - \beta_{t\tau} \le p_{t\tau}, \quad \forall 1 \le t,\tau \le T, \\ & \beta_{t\tau} \ge 0, \quad \forall 1 \le t,\tau \le T. \end{aligned} \tag{3}$$

Let $\mathcal{S}(d)$ be the set of optimal solutions for problem (3) and A_1 be the set of allocations induced by the optimal dual solutions, referred to as the *dual set* of problem (3), i.e.,

$$A_1 = \{l = (l_1, \cdots, l_n) : \text{there exits } \ (\bar{b}, \bar{\beta}) \in \mathcal{S}(d^N), \ \text{ such that } \ l_r = \sum_{t=1}^{T} \bar{b}_t d_t^r, \ \text{ for any } \ 1 \le r \le n\}.$$

Theorem 1 below is first proved by Tamir (1992).

Theorem 1. *For the ELS game with setup cost, the dual set A_1 is a subset of the core. This implies that the core of this game is non-empty and a core allocation can be computed in polynomial time.*

Notice that the dual set A_1 is described by a system of linear inequalities whose size is polynomial in terms of the input size of the ELS game with setup cost. Therefore, if A_1 would always coincide with the core, then we would be able to check in polynomial time whether any given cost allocation is in the core or not for the game. However, it has been pointed out to us by van den Heuvel (2008) that the problem of checking core membership is NP-hard for this game. Indeed, it is not hard to construct an example for which the dual set A_1 is a true subset of the core.

For any $\bar{b} \in \mathcal{S}(d^N)$, the dual variable $\bar{b}_t$ can be interpreted as the price of satisfying one unit of demand at period t. For any $l \in A_1$, l_r, i.e., the cost allocated to retailer r is proportional to its demand. A nice feature of such an allocation is that, the price of the demand, i.e., $\bar{b}$, is dependent only on the aggregated demand d^N. If we change the demand profile of the game, $(d^1, d^2, \ldots, d^n)$, without changing its aggregated demand, then the price per unit demand should keep the same. The dual set of the new game is still a subset of the core. This motivates us to consider the set of 'demand-profile-independent' prices that give rise to core allocations. We show that it coincides with $\mathcal{S}(d)$.

Proposition 1. *Let $\mathcal{S}^*(d)$ be the set of $b = (b_1, b_2, \ldots, b_T)$ such that for any demand profile $(d^1, d^2, \ldots, d^n)$ with $d^N = d$, $l = (l_1, l_2, \ldots, l_n)$ defined by $l_r = \sum_{t=1}^{T} \bar{b}_t d_t^r$ is in the core of the corresponding cooperative game. Then $\mathcal{S}^*(d) = \mathcal{S}(d)$.*

Proof. It is clear that $\mathcal{S}(d) \subseteq \mathcal{S}^*(d)$. We shall prove $\mathcal{S}^*(d) \subseteq \mathcal{S}(d)$ by contradiction. Assume that there exists $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ in $\mathcal{S}^*(d)$, but not in $\mathcal{S}(d)$, we can show that there exists a demand profile $(d^1, d^2, \ldots, d^n)$ with $d^N = d$ such that the allocation $l = (l_1, l_2, \ldots, l_n)$ defined by $l_r = \sum_{t=1}^{T} \bar{b}_t d_t^r$ is not in the core of the corresponding cooperative game. First observe that if $\sum_{1 \leq \tau \leq T} d_\tau^N \bar{b}_\tau \neq F(N)$, then obviously the allocation l is not in the core. We now assume $\sum_{1 \leq \tau \leq T} d_\tau^N \bar{b}_\tau = F(N) = C(d)$. Then $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T) \notin \mathcal{S}(d)$ implies that $\bar{b}$ does not satisfy some of the constraints of problem (3), which are equivalent to the following inequalities:

$$\sum_{1 \leq \tau \leq T} d_\tau \max(b_\tau - p_{t\tau}, 0) \leq K_t, \quad \forall t = 1, \ldots, T.$$

Thus, there exists an index t such that $\sum_{1 \leq \tau \leq T} d_\tau \max(\bar{b}_\tau - p_{t\tau}, 0) > K_t$. We construct a demand profile $(d^1, d^2, \ldots, d^n)$ such that

$$d_\tau^1 = \begin{cases} d_\tau, & \text{if } \bar{b}_\tau > p_{t\tau}, \\ 0, & \text{otherwise,} \end{cases}$$

and $(d^2, \ldots, d^n)$ is arbitrarily chosen as long as $d^N = d$ and $d^r \geq 0$. Now the cost allocation to retailer one is given by

$$\begin{aligned} l_1 &= \textstyle\sum_{1\leq\tau\leq T} d^1_\tau \bar{b}_\tau \\ &= \textstyle\sum_{1\leq\tau\leq T} d^1_\tau(\bar{b}_\tau - p_{t\tau}) + \sum_{1\leq\tau\leq T} d^1_\tau p_{t\tau} \\ &= \textstyle\sum_{1\leq\tau\leq T} d_\tau \max(\bar{b}_\tau - p_{t\tau}, 0) + \sum_{1\leq\tau\leq T} d^1_\tau p_{t\tau} \\ &> K_t + \textstyle\sum_{1\leq\tau\leq T} d^1_\tau p_{t\tau} \\ &\geq F(\{1\}), \end{aligned}$$

where the third equality follows from the definition of d^1, and the last inequality holds since ordering at period t to satisfy the demand at all periods of player one incurs a cost $K_t + \sum_{1\leq\tau\leq T} d^1_\tau p_{t\tau}$. This implies that l is not in the core. $\square$

4 Economic Lot-Sizing Game with General Concave Ordering Cost

In this section, we analyze the ELS game with general concave ordering cost function. We first show that it can be reduced to an ELS game with setup cost and thus has a nonempty core.

For each $t : 1 \leq t \leq T$ and each $j = 0, 1, 2, \cdots$, define

$$c_{t(j)} = c_t(j+1) - c_t(j) \geq 0 \quad \text{and} \quad K_{t(j)} = c_t(j) - c_{t(j)}j$$

It is clear that, by the concavity of $c_t(\cdot)$, for any nonnegative integer z,

$$c_t(z) = \min_{j=0,1,2,\cdots} \{K_{t(j)} + c_{t(j)}z\},$$

and the minimum is achieved at $j = z$. Thus, for the ELS game with non-decreasing concave ordering cost, the characteristic cost function $F(S)$, for any coalition S, is the optimal value of the following integer program:

$$\begin{aligned} F(S) = \quad \min \quad & \sum_{1\leq t\leq T} \sum_{j\in OP} \sum_{1\leq\tau\leq T} d^S_\tau p_{t(j),\tau} \lambda_{t(j),\tau} + \sum_{1\leq t\leq T} \sum_{j\in OP} K_{t(j)} y_{t(j)} \\ \text{s.t.} \quad & d^S_\tau \sum_{1\leq t\leq T} \sum_{j\in OP} \lambda_{t(j),\tau} = d^S_\tau, \quad \forall \tau = 1, \ldots, T, \\ & \lambda_{t(j),\tau} \leq y_{t(j)}, \quad \forall 1 \leq t, \tau \leq T, j \in OP \\ & \lambda_{t(j),\tau}, y_{t(j)} \in \{0,1\}, \quad \forall 1 \leq t, \tau \leq T, j \in OP \end{aligned} \tag{4}$$

where $OP = \{j | j = 1, 2, \cdots, D^N = \sum_{t=1}^{T} d_t^N\}$, $p_{t(j),\tau} = c_{t(j)} + \sum_{i=t}^{\tau-1} h_t^+$ if $t \leq \tau$, and $p_{t(j),\tau} = c_{t(j)} + \sum_{i=\tau}^{t-1} h_t^-$ if $t > \tau$.

Therefore, we conclude that the ELS game with non-decreasing concave ordering cost can be reduced to the ELS game with setup cost. (It is known that a facility location problem with concave facility cost can be reduced to a facility location problem with setup cost; see, for example, Mahdian et al. 2006). Theorem 1 implies that

Corollary 1. *The core of the ELS game with non-decreasing concave ordering cost is non-empty.*

The reduction, unfortunately, ends up with a formulation which does not have a polynomial size in terms of the input of the game and does not lead to a polynomial time algorithm to compute a core allocation. This is because the size of the set OP is not polynomial. A natural question is whether it is possible to reduce the size of OP. For the ELS problem with demand $d = (d_1, \cdots, d_T)$, it is known that there exists an optimal solution so that the size of each order is equal to the total demand of a number of consecutive periods. This implies that for each concave function $c_t(z_t)$, it suffices to focus on values $z_t \in OP(d) := \{z | z = \sum_{t=i}^{j} d_t \quad \text{for} \quad 1 \leq i \leq j \leq T\}$. It is clear that the size of $OP(d)$ is $O(T^2)$. Thus, for each coalition S of retailers, the size of OP in formulation (4) can be reduced to $O(T^2)$. However, notice that the set $OP(d)$ is dependent on the demand vector $d = (d_1, \cdots, d_T)$. Thus, for each coalition S, the set $OP(d^S)$ is dependent on S. Therefore, if we solve problem (4) for the grand coalition N, with OP replaced by $OP(d^N)$, and get an optimal solution, it is not necessarily true that this optimal solution is always feasible to problem (4) with OP replaced by $OP(d^S)$ for every coalition $S \subset N$. Indeed, it is not hard to construct an example to show that this is not true in general. Thus, the proof of Theorem 1 does not go through.

To find a core allocation in polynomial time, we analyze an alternative LP formulation for the ELS problem (rather than the facility location based formulation). Here is the outline of our approach. First, we construct a natural $0-1$ integer programming (IP) formulation for problem (1) whose LP relaxation has an optimal integral solution. The size of this LP relaxation is polynomial in the size of the input. Second, we construct the dual of this LP relaxation and use the optimal dual solutions to define cost allocations of the ELS game. Third, we illustrate through counterexamples that these allocations may not be in the core of the ELS game. Fourth, building upon the insight gained from the counterexamples, we strengthen the dual by adding additional inequalities, and show that allocations derived from the optimal solutions of the strengthened dual are in the core. It then follows that a core allocation can be computed in polynomial time. In fact, it is possible to obtain such a core allocation without solving a linear program.

4.1 Integer Programming Formulation

For problem (1) and any $1 \leq i \leq j \leq T$, we define $d_{ij} = \sum_{t=i}^{j} d_t$,

$$C_{ij}(d) = \min_{l: i \leq l \leq j} \left\{ c_l(d_{ij}) + \sum_{t=i}^{l-1} h_t^- d_{it} + \sum_{t=l+1}^{j} h_{t-1}^+ d_{tj} \right\}. \tag{5}$$

Our alternative integer program is then formulated as follows.

$$\begin{aligned} \min \quad & \sum_{1 \leq i \leq j \leq T} C_{ij}(d) x_{ij} \\ \text{s.t.} \quad & d_l \sum_{i \leq l \leq j} x_{ij} = d_l, \forall\, l = 1, \ldots, T, \\ & x_{ij} \in \{0, 1\}. \end{aligned} \tag{6}$$

Proposition 2. *If $c_t(\cdot)$ is a non-decreasing concave function for every $1 \leq t \leq T$, then problem (6) and problem (1) have the same optimal objective values.*

The above IP formulation can be derived from the dynamic programming formulation (Zangwill 1969) of the ELS problem following a standard approach that has been used in the literature (see, for example, Barany et al. 1996).

If we replace the constraint $x_{ij} \in \{0, 1\}$ by $x_{ij} \geq 0$, we obtain an LP relaxation of problem (6), whose dual is

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} b_t d_t \\ \text{s.t.} \quad & \sum_{t=i}^{j} b_t d_t \leq C_{ij}(d), \forall 1 \leq i \leq j \leq T. \end{aligned} \tag{7}$$

Again, the variable b_t can be interpreted as the price of satisfying one unit demand at period t.

It turns out that the dual problem (7) admits a closed-form optimal solution. In order to present this optimal dual solution, we introduce another notation. Given T and the demand vector $d = (d_1, \cdots, d_T)$, for each $1 \leq t \leq T$, we defined a T dimensional vector $g^t(d)$ as follows:

$$g^t(d)_i = \begin{cases} d_i, & \text{if } \ 1 \leq i \leq t; \\ 0, & \text{if } \ t+1 \leq i \leq T. \end{cases}$$

It is clear that $g^T(d) = d$. We also let $g^0(d) = 0$. For each $1 \leq t \leq T$, we define

$$b_t^* = \frac{C(g^t(d)) - C(g^{t-1}(d))}{d_t}, \tag{8}$$

where the function $C(\cdot)$ is defined in the optimization problem (1). By definition, $C(g^t(d))$ is the minimum cost to satisfy the demand from period 1 to period t. It is clear that $b_t^* \geq 0$ for every $1 \leq t \leq T$.

Similarly, define $g_t(d) = d - g^t(d)$ for each $0 \leq t \leq T$, and for each $1 \leq t \leq T$,

$$b_t^{**} = \frac{C(g_{t-1}(d)) - C(g_t(d))}{d_t}. \tag{9}$$

Notice that $g_0(d) = d$ and $b_t^{**} \geq 0$ since $g_{t-1}(d) \geq g_t(d)$ for $1 \leq t \leq T$. The following result, whose proof is presented in the appendix, essentially implies that there exists an optimal 0-1 solution to the LP relaxation of problem (6).

Lemma 1. *The solutions* (b_t^*) *and* (b_t^{**}) *defined by (8) and (9), respectively, are optimal to the dual problem (7), and the corresponding optimal dual value is equal to* $C(d)$.

4.2 Dual-Based Allocation

Now we solve the dual problem (7) for the grand coalition, referred to as the grand dual problem:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} b_t d_t^N \\ \text{s.t.} \quad & \sum_{t=i}^{j} b_t d_t^N \leq C_{ij}(d^N), \forall 1 \leq i \leq j \leq T. \end{aligned} \tag{10}$$

The grand dual problem may have multiple optimal solutions. Assume that $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ is an optimal solution to problem (10). We define an allocation $l = (l_1, \cdots, l_N)$ as follows:

$$l_r = \sum_{t=1}^{T} \bar{b}_t d_t^r. \tag{11}$$

Given the conceptual similarity of our approach to the LP duality approach to linear production games by Owen (1975) and the result in Section 3, one may expect that l derived from any optimal dual solution $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ is an allocation in the core. Interestingly, the following example illustrates that for the economic lot-sizing game with concave ordering cost, some of the optimal dual solutions do give rise to core allocations while some other optimal dual solutions do not.

Example 1. We consider the game with three periods and two players, i.e., $T = 3$ and $N = \{1, 2\}$. The demands of the players are given as

$$d^1 = (10, 0, 6), \quad d^2 = (0, 2, 0)$$

We assume that backlogging is not allowed. We let the holding cost for every period be zero, and the ordering costs are defined by $K_t\delta(z_t) + c_t z_t$, where

$$K_1 = 5, K_2 = 9, K_3 = 8, c_1 = 5, c_2 = 1, \text{ and } c_3 = 8.$$

It is easy to verify that $\bar{b} = (5.5, 5, 1\frac{1}{6})$ is optimal to problem (10) and provides an allocation $(62, 10)$, which is in the core. However, $\bar{b} = (5.5, 0, 2\frac{5}{6})$ is optimal to problem (10) and defines an allocation $(72, 0)$, which is clearly not in the core.

The above example shows that, unlike many other cooperative games studied in the literature, we can not use an arbitrary dual optimal solution to find an allocation in the core. Fortunately, as we shall prove below, one of the optimal dual solutions to (10) does define an allocation in the core. To derive our main results, we notice that for every $i \le j$, the dual constraint

$$\sum_{t=i}^{j} b_t d_t \le C_{ij}(d) \tag{12}$$

is equivalent to the following $j - i + 1$ constraints:

$$\sum_{t=i}^{j} b_t d_t \le c_l(d_{ij}) + \sum_{t=i}^{l-1} h_t^- d_{it} + \sum_{t=l+1}^{j} h_{t-1}^+ d_{tj} \quad \forall l : i \le l \le j. \tag{13}$$

We can further simplify (13) by introducing some notations.

Given a vector $b = (b_1, \cdots, b_T)$, for any $1 \le i \le l \le j \le T$, define

$$b_t^-(l) = b_t - \sum_{l=t}^{l-1} h_l^- \quad \text{for each} \quad t \le l,$$

and

$$b_t^+(l) = b_t - \sum_{l=l}^{t-1} h_l^+ \quad \text{for each} \quad t \ge l.$$

It is clear that $b_l^-(l) = b_l^+(l) = b_l$ for any $1 \le l \le T$. Then it is straightforward to verify that, (13), and thus (12) are equivalent to the following $j - i + 1$ constraints:

$$\sum_{t=i}^{l-1} b_t^-(l) d_t + \sum_{t=l}^{j} b_t^+(l) d_t \le c_l(d_{ij}) \quad \forall l : i \le l \le j. \tag{14}$$

It will become clear shortly that the above constraints can be handled by analyzing a continuous piecewise linear function and a concave function. For this purpose, consider two n-dimensional vectors τ and β with $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n$, and define $u(x; \tau, \beta)$ to be the continuous piecewise linear function on the interval $[0, \tau_n]$ with break points τ_κ and slope β_κ over the interval $(\tau_{\kappa-1}, \tau_\kappa)$. Notice that $u(x; \tau, \beta)$ is uniquely defined once its value at the initial point $x = 0$ is fixed, and $u(x; \tau, \beta)$ is concave if β_κ is non-increasing in κ. On the other hand, if $u(x; \tau, \beta)$ is concave, we can assume without loss of generality that β_κ is non-increasing in κ.

Lemma 2. *Assume that the piecewise linear function $u(x; \tau, \beta)$ is bounded above by a concave function $v(x)$ for all $x \in [0, \tau_n]$. We have the following results.*

- *(a) If $u(x; \tau, \beta)$ is concave with β_κ non-increasing in κ, then the new concave piecewise linear function $u(x; \tau', \beta)$ with $\tau' \leq \tau$, $0 \leq \tau'_1 \leq \ldots \leq \tau'_n$ and $u(0; \tau', \beta) = u(0; \tau, \beta)$ is also bounded above by $v(x)$ for all $x \in [0, \tau'_n]$.*
- *(b) If $u(x; \tau, \beta)$ is not concave, then there exists another vector β' with β'_κ non-increasing in κ such that $u(x; \tau, \beta) \leq u(x; \tau, \beta') \leq v(x)$ for all $x \in [0, \tau_n]$ and $u(x; \tau, \beta) = u(x; \tau, \beta')$ for $x = 0, \tau_n$.*

Proof. To prove part (a), observe that the slope of $u(x; \tau', \beta)$ over the interval $(\tau'_{\kappa-1}, \tau'_\kappa)$, β_κ, is no more than the slope(s) of $u(x; \tau, \beta)$ over the same interval since $\tau' \leq \tau$ and β_κ is non-increasing in κ. Therefore, $u(x; \tau', \beta) \leq u(x; \tau, \beta) \leq v(x)$ for all $x \in [0, \tau'_n]$.

We now prove part (b). Without loss of generality, we can assume that τ_κ ($\kappa = 1, 2, \ldots, n$) are distinct. Since $u(x; \tau, \beta)$ is not concave, there must exists an index l such that $\beta_l < \beta_{l+1}$. Let l_0 be the smallest such l, referred to as the violation index, and $i_0 = \max\{i \leq l_0 : \beta_{i+1} < \beta_i\}$, referred to as the inconstant index, or $i_0 = 0$ if the set is empty. It is clear that $\beta_{i+1} = \beta_i$ for $i_0 < i < l_0$. In addition, the function $u(x; \tau, \beta)$ has one linear piece for $x \in (\tau_{i_0}, \tau_{l_0})$ and another linear piece for $x \in (\tau_{l_0}, \tau_{l_0+1})$. We now modify $u(x; \tau, \beta)$ on the interval $(\tau_{i_0}, \tau_{l_0+1})$ (the remaining part of u is kept unchanged): we increase the the function value of $u(x; \tau, \beta)$ at τ_{l_0} and maintain the two linear pieces over the two intervals (τ_{i_0}, τ_{l_0}) and $(\tau_{l_0}, \tau_{l_0+1})$. It implies that β_i is increased by the same amount, say δ, to $\tilde{\beta}_i$, for $i_0 < i \leq l_0$ and β_{l_0+1} is decreased correspondingly to $\tilde{\beta}_{l_0+1}$. We increase δ until either $\beta_{i_0} < \tilde{\beta}_{i_0+1} = \ldots = \tilde{\beta}_{l_0} = \tilde{\beta}_{l_0+1}$ or $\beta_{i_0} = \tilde{\beta}_{i_0+1} = \ldots = \tilde{\beta}_{l_0} \leq \tilde{\beta}_{l_0+1}$ holds, whichever happens first. In the former case, we increase the smallest violation index by 1, while in the latter case, we decrease the inconstant index by 1. In either case, we can repeat the same process until there is no more violation index. At this point, the resulting function $u(x;\tau,\beta')$ must satisfy the conditions in part (b). $\square$

Lemma 3. *Assume $b_t - h_t^- \le b_{t+1} \le b_t + h_t^+$ for every $1 \le t \le T-1$. If (12) holds for $d = d^N$ for any $1 \le i \le j \le T$, then it also holds for any $d = d' \le d^N$.*

Proof. It is clear that the assumption $b_t - h_t^- \le b_{t+1} \le b_t + h_t^+$ for every $1 \le l \le T-1$ is equivalent to

$$b_t^+(l) \ge b_{t+1}^+(l) \text{ for } t \ge l \quad \text{ and } \quad b_t^-(l) \le b_{t+1}^-(l) \text{ for } t \le l$$

for $1 \le l \le T$.

For any i and l with $i \le l$, define

$$u_l^i(x;d,b) = \sum_{k=i}^{l-1} b_k^-(l)d_k + \sum_{k=l}^{j-1} b_k^+(l)d_k + b_j^+(l)(x - d_{l,j-1}), \text{ for } x \in [d_{l,j-1}, d_{l,j}], j \ge l, \tag{15}$$

and

$$v_l^i(x;d) = c_l(d_{i,l-1} + x). \tag{16}$$

Our assumption implies that $u_l^i(x;d,b)$ is a concave piecewise linear function for $x \in [0, d_{lT}]$ with break points d_{lj} $(j = l, \dots, T)$. In addition, $u_l^i(x;d^N) \le v_l^i(x;d^N)$ for $x = d_{lj}$ $(j = l, \dots, T)$. Therefore, $u_l^i(x;d^N) \le v_l^i(x;d^N)$ for $x \in [d_{ll}^N, d_{lT}^N]$. For any vector d' with $d'_j = d_j$ for $j = 1, \dots, l$ and $d'_j \in [0, d_j^N]$ for $j > l$, applying Lemma 2, we can show that $u_l^i(x;d',b) \le v_l^i(x;d')$ for $x \in [d'_{ll}, d'_{lT}]$.

Similarly, for any j and l with $j \ge l$, if we define

$$\tilde{u}_l^j(x;d,b) = \sum_{k=l}^{j} b_k^+(l)d_k + \sum_{k=i+1}^{l-1} b_k^-(l)d_k + b_i^-(l)(x - d_{i+1,l-1}), \text{ for } x \in [d_{i+1,l-1}, d_{i,l-1}] \tag{17}$$

and

$$\tilde{v}_l^j(x;d) = c_l(x + d_{lj}), \tag{18}$$

we can show that $\tilde{u}_l^j(x;d',b) \le \tilde{v}_l^j(x;d')$ for $x \in [0, d'_{1,l-1}]$ for any vector d' with $d'_i = d_i$ for $i \ge l$ and $d'_i \in [0, d_j^N]$ for $i = 1, \dots, l-1$.

If $d^N_{i,l-1} \geq d^N_l$, we can define a new vector d' such that $d'_i = d_i$ for $i \geq l$, $d'_i \in [0, d^N_j]$ for $i = 1, \ldots, l-1$, and $d^N_{i,l-1} = d'_{i,l-1} + d^N_l$. Our above discussion implies that

$$\begin{aligned} u^i_l(0; d^N, b) &= \textstyle\sum_{k=i}^{l} b^-_k(l) d_k - b^-_l(l) d_l \\ &= \textstyle\sum_{k=i}^{l} b^-_k(l) d'_k + \sum_{k=i}^{l-1} (b^-_k(l) - b^-_l(l))(d_k - d'_k) \\ &\leq \tilde{u}^l_l(d'_{i,l-1}; d', b) \\ &\leq \tilde{v}^l_l(d'_{i,l-1}; d') \\ &= v^i_l(0; d^N), \end{aligned}$$

where the first inequality follows from the monotonicity of $b^-_k(l)$ and the second inequality from our above discussion. $\square$

Lemma 4. *Assume that $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_T)$ is an optimal solution to problem (10). If $\bar{b}_t - h^-_t \leq \bar{b}_{t+1} \leq \bar{b}_t + h^+_t$ for every $1 \leq t \leq T-1$, then the allocation $l = (l_1, \cdots, l_N)$ defined by*

$$l_r = \sum_{t=1}^{T} \bar{b}_t d^r_t$$

is in the core of the ELS game (N, F).

Proof. Notice that for any $S \subseteq N$,

$$\sum_{r \in S} l_r = \sum_{r \in S} \sum_{t=1}^{T} \bar{b}_t d^r_t = \sum_{t=1}^{T} \bar{b}_t d^S_t.$$

Therefore, by the definition of $\bar{b}$, $\sum_{r \in N} l_r$ is equal to the optimal value of (10), and thus equal to $F(N)$ by using Lemma 1.

It remains to show $\sum_{r \in S} l_r \leq F(S)$ for any $S \subseteq N$. By weak duality of linear programming, it suffices to prove that $\bar{b}$ is feasible to

$$\sum_{t=i}^{j} \bar{b}_t d^S_t \leq C_{ij}(d^S),$$

which is implied by Lemma 3. The proof is complete. $\square$

As stated in the lemma, the inequalities $\bar{b}_t - h^-_t \leq \bar{b}_{t+1} \leq \bar{b}_t + h^+_t$ for $1 \leq t \leq T-1$ are crucial to our analysis, although by no means it is a necessary condition. We now provide some intuition on these inequalities. Recall that the dual variable b_i may be interpreted as the price of satisfying one unit of demand at period i. Notice that the demand at period $i+1$ can be either satisfied by orders after period i or by an order before period $i+1$. In the latter case, period $i+1$ demand is satisfied by the inventory carried over from period i to period $i+1$ by paying a unit inventory holding cost h_i^+. Thus, it is reasonable to expect that the charge of a unit demand at period $i+1$ is no more than the charge of a unit demand at period i plus the unit inventory holding cost h_i^+. The inequality $\bar{b}_{i+1} \geq \bar{b}_i - h_i^-$ can be explained in a similar way.

The next question that we shall answer is whether or not there exists any optimal dual solution that satisfies these inequalities. We first check whether or not the optimal solutions b^* and b^{**} defined by (8) and (9) satisfy these inequalities, and whether or not the cost allocations defined by these optimal solutions are in the core.

Indeed, one can show that $b^*_{t+1} \leq b^*_t + h^+_t$ for every $t : 1 \leq t \leq T-1$ (we omit its proof here as it is similar to the proof of Proposition 2 in Barany et al. (1984), and not crucial to our main result). However, as illustrated by the following example, we may not have $b^*_{t+1} \geq b^*_t - h^-_t$ for every $t : 1 \leq t \leq T-1$ and the cost allocation defined by b^* may not be in the core.

Example 2. We consider a game with setup cost and three periods. Let the aggregated demand $d = (1, 5, 1)$. In addition, let

$$K = (2, 4, 1), c = (1, 0, 0), h^+ = (1, 1, 0), h^- = (1, 1, 0).$$

One can compute that $b^* = (3, 0.4, 1)$. It is straightforward to verify that $b^*_{t+1} \leq b^*_t + h^+_t$ for every $t : 1 \leq t \leq T-1$. However, $b^*_2 < b^*_1 - h^-_1$. We can also verify that $b^* = (3, 0.4, 1)$ is not feasible for problem (3). Indeed, for $t = 3$, $\sum_{1 \leq \tau \leq T} d_\tau \max(b_\tau - p_{t\tau}, 0) = 2 < 1 = K_t$ and thus it is impossible to find $\{\beta_{t\tau}\}$ such that $b^* = (3, 0.4, 1)$ is feasible for problem (3). Our discussion at the end of Section 3 implies that there exists a demand profile such that the cost allocation induced by b^* is not in the core of the corresponding game.

Nonetheless, if backlogging is not allowed, then $h^-_t = \infty$ for every t. Thus, solution b^* automatically satisfies the condition required by Lemma 4. This leads to the following result

Corollary 2. *If backlogging is not allowed, then the ELS game with non-decreasing concave cost has a non-empty core. In particular, the allocation defined by (11) with $\bar{b} = b^*$ is in the core.*

Similarly, one can show that for solution b^{**}, $b^{**}_{t+1} \geq b^{**}_t - h^-_t$ for every $t : 1 \leq t \leq T-1$. However, $b^{**}_{t+1} \leq b^{**}_t + h^+_t$ may not hold for all t, and the cost allocation defined by b^{**} may not be in the core.

Even though b^* and b^{**} may not lead to core allocations, Lemma 5 below says that we can always construct in polynomial time an optimal dual solution $\bar{b}$, starting from either b^* or b^{**} (indeed any optimal dual solution for problem (7)), that satisfies the condition required by Lemma 4. The lemma is the key to our main result. It might be of independent interest as well.

Lemma 5. *Given any feasible solution $b = (b_1, \cdots, b_T)$ to problem (7), we can construct in polynomial time another feasible solution $\hat{b} = (\hat{b}_1, \cdots, \hat{b}_T)$ with the same objective value such that $-h^-_t \leq \hat{b}_{t+1} - \hat{b}_t \leq h^+_t$ for every $t : 1 \leq t \leq T-1$.*

The proof is carried out in two steps. In the first step, we construct a feasible solution b' with the same objective value and $b'_{t+1} \leq b'_t + h^+_t$ for every $t : 1 \leq t \leq T-1$. In the second step, we convert b' to $\hat{b}$ with the same objective value such that $-h^-_t \leq \hat{b}_{t+1} - \hat{b}_t \leq h^+_t$ for every $t : 1 \leq t \leq T-1$. Both steps can be done in polynomial time. Notice that b^* and b^{**} can be found in polynomial time via dynamic programming. Furthermore, if we start with solution b^* (b^{**} respectively), then we do not have to perform step 1 (step 2 respectively). Since the proof of Lemma 5 is rather long, we present it in the appendix.

Lemma 5 and Lemma 4 immediate lead to our main result.

Theorem 2. *For the ELS game (N, F) with backlogging, and with general nondecreasing concave ordering cost, linear inventory holding cost, and linear backlogging cost, the core is always nonempty and an allocation in the core can be found in polynomial time.*

In fact, we have picked a subset of the dual optimal solutions that lead to core allocations. These solutions are the same as the optimal solutions to the following linear program.

$$
\begin{aligned}
\max \quad & \sum_{t=1}^{T} b_t d^N_t \\
\text{s.t.} \quad & \sum_{t=i}^{j} b_t d^N_t \leq C_{ij}(d^N), \quad \forall 1 \leq i \leq j \leq T \\
& b_t \geq b_{t+1} - h^+_t, \quad \forall 1 \leq t \leq T-1 \\
& b_t \leq b_{t+1} + h^-_t, \quad \forall 1 \leq t \leq T-1
\end{aligned}
\tag{19}
$$

However, Lemma 5 and Lemma 4 imply that to find one of the core allocations, we do not really need to solve the above linear program.

An interesting and open question is whether the core allocations derived from our approach allow us to construct a population monotonic allocation scheme, which roughly speaking says that adding one more player to the grand coalition reduces the cost allocations of existing players (see Sprumont 1990 for the exact definition). Unfortunately, since problem (19) may have multiple optimal solutions, it is not clear whether and how one can choose the optimal dual solutions appropriately to construct a population monotonic allocation scheme.

Finally, we compare different formulations for the ELS game with setup cost. In this case, both optimal solutions to (19) and (3) define allocations in the core. Let A_1, A_2, and A_3 be the sets of optimal solutions to problems (3), (10), and (19) with $d = d^N$, respectively. We can show that $A_3 \subsetneqq A_1 \subsetneqq A_2$. The proof is straightforward and thus omitted.

5 Conclusion

In this paper, we study the ELS game with backlogging and with non-decreasing concave ordering cost. We show that there exists an allocation in the core, which can be computed in polynomial time.

There are several other directions for future research. First, our focus in this paper is the core. However, there are several other important concepts in cooperative game such as the Shapley value (Peleg and Sudhölter 2003) and the nucleolus (Deng and Papadimitriou 1994). It would be interesting to design efficient algorithms to compute them.

Second, we have essentially assumed that the retailers have uniform inventory holding costs. In addition, the inventory holding cost and backlogging cost are assumed to be linear functions. We are currently investigating whether some of these assumptions can be relaxed.

Third, the demand is assumed to be exogenous in this paper. However, in many practical situations, demand could be a function of the price of the product, and the price can be optimized, together with inventory decisions, in order to maximize profit (see, for instance, Chen and Simchi-Levi 2004a, 2004b). It remains a challenge to incorporate pricing decisions into ELS games.

Finally, we hope the methodology employed here can encourage more research developing effective cost allocation mechanisms essential for sustainable collaborations in many not for profit operations.

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Appendix

Proof of Lemma 1.

We will show that (b_t^*) is optimal to problem (7). The proof of the optimality of (b_t^{**}) is similar. First of all, we show that (b_t^*) is a feasible solution to problem (7). To that end, we notice that for any $1 \le i \le j \le d$,

$$\begin{aligned}\sum_{t=i}^{j} b_t^* d_t &= \sum_{t=i}^{j} C(g^t(d)) - C(g^{t-1}(d)) \\ &= C(g^j(d)) - C(g^{i-1}(d)) \\ &\le C_{ij}(d),\end{aligned}$$

where the last inequality holds since

$$C(g^j(d)) \le C(g^{i-1}(d)) + C_{ij}(d), \tag{20}$$

which in turn follows from the fact that the right hand side of (20) is the cost of a feasible ordering policy to satisfy the demand from period 1 to period j (i.e., by ordering optimally to satisfy the demand from period 1 and period $i-1$, and ordering at a period in between i and j to satisfy the total demand from period i to period j), while the left hand side of (20) is the minimum cost to achieve that.

Moreover, the objective value of problem (7) associated with the solution (b_t^*) is

$$\sum_{t=1}^{T} b_t^* d_t = C(g^T(d)) = C(d),$$

which is the optimal objective value of the integer program (6). Then it follows from the weak duality of LP, (b_t^*) must be an optimal solution to problem (7) with an objective value of $C(d)$.

Proof of Lemma 5

For a given function $f : \Re \to \Re$, define its left-side directional derivative as

$$f'_-(x) = \lim_{\delta \to 0-} \frac{f(x+\delta) - f(x)}{\delta}.$$

Then we have,

Lemma 6. *Assume that $u, v : X \subset \Re \to \Re$ are two functions on an interval X and $u(x) \le v(x)$ for any $x \in X$. Then for any x with $u(x) = v(x)$, $u'_-(x) \ge v'_-(x)$ as long as x is different from X's left end point and the directional derivatives are well defined.*

The proof of Lemma 6 is straightforward and thus omitted.

Given a feasible solution $b = (b_1, \cdots, b_T)$ to problem (7), we convert $b = (b_1, \cdots, b_T)$ to another feasible solution $\tilde{b} = (\tilde{b}_1, \cdots, \tilde{b}_T)$ with the same objective value such that $-h_t^- \le \tilde{b}_{t+1} - \tilde{b}_t \le h_t^+$ for every $1 \le t \le T-1$.

Though the proof is lengthy, the idea is similar to the observation made in Lemma 2 part (b). The additional complexity comes from the fact that we have to maintain several inequalities simultaneously while changing both the function values at the initial point and the slopes of the relevant piecewise linear functions.

To simplify our analysis, we first assume that $d_j > 0$ for $j = 1, \ldots, T$.

Assume that there exists some t such that $h_t^- \le b_{t+1} - b_t \le h_t^+$ is violated. Let l_0 be the violation index, i.e., the smallest such t. We will show that we can change the values of b_i for $i \le l_0 + 1$ so that the violation index will increase by at least 1. In addition, the new solution is still feasible with the objective value unchanged.

We distinguish between two different cases.

Case 1. $b_{l_0+1} > b_{l_0} + h_{l_0}^+$.

Let i_0 be the inconstant index, i.e.,$i_0 = \max\{i \le l_0 : b_{i+1} - b_i < h_i^+\}$ or $i_0 = 0$ if the set is empty. It is clear that $i_0 + 1 \le l_0$. Furthermore, for any $i_0 + 1 \le i < l_0$, $b_{i+1} - b_i = h_i^+$. For a given $\delta > 0$, we define $\tilde{b}$ such that for every j,

$$\tilde{b}_j = \begin{cases} b_j + \delta, & \text{if } \ i_0 + 1 \le j \le l_0; \\ b_j - \frac{d_{i_0+1,l_0}}{d_{l_0+1}}\delta, & \text{if } \ j = l_0 + 1; \\ b_j, & \text{otherwise.} \end{cases} \tag{21}$$

It is clear that $\tilde{b}$ and b have the same objective value. We shall show that there exists some small $\delta > 0$ such that $\tilde{b}$ is feasible. That is, for any $i \le l \le j$,

$$\sum_{k=i}^{l-1} \tilde{b}_k^-(l) d_k + \sum_{k=l}^{j} \tilde{b}_k^+(l) d_k \le c_l(d_{ij}), \tag{22}$$

where $\tilde{b}_k^-(l) = \tilde{b}_k - \sum_{t=k}^{l-1} h_k^-$ for $k \le l-1$ and $\tilde{b}_k^+(l) = \tilde{b}_k - \sum_{t=k}^{j-1} h_k^+$ for $k \ge l$.

If $l > l_0$, or $j \le i_0$, or $l \le l_0$ and $j \ge l_0 + 1$, the inequality (22) obviously holds for $\tilde{b}$. Thus, we assume that $\max(l, i_0 + 1) \le j \le l_0$.

Recall the functions $u_l^i(x; d, b)$ and $v_l^i(x; d)$ defined in (15) and (16). For simplicity, denote

$u_l^i(x) = u_l^i(x; d, b)$ and $v_l^i(x) = v_l^i(x; d)$. Since the vector b is feasible, we have that $u_l^i(x) \leq v_l^i(x)$ for any $x = d_{lj}$ $(j = l, \ldots, T)$ and $i \leq l$. On the other hand, the definition of l_0 implies that $b_1^-(l) \leq b_2^-(l) \leq b_l^-(l)$ and thus $u_l^i(0) \leq v_l^i(0)$ from the proof in Lemma 3. Since u_l^i is piecewise linear with break points d_{lk} $(k = l, \ldots, T)$ and v_l^i is concave, we have that $u_l^i(x) \leq v_l^i(x)$ for any $x \in [0, d_{lT}]$ and $i \leq l$.

To define δ, we claim that $u_l^i(d_{lj}) < v_l^i(d_{lj})$ for any $i \leq l \leq \max(l, i_0 + 1) \leq j \leq l_0$. Otherwise, there must exist i, l, j such that $i \leq l \leq \max(l, i_0 + 1) \leq j \leq l_0$ and $u_l^i(d_{lj}) = v_l^i(d_{lj})$. Then by Lemma 6,

$$\tilde{b}_j^+(l) = \left\{ \frac{du_{l-}^i(x)}{dx} \right\} |_{x=d_{lj}} \geq \left\{ \frac{dv_{l-}^i(x)}{dx} \right\} |_{x=d_{lj}} \geq \frac{dv_{l-}^i(x)}{dx}, \text{ for } x \in [d_{lj}, d_{lT}], \tag{23}$$

where the second inequality follows from the concavity of function $v_l^i(x)$. On the other hand, it is clear that

$$\tilde{b}_{l_0+1}^+(l) > \tilde{b}_{l_0}^+(l) = \tilde{b}_j^+(l). \tag{24}$$

Since the function $u_l^i(x)$ is linear in $(d_{l,j-1}, d_{l,l_0}]$ with a slope of $b_{l_0}^+(l)$ and linear in $(d_{l,l_0}, d_{l,l_0+1}]$ with a slope of $b_{l_0+1}^+(l)$, and $v_l^i(x)$ is concave, (23) and (24) imply that $u_l^i(x) > v_l^i(x)$ for $x \in (d_{l,j}, d_{l,l_0+1}]$, which contradicts the feasibility of b. Thus, $u_l^i(d_{lj}) < v_l^i(d_{lj})$ for any $i \leq l \leq \max(l, i_0+1) \leq j \leq l_0$.

We can now choose δ such that

$$\delta = \min \left\{ \min_{i,l,j: i \leq l \leq \max(l, i_0+1) \leq j \leq l_0} \frac{v_l^i(d_{lj}) - u_l^i(d_{lj})}{d_{\max(i,i_0+1),j}}, (b_{l_0+1} - b_{l_0} - h_{l_0}^+) \frac{d_{l_0+1}}{d_{i_0+1,l_0+1}}, b_{i_0} + h_{i_0}^+ - b_{i_0+1} \right\} > 0, \tag{25}$$

if $i_0 \geq 1$, and

$$\delta = \min \left\{ \min_{i,l,j: i \leq l \leq \max(l, i_0+1) \leq j \leq l_0} \frac{v_l^i(d_{lj}) - u_l^i(d_{lj})}{d_{\max(i,i_0+1),j}}, (b_{l_0+1} - b_{l_0} - h_{l_0}^+) \frac{d_{l_0+1}}{d_{i_0+1,l_0+1}} \right\} > 0,$$

if $i_0 = 0$.

From our construction, we have that $\tilde{b}$ satisfies (22) and thus is still feasible to problem (7). Since $\delta \leq (b_{l_0+1} - b_{l_0} - h_{l_0}^+) \frac{d_{l_0+1}}{d_{i_0+1,l_0+1}}$, we have $\tilde{b}_{l_0+1} \geq \tilde{b}_{l_0} + h_{l_0}^+$ and therefore $\tilde{b}_{i+1} \geq \tilde{b}_i - h_i^-$ for $i \leq l_0$.

If $\delta = \min_{i,l,j: i \leq l \leq \max(l, i_0+1) \leq j \leq l_0} \frac{v_l^i(d_{lj}) - u_l^i(d_{lj})}{d_{\max(i,i_0+1),j}}$, we claim that $\tilde{b}_1^+(1) \geq \ldots \geq \tilde{b}_{l_0}^+(1) = \tilde{b}_{l_0+1}^+(1)$. If this is not true, it is clear that l_0 is the violation index associated with $\tilde{b}$, i.e., the smallest l such that $\tilde{b}_{l+1}^+(1) > \tilde{b}_l^+(1)$. Similar to the argument we used before, we can show that for

$i \leq l \leq \max(l, i_0+1) \leq j \leq l_0$, $u_l^i(d_{lj}) < v_l^i(d_{lj})$, where in the definition of $u_l^j(\cdot)$, b is replaced by $\tilde{b}$. But this contradicts the definition of δ.

If $\delta = b_{i_0}^+(1) - b_{i_0+1}^+(1)$, then $\tilde{b}_{i_0}^+(1) = \tilde{b}_{i_0+1}^+(1) = \ldots = \tilde{b}_{l_0}^+(1)$. Therefore, inconstant index is decreased by at least one. In this case, if l_0 remains the violation index for $\tilde{b}$, we repeat the above argument by at most i_0 times and each time the value of i_0 will be reduced by at least one. Eventually, we end up with $\tilde{b}_1^+(1) \geq \ldots \geq \tilde{b}_{l_0}^+(1) = \tilde{b}_{l_0+1}^+(1)$.

Case 2. $b_{l_0+1} < b_{i_0} - h_{l_0}^-$.

Let i_0 be the inconstant index, i.e.,$i_0 = \max\{i \leq l_0 : b_{i+1} - b_i > -h_i^-\}$ or $i_0 = 0$ if the set is empty. It is clear that $i_0 + 1 \leq l_0$. Furthermore, for any $i_0 + 1 \leq i < l_0$, $b_{i+1} - b_i = -h_i^-$. For a given $\delta > 0$, we define $\tilde{b}$ such that for every j,

$$\tilde{b}_j = \begin{cases} b_j - \delta, & \text{if } \; i_0 + 1 \leq j \leq l_0; \\ b_j + \frac{d_{i_0+1,l_0}}{d_{l_0+1}}\delta, & \text{if } \; j = l_0 + 1; \\ b_j, & \text{otherwise.} \end{cases} \tag{26}$$

It is clear that $\tilde{b}$ and b have the same objective value. We shall show that there exists some small $\delta > 0$ such that $\tilde{b}$ is feasible, i.e., for any $i \leq l \leq j$, inequality (22) holds.

If $l \leq l_0$, or $i > j_0$, or $l > l_0$ and $i \leq l_0$, the inequality (22) obviously holds for $\tilde{b}$. Thus, we assume that $l_0 + 1 \leq i \leq \min(j_0, l)$. It is clear that we can simply focus on i with $d_i > 0$.

We know that $u_l^j(x) \leq v_l^j(x)$ for any $x \in [0, d_{1,l-1}]$, where for simplicity, denote $u_l^j(x) = \tilde{u}_l^j(x; d, \tilde{b})$ and $v_l^j(x) = \tilde{v}_l^j(x; d)$, with $\tilde{u}_l^j$ and $\tilde{v}_l^j$ being defined in (17) and (18), respectively. To define δ, we claim that $u_l^j(d_{i,l-1}) < v_l^j(d_{i,l-1})$ for any $l_0 + 1 \leq i \leq \min(j_0, l)$. Otherwise, assume that there exists i, l, and j such that $u_l^j(d_{i,l-1}) = v_l^j(d_{i,l-1})$. Then we can show, similar to (23) and (24),

$$\tilde{b}_{l_0}^- > \tilde{b}_{l_0+1}^- = \tilde{b}_i^- = \left\{\frac{du_{l-}^j(x)}{dx}\right\}|_{x=d_{il-1}} \geq \left\{\frac{dv_{l-}^j(x)}{dx}\right\}|_{x=d_{i,l-1}} \geq \frac{dv_{l-}^j(x)}{dx}, \text{ for } x \in [d_{i,l-1}, d_{1,l-1}].$$

This implies that $u_l^j(x) > v_l^j(x)$ for $x \in (d_{i,l-1}, d_{l_0+1,l-1}]$, which contradicts the feasibility of $\tilde{b}$. Thus, $u_l^j(d_{i,l-1}) < v_l^j(d_{i,l-1})$ for any $l_0 + 1 \leq i \leq \min(j_0, l)$. We can now choose δ such that

$$\delta = \min\left\{\min_{i,l,j: l_0+1 \leq i \leq \min(j_0,l) \leq l \leq j} \frac{v_l^j(d_{i,l-1}) - u_l^j(d_{i,l-1})}{d_{l_0+1,\min(j_0,l)}}, -(\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}}, \tilde{b}_{j_0+1} - \tilde{b}_{j_0} + h_{j_0}^-\right\} > 0,$$

if $j_0 < T$, and

$$\delta = \min\left\{\min_{i,l,j:l_0+1\le i\le \min(j_0,l)\le l\le j} \frac{v_l^j(d_{i,l-1}) - u_l^j(d_{i,l-1})}{d_{l_0+1,\min(j_0,l)}}, -(\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}}\right\} > 0$$

if $j_0 = T$. Then $\tilde{b}$ satisfies (22) and thus is still feasible to problem (7).

Now we consider two cases.

Case 1. $\delta = -(\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}}$. In this case, we have that $\hat{b}_{l_0+1} = \hat{b}_{l_0} - h_{l_0}^-$. Therefore, the largest integer such that $\hat{b}_{t+1} < \hat{b}_t - h_t^-$ has decreased by at least 1. This is exactly what we need.

Case 2. $\delta < -(\tilde{b}_{l_0+1} - \tilde{b}_{l_0} + h_{l_0}^-)\frac{d_{l_0}}{d_{l_0,j_0}}$. In this case, $\hat{b}_{l_0+1} < \hat{b}_{l_0} - h_{l_0}^-$. It implies that l_0 is the largest integer such that $\hat{b}_{t+1} < \hat{b}_t - h_t^-$. Then by the same argument as above, $u_l^j(d_{i,l-1}) < v_l^j(d_{i,l-1})$ for any $l_0 + 1 \le i \le \min(j_0, l)$, where in the definition of $u_l^j(\cdot)$ and $v_l^j(\cdot)$, we replace $\tilde{b}$ with $\hat{b}$. This would imply that $\delta < \frac{v_l^j(d_{i,l-1}) - u_l^j(d_{i,l-1})}{d_{l_0+1,\min(j_0,l)}}$. It then follows that $\delta = \tilde{b}_{j_0+1} - \tilde{b}_{j_0} + h_{j_0}^-$. Hence we have $\hat{b}_{j_0+1}^- = \cdots = \hat{b}_{j_0}^-$. Therefore, the value of $\min\{j \ge l_0 + 1 : \hat{b}_{j+1} - \hat{b}_j > -h_j^-\}$ is no less than $j_0 + 1$. We can repeat the procedure by at most $T - j_0$ times until Case 2 will not happen (and thus eventually we are in Case 1).

The only thing that is left to verify is that $\hat{b}_{t+1} \le \hat{b}_t + h_t^+$ for every $1 \le t \le T - 1$. This inequality can possibly be violated if during the process one of the following situations happens. (Notice that for the variables increased, they are increased by the same amount.)

(1) $\tilde{b}_{t+1}$ is increased by δ, but $\tilde{b}_t$ is unchanged, or $\tilde{b}_t$ is unchanged, but $\tilde{b}_{t-1}$ is decreased by a positive amount. It is clear that this can never happen in our procedure.

(2) $\tilde{b}_{t+1}$ is increased by δ, but $\tilde{b}_t$ is decreased by a positive amount. This happens only if $t = l_0$. But after the change, $\hat{b}_{l_0+1} = \tilde{b}_{l_0+1} + \delta$, and $\hat{b}_{l_0} = \tilde{b}_{l_0} - \frac{d_{l_0+1,j_0}}{d_{l_0}}\delta$. By the definition of δ, $\hat{b}_{l_0+1} = \hat{b}_{l_0} - h_{l_0}^-$ and thus $\hat{b}_{l_0+1} \le \hat{b}_{l_0} + h_{l_0}^+$.

We now illustrate the above procedure on Example 2 starting with $\tilde{b} = b^* = (3, 0.4, 1)$. As we already pointed out, only Step 2 is needed. In this case, it is easy to see that $l_0 = 1$ and $j_0 = 2$. Careful calculation gives $\delta = 0.8/3 = -(\tilde{b}_2 - \tilde{b}_1 + h_1^-)\frac{d_1}{d_{12}}$ and $\hat{b} = (\frac{5}{3}, \frac{2}{3}, 1)$. Clearly, $-h_t^- \le \hat{b}_{t+1} - \hat{b}_t \le h_t^+$ for $t = 1, 2$.