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Technical Note—Dynamic Pricing with Gain-Seeking Reference Price Effects

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We study a dynamic pricing problem of a firm facing reference price effects at an aggregate demand level, where demand is more sensitive to gains than losses. We find that even the myopic pricing strategy belongs to one type of discontinuous maps, which can exhibit complex dynamics over time. Our numerical examples show that, in general, the optimal pricing strategies may not admit any simple characterizations and the resulting reference price/price dynamics can be very complicated. We then show for a special case that a cyclic skimming pricing strategy is optimal, and we provide conditions to guarantee the optimality of high-low pricing strategies.

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1. Introduction

The concept of *reference price*, developed in the economics and marketing literature, has attracted considerable attention in recent years. It argues that consumers form price expectations and use them to judge the current selling price. A purchasing instance is then perceived by consumers as gains (losses) if the selling price is below (above) the reference price. Consumers are categorized as loss averse or gain seeking depending on whether they are more sensitive to losses or gains. Similarly, an aggregate-level demand is categorized as loss averse or gain seeking depending on whether the demand is more sensitive to the negative part or the positive part of the difference between reference price and price. In this note, we focus on the dynamic pricing problem of a firm facing gain-seeking demand at the aggregate level with an objective of maximizing the total discounted profit over an infinite time horizon.

There are empirical evidences that support gain-seeking reference price effects at both the individual level (Krishnamurthi et al. 1992, Kopalle et al. 2012) and the aggregate level (Greenleaf 1995, Raman and Bass 2002). We refer readers to Table 2 in Slonim and Garbarino (2009) for a summary of the evidence that supports each direction of asymmetric reference price effects in different product categories. In response to this evidence, several explanations are offered in the literature. On the basis of discrete choice models, Kallio and Halme (2009) analytically show that if a product is purchased with low probability (possibly as a result of a large number of promotion-driven consumers), then the asymmetry in consumers' utility (value) under gains or losses can be reversed while transforming utility to demand through a multinomial logit model. Although the model in Kallio and Halme is not directly comparable to ours, their results nevertheless offer a potential explanation to the gain-seeking demand. In the line of our model, Greenleaf (1995) argues that when the market consists of "light" households who are more price sensitive and "heavy" households who are less price sensitive, the market demand can appear to be more sensitive to gains than losses. Slonim and Garbarino reinterpret the argument provided by Greenleaf as stockpiling behavior of consumers. They further provide an extensive simulation study, which shows that when a product is highly stockable, the demand tends to be estimated as gain seeking even if consumers are driven by loss aversion. It is not our intention to argue in this note what is the most plausible explanation behind the gain-seeking demand and how to disentangle the possible confounding effects. Instead, we emphasize here its existence, especially when there are many promotion-driven consumers in the market and when the product is highly stockable—for example, liquid detergent, toilet tissue (Briesch et al. 1997), or peanut butter (Greenleaf 1995).

This note strives to provide prescriptive solutions to the dynamic pricing problems when demand is estimated to be gain seeking. The resulting optimization problems are challenging because they are nonconvex and nonsmooth because of the gain-seeking assumption. Even the myopic pricing strategy, where the firm ignores the effect of current prices on future revenues, belongs to one type of discontinuous map, which may lead to complicated dynamics over time. We derive conditions under which the dynamics resulted from the myopic pricing strategy have simple characterizations. The high-low pricing, proposed in several papers (Kopalle et al. 1996, Popescu and Wu 2007, Geng et al. 2010), is generally not optimal, and the characterizations as well as the dynamics of the optimal pricing strategy can be very complex. Interestingly, under the assumptions that consumers only remember the most recent price and the aggregate demand is insensitive to the losses, we prove that the optimal pricing strategy is a cyclic skimming pricing strategy. In other words, when consumers have a low reference price, the firm should charge a high price and then gradually offer deeper and deeper discounts until the consumers' reference price drops low again, and then repeat the cycle. We further provide sufficient conditions for the high-low pricing strategy to be optimal.

Our results are significantly different from previous studies (Kopalle et al. 1996, Fibich et al. 2003, Popescu and Wu 2007, Nasiry and Popescu 2011), which primarily focus on the dynamic pricing problem with loss-averse demands and establish the optimality of a constant pricing strategy. There are a few papers that touch upon the dynamic pricing problem with gain-seeking demands. Kopalle et al. (1996) first prove that when gain seeking is present, a constant pricing strategy is not optimal. Popescu and Wu (2007) generalize this result to general demand functions and postulate that "high-low pricing is provably optimal if consumers are focused on gains" (p. 420). Geng et al. (2010) further show that the high-low pricing strategy outperforms the constant pricing strategy for the retailer in a supply chain consisting of a single manufacturer and a single retailer. They derive an explicit expression for the optimal promotion frequency and extend the analysis to the case with uncertain demand. Our work distinguishes from previous literature in the sense that we focus on identifying the structure of the optimal strategy, whereas previous studies illustrate the nonoptimality of constant pricing strategies when facing gain-seeking demand. Specifically, under some conditions, we show the optimality of cyclic skimming pricing strategy and formally establish verifiable conditions under which high-low pricing strategy is optimal.

Cyclic pricing strategies are also found to be optimal when consumers have strategic (Conlisk et al. 1984, Besbes and Lobel 2015) or nonstrategic (Ahn et al. 2007, Liu and Cooper 2015) waiting behavior. We emphasize here some key distinctions of our work from these papers. In terms of modeling, our argument behind reference price models is that the demands are affected by the prices in the past rather than anticipated prices in the future. In terms of proofs, we form a dynamic programming problem and identify the cyclic pricing strategy by analyzing properties of the value function, whereas the above papers all tackle their problems directly by utilizing the notion of regeneration points.

Our work is also closely related to the ongoing research in the one-dimensional discontinuous map in the dynamic system and chaos community (see Jain and Banerjee 2003). Recently, Rajpathak et al. (2012) analyze in detail the stable periodic orbits of one type of discontinuous map. It turns out that the myopic pricing strategy in our work can be reduced to this type of discontinuous map. However, to the best of our knowledge, the class of discontinuous maps with multiple discontinuous points, into which our optimal pricing strategy typically falls, has not been considered in the previous literature.

2. Model

We adopt a memory-based reference price model, which is commonly used and empirically validated on retail data for a variety of products (see, for example, Greenleaf 1995). In this model, reference price is generated by exponentially weighting past prices. Specifically, starting with a given initial reference price r_0, the reference price at period t, denoted by r_t, evolves as

$$r_{t+1} = \alpha r_t + (1-\alpha) p_t, \quad t \geqslant 0.$$

In the above evolution equation, $p_t \in [0, U]$ is the price charged by the firm at period t, where U is the upper bound on feasible prices. The parameter $\alpha \in [0, 1)$ is called the memory factor, and as α increases, consumers adapt to the new price information at a slower rate. We restrict $\alpha < 1$ because reference prices will otherwise remain constant over the whole planning horizon; consequently, a constant pricing strategy is optimal irrespective of gain-loss asymmetry. As reference prices are generated from historical prices, it is also reasonable to assume that $r_0 \in [0, U]$.

Following Greenleaf (1995), Kopalle et al. (1996), Fibich et al. (2003), and Nasiry and Popescu (2011), the demand depends on the price p and reference price r via the model

$$D(r, p) = b - ap + \eta^+ \max\{r-p, 0\} + \eta^- \min\{r-p, 0\},$$

where $b, a > 0$ and $\eta^+, \eta^- \geqslant 0$. Here, $D(p, p) = b - ap$ is the base demand independent of reference prices, and $\eta^+(r-p)$ or $\eta^-(r-p)$ is the additional demand or demand loss induced by the reference price effect. To avoid negative demand, we further assume that $D(0, U) \geqslant 0$; i.e., $U \leqslant b/(a+\eta^-)$. The difference between reference price and selling price is referred to as gain if $r > p$ and as loss if $r < p$. The aggregate-level demands are classified as loss averse, loss/gain neutral, or gain seeking depending on whether $\eta^+ < \eta^-$, $\eta^+ = \eta^-$, or $\eta^+ > \eta^-$, respectively. In this note we focus on the case when gains have greater impact than losses; i.e., $\eta^+ > \eta^-$.

The firm's one-period profit is denoted as $\Pi(r, p) = pD(r, p)$. Here, the marginal cost is assumed to be 0 for simplicity. All our results can be extended to cases with a nonzero marginal cost. We assume $U \geqslant b/(2a)$ such that $\Pi(p, p)$, called the base profit, is not monotone in $p \in [0, U]$. This assumption allows us to "rule out pathological boundary steady states" (Popescu and Wu 2007, p. 418), but our analysis can be carried over similarly by distinguishing those boundary steady states when this assumption fails. Note that the assumptions $\eta^+ > \eta^-$ and $p \geqslant 0$ allow us to rewrite the one-period profit as

$$\Pi(r, p) = \max\{\Pi^+(r, p), \Pi^-(r, p)\}, \tag{1}$$

where $\Pi^+(r, p) = p[b - ap + \eta^+(r - p)]$ and $\Pi^-(r, p) = p[b - ap + \eta^-(r - p)]$. Contrary to the loss-averse case, the one-period profit function is no longer a concave function in p.

Given an initial reference price r_0, the firm's long-term profit maximization problem is then

$$V(r_0) = \max_{p_t \in [0, U]} \sum_{t=0}^{\infty} \gamma^t \Pi(r_t, p_t), \tag{2}$$

where $\gamma \in [0, 1)$ is a discount factor, and we interpret $0^0 = 1$.[1] The infinite horizon problem is of particular interest in the literature since it is often more tractable than the finite horizon counterpart and provides valuable insights into the long-run behavior of the optimal pricing strategy, which in turn may shed light on the development of efficient heuristics for finite horizon models.

The Bellman equation for problem (2) is

$$V(r) = \max_{p \in [0, U]} \big\{\Pi(r, p) + \gamma V(\alpha r + (1-\alpha)p)\big\}. \tag{3}$$

A *pricing strategy* $p(r)$ is a function from $[0, U]$ to $[0, U]$ that specifies a feasible solution to (3) for a given reference price r. Given any pricing strategy $p(r)$, the sequence $\{r_t\}$ of reference prices that evolve according to $r_{t+1} = \alpha r_t + (1-\alpha)p(r_t)$ is referred to as the *reference price path* of the pricing strategy $p(r)$. We say $p(r)$ has a *periodic orbit* of period n or is a *cyclic pricing strategy* with cycle length n if there exists $r_0 \in [0, U]$ such that the reference price path of $p(r)$ satisfies $r_n = r_0$ and $r_t \neq r_0$ for all $0 < t < n$. Clearly, if $r_n = r_0$, then by $r_{t+1} = \alpha r_t + (1-\alpha)p(r_t)$, the sequence $\{r_0, \ldots, r_{n-1}\}$ is repeated infinitely over time; this sequence is referred to as the *periodic orbit* of $p(r)$. In particular, when $n = 1$, we say $p(r)$ admits a *steady state* r_0, and when $n = 2$, $p(r)$ is a *high-low pricing strategy*. If there exists a periodic orbit that has the additional property that $r_0 < r_1 < \cdots < r_{n-1}$, then we refer to $p(r)$ as a *cyclic penetrating pricing strategy*. If, on the other hand, $r_0 > r_1 > \cdots > r_{n-1}$, then we refer to $p(r)$ as a *cyclic skimming pricing strategy*. Note that in the special case when $\alpha = 0$, $r_0 < r_1 < \cdots < r_{n-1}$ if and only if $p(r_{n-1}) < p(r_0) < \cdots < p(r_{n-2})$. (Similarly, $r_0 > r_1 > \cdots > r_{n-1}$ if and only if $p(r_{n-1}) > p(r_0) > \cdots > p(r_{n-2})$.) That is, the monotonicity of reference prices is equivalent to the monotonicity of charged prices. However, for $\alpha > 0$, it is possible to have monotone reference prices with nonmonotone charged prices. Recall that in practice, a skimming (penetrating) pricing strategy is used to describe pricing strategy with a decreasing (increasing) price path over time. Here, we use the term "skimming" ("penetrating") to reflect the fact that a skimming (penetrating) pricing strategy is usually designed to capture consumers with decreasing (increasing) valuations (an analogy of the notion "reference price" in our model) over time. Since $[0, U]$ is compact and the objective function can be easily shown to be continuous, the *optimal pricing strategy* that solves (3) exists and is denoted by $p^*(r)$. As mentioned in §1, Kopalle et al. (1996) and Popescu and Wu (2007) prove that $p^*(r)$ does not admit a steady state. That is, for any $r \in [0, U]$, $p^*(r) \neq r$.

3. Main Results

In §3.1, we first establish the underlying connection between our problem and discontinuous maps by analyzing the myopic pricing strategy. The analysis of the optimal pricing strategy is then presented in §3.2.

3.1. Myopic Pricing Strategy

By definition, the myopic pricing strategy $p^m(r)$ is given by solving the following problem:

$$p^m(r) = \arg\max_{p \in [0, U]} \Pi(r, p).$$

Here, we assume, without loss of generality, that $p^m(r)$, and the optimal solutions for other optimization problems in this section always take the largest among multiple solutions.

Define the constant

$$R = \frac{b}{a + \sqrt{(a+\eta^+)(a+\eta^-)}}. \tag{4}$$

Lemma 1. *Let $R_U = (2(a+\eta^-)U - b)/\eta^-$. If $R \leqslant R_U$,*

$$p^m(r) = \begin{cases} \dfrac{\eta^- r + b}{2(a+\eta^-)}, & r \leqslant R, \\[2ex] \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & r > R. \end{cases} \tag{5}$$

If $R > R_U$, then there will be an additional piece where $p^m(r) = U$. To keep the presentation clear and simple, we assume for the rest of this subsection that $R \leqslant R_U$. That is, $p^m(r)$ is determined by (5). The analysis presented in this subsection can also be extended to the case $R > R_U$ with additional discussions on whether or not U will appear on the periodic orbit.

Note that $p^m(r)$ is not a continuous function. As a result, the dynamics of reference prices under the myopic pricing strategy, $r_{t+1}(r_t) = \alpha r_t + (1-\alpha)p^m(r_t)$, ends up with a *discontinuous map* from $[0, U]$ to $[0, U]$, whose analysis is considered to be "a very complicated research subject and we can obtain useful and interesting results only for various special classes of maps" (Sharkovsky and Chua 1993, p. 730).

In the following, we identify necessary and sufficient conditions for the myopic pricing strategy to admit a cyclic penetrating pattern and a cyclic skimming pattern with cycle length n. To simplify the expressions of our conditions, define constants

$$\mu = \frac{a+\eta^+ - \sqrt{a^2 + a\eta^+ + a\eta^- + \eta^+\eta^-}}{\eta^+ - \eta^-},$$

$$A = 1+\alpha-\alpha\frac{\eta^-}{2(a+\eta^-)}, \quad B = 1+\alpha-\alpha\frac{\eta^+}{2(a+\eta^+)},$$

and denote for $n \geqslant 0$ the sum of geometric series $\sum_{i=0}^{n} x^i$ by S_n^x (for $n<0$, let $S_n^x = 0$).

Proposition 1. *For $n \geqslant 2$, $p^m(r)$ is a cyclic penetrating pricing strategy with cycle length n if and only if the following inequalities hold*:

$$\frac{A^{n-1}}{S_{n-1}^A} < \mu \leqslant \frac{A^{n-2}}{A^{n-2}B + S_{n-2}^A}. \tag{6}$$

On the other hand, $p^m(r)$ is a cyclic skimming pricing strategy with cycle length n if and only if the following inequalities hold:

$$\frac{AB^{n-2} + S_{n-3}^B}{AB^{n-2} + S_{n-2}^B} < \mu \leqslant \frac{S_{n-2}^B}{S_{n-1}^B}. \tag{7}$$

Note that when $n = 2$, both (6) and (7) reduce to $A/(1+A) < \mu \leqslant 1/(1+B)$, which is always satisfied when consumers only remember the last period price ($\alpha = 0$) and the direct price effect weakly dominates the reference price effect ($4a \geqslant \eta^+$). In this case, $p^m(r)$ is a high-low pricing strategy, and the computation in evaluating the performance of $p^m(r)$ can be significantly reduced. For n large enough, there exist ranges of parameters other than (6) and (7), resulting in periodic orbits of period n that are neither penetrating nor skimming. Readers are referred to Rajpathak et al. (2012) for a characterization of other patterns of periodic orbits and the conditions under which they may occur for dynamic systems of discontinuous maps with two linear pieces. The type of characterization may facilitate the effective computation of the performance of the myopic strategy, commonly used to benchmark more complicated strategies.

The above connection between the myopic pricing strategy and the discontinuous maps provides a prelude to the behaviors of the optimal pricing strategy, which we will be discussing in the following. In addition, revealing such a relationship could be potentially useful in analyzing other nonconvex dynamic optimization problems as well.

3.2. Optimal Pricing Strategy

Unlike the myopic pricing strategy, we do not have an explicit solution for the optimal pricing strategy, which makes the analysis significantly more challenging. To build more intuition, we first present a few properties on the value function and optimal solution.

By (1), the Bellman equation (3) can be correspondingly rewritten as

$$V(r) = \max_{p\in[0,U]} \{\max\{\Pi^+(r,p), \Pi^-(r,p)\} + \gamma V(\alpha r + (1-\alpha)p)\}. \tag{8}$$

It is worth noting here that two assumptions commonly imposed on optimization problems, differentiability and concavity in the decision variables, are both absent in problem (8) because of the term $\max\{\Pi^+(r,p), \Pi^-(r,p)\}$, which makes the analysis quite challenging. To circumvent this difficulty, we consider the following two problems:

$$V^+(r) = \max_{p\in[0,U]} \{\Pi^+(r,p) + \gamma V(\alpha r + (1-\alpha)p)\}, \tag{9a}$$

$$V^-(r) = \max_{p\in[0,U]} \{\Pi^-(r,p) + \gamma V(\alpha r + (1-\alpha)p)\}. \tag{9b}$$

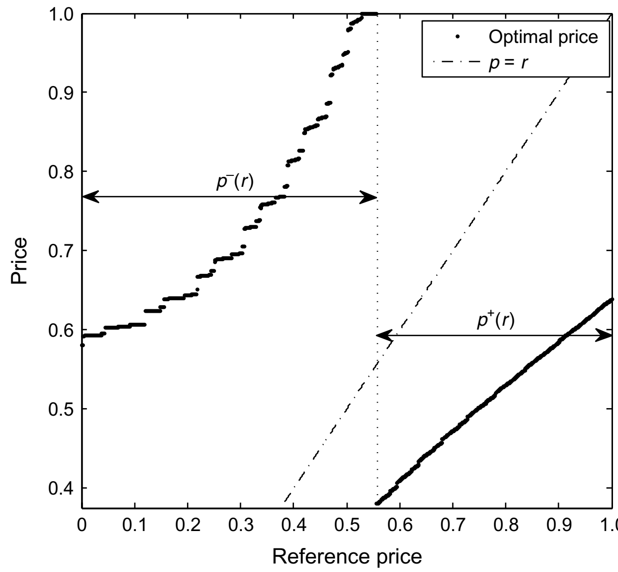
The solutions of (9a) and (9b) are denoted respectively as $p^+(r)$ and $p^-(r)$. An observation here is that $V(r) = \max\{V^+(r), V^-(r)\}$ and $p^*(r) \in \{p^+(r), p^-(r)\}$. We next characterize properties of $V^\pm(r)$ and $p^\pm(r)$.

Lemma 2. *Both $V^+(r)$ and $V^-(r)$ are increasing and convex in r, whereas $p^+(r)$ and $p^-(r)$ are increasing in r.*

We are interested in how $p^+(r)$ and $p^-(r)$ relate to the optimal pricing strategy $p^*(r)$ and whether it is possible to obtain simple characterizations for $p^+(r)$ and $p^-(r)$. For this purpose, we draw in Figure 1 the (numerically approximated) optimal pricing strategy, where the parameters used in the figure come from an empirical example in Hu (2015). In Figure 1, we observe that there exists a point $\hat{r} \in [0, U]$ such that $p^*(r)$ is given by $p^-(r)$ for $r \leqslant \hat{r}$ and $p^+(r)$ for $r > \hat{r}$. Although this observation seems to hold in all our numerical experiments, we do not have a proof for general parameter configurations. Of course, even if this observation is indeed true, Figure 1 suggests that both $p^+(r)$ and $p^-(r)$ can have numerous discontinuous points and may not admit any simple characterizations.

In Figure 2, we further illustrate two possible dynamics under different levels of α while keeping all other parameters the same as in Figure 1. In Figure 2, the bold lines represent the discontinuous map $r_{t+1}(r_t) = \alpha r_t + (1-\alpha)p_t^*(r_t)$, which maps r_t to r_{t+1}. The arrowed lines illustrate a periodic orbit. Specifically, the vertical arrowed lines indicate that the trajectory evolves from r_t to r_{t+1}, and the horizontal arrowed lines visually aid us in perceiving the function value r_{t+1} as an argument of the next map. In each panel, the arrowed lines form a closed loop, so it is a periodic

Figure 1. Optimal pricing strategy when $b = 582.0$, $a = 569.4$, $\eta^+ = 2{,}671.2$, $\eta^- = 0$, $\alpha = 0.8$, and $\gamma = 0.9$.



orbit, but one can see that it is quite complicated. By comparing the two panels, one can see that a mere increment of 0.05 in α can result in dramatic changes in the dynamics, a phenomenon termed as *border collision bifurcation* in the dynamic system and chaos community (Jain and Banerjee 2003).

Thus, for the rest of this section, we focus on a special case satisfying the following assumption.

Assumption 1. *Consumers only remember the most recent price* (*i.e.*, $\alpha = 0$), *and the demand is insensitive to the loss* (*i.e.*, $\eta^- = 0$).

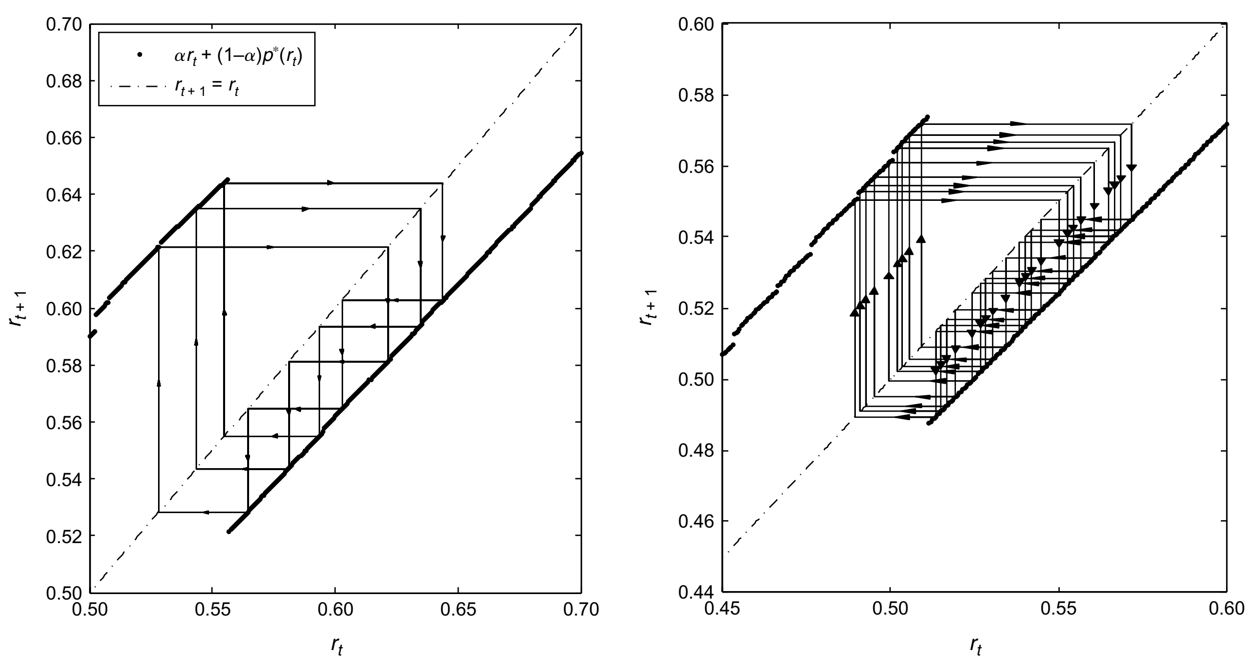
Assumption 1 seems restrictive but has very plausible explanations and can provide good approximation to some practical scenarios. First of all, consumers are unlikely to remember many historical prices and form a reference price by averaging them. Several papers—for example, Krishnamurthi et al. (1992), Mayhew and Winer (1992), and Raman and Bass (2002)—also assume that $\alpha = 0$. Second, $\eta^- = 0$ models the market of promotion-driven products, where the product demand consists of a base demand $b - ap$ and a promotion-stimulated demand $\eta^+ \max\{r - p, 0\}$. It provides a good approximation to a market that consists of many promotion-driven consumers (Kallio and Halme 2009) or a market of highly stockable products Slonim and Garbarino (2009). Finally, it is found in several empirical examples in Hu (2015) that imposing Assumption 1 does not result in much loss in the goodness of fit of the model.

An immediate consequence from Assumption 1 is that both $V^-(r)$ and $p^-(r)$ are now constant functions. In the sequel, we will use constants p^- and V^- to denote the function values of $p^-(r)$ and $V^-(r)$. That $V^-(r)$ is a constant function is critical for the simplification of the problem, as it allows us to relate $p^+(r)$ and $p^-(r)$ with $p^*(r)$ in a simple way, as demonstrated in the following lemma.

Lemma 3. *Under Assumption 1, there exists* $R_0 \in (0, U)$ *such that if* $r \leqslant R_0$, *then* $V(r) = V^-$ *and* $p^*(r) = p^- > r$. *If* $r > R_0$, *then* $V(r) = V^+(r)$ *and* $p^*(r) = p^+(r) < r$.

Lemma 3 gives us a broad picture of what $p^*(r)$ looks like. That is, when $r \leqslant R_0$, $p^*(r)$ is a constant function and

Figure 2. Periodic orbits for the optimal pricing strategies when $\alpha = 0.8$ (left) and $\alpha = 0.85$ (right).



is always above r. At the point R_0, there is a "downward jump" from $p^- > R_0$ to $p^+(R_0) < R_0$. When $r > R_0$, $p^*(r)$ is then monotonically increasing in r. Using Lemma 3, we give in the following proposition a complete characterization of the optimal pricing strategy $p^*(r)$. Let $m_1 = 0$, and for $k > 1$, $m_k = \gamma\eta^+/(2(a+\eta^+) - m_{k-1}\eta^+)$.

Proposition 2. *Under Assumption 1, there exists an integer $N \geqslant 0$ and $0 < R_0 < R_1 < \cdots < R_N < U = R_{N+1}$ such that*

$$p^*(r) = \begin{cases} p^-, & 0 \leqslant r \leqslant R_0, \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & R_0 < r < R_1, \\ \dfrac{\eta^+ r + b + \sum_{i=0}^{k}(\prod_{j=0}^{i} m_{k+1-j})b}{2(a+\eta^+) - m_{k+1}\eta^+}, & \\ & R_k \leqslant r < R_{k+1},\ k = 1, \ldots, N. \end{cases}$$

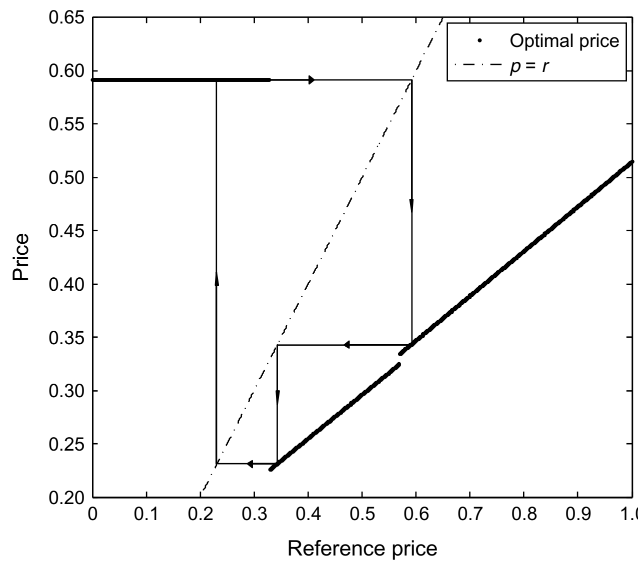
Now we have a complete picture of the optimal pricing strategy $p^*(r)$. After a downward jump at R_0, $p^*(r)$ follows a piecewise linear function with finitely many "upward jumps" at R_k for $k = 1, \ldots, N$. Moreover, the slopes of these linear pieces increase after each "upward jump" by the expression of $p^*(r)$ given in Proposition 2.

Another important insight from our analysis is that for $k = 1, \ldots, N$, $p^*(r)$ maps $[R_k, R_{k+1})$ to $[R_{k-1}, R_k]$, and finally $p^*(r)$ maps (R_0, R_1) to $[0, R_0]$. This leads us to the study of the dynamics of $p^*(r)$, which is a discontinuous map with more than one discontinuous point. Even though the dynamics of a discontinuous map with only one discontinuous point is already complicated, the characterization presented in Proposition 2 allows us to find simple dynamics for $p^*(r)$. Denote $p_1^*(r) = p^*(r)$ and $p_i^*(r) = p^*(p_{i-1}^*(r))$ for $i > 1$. To emphasize the role of the constant p^- in Proposition 2 in characterizing dynamics, in the following, we alternatively use r^* to denote p^-.

Proposition 3. *Let N be the integer in Proposition 2. Under Assumption 1, there exists an integer n with $2 \leqslant n \leqslant N+2$ such that $p_n^*(r^*) = r^*$, and for all $r_0 \in [0, U]$, the optimal reference price path r_t^* converges in at most $N+2$ periods to the unique periodic orbit: $\{r^*, p_1^*(r^*), \ldots, p_{n-1}^*(r^*)\}$; i.e., there exists $0 \leqslant \tau \leqslant N+2$ such that $r_\tau^* = r^*$. Moreover, the periodic orbit has the property $r^* > p_1^*(r^*) > \cdots > p_{n-1}^*(r^*)$; i.e., $p^*(r)$ is a cyclic skimming pricing strategy.*

Proposition 3 suggests the following pricing strategy for practitioners when the demand they face is gain seeking and promotion driven: when consumers' initial reference price is below some threshold, the firm should use a regular price. Then the firm applies a skimming pricing strategy by gradually discounting the regular price over time until the consumers' reference price falls below the threshold again, and it repeats such a pricing strategy. The intuition behind this is easy to understand. When consumers have a low reference price, because the demand is gain seeking, it will not hurt the firm too much by setting a high price in order to drag the consumers' reference price to a higher level. After such a manipulation, the firm will benefit greatly by offering discounts since demand is sensitive to gains and there will be a boost in demand. An illustration of the optimal pricing strategy and the periodic orbit is provided in Figure 3. One can see that, indeed, there is more than one discontinuous point, and in this particular example, the periodic orbit has three periods ($n = 3$).

Figure 3. Optimal pricing strategy and periodic orbit when $b = 582.0$, $a = 569.4$, $\eta^+ = 2{,}671.2$, $\eta^- = 0$, $\alpha = 0$, and $\gamma = 0.1$.



Next, we identify conditions on parameters such that a high-low pricing strategy is optimal. Define the constant $K = (a + \eta^+ - \sqrt{(a+\eta^+)^2 - \gamma(\eta^+)^2})/\eta^+$ and recall the constant R defined in (4).

Proposition 4. *Under Assumption 1, if the following inequality also holds:*

$$\frac{\eta^+ U + (1/(1-K))b}{2(a+\eta^+) - K\eta^+} \leqslant R, \tag{10}$$

then a high-low pricing strategy $\{p_H, p_L\}$ is optimal, where $p_H = r^*$ and $p_L = (\eta^+ r^* + b)/(2(a+\eta^+))$.

Proposition 4 above formally settles the conjecture of Popescu and Wu (2007) in the sense that it provides a verifiable condition from problem parameters that guarantees the optimality of a high-low pricing strategy. One direct implication of condition (10) is that a high-low pricing strategy is always optimal if the feasible prices are not too high; i.e., U is sufficiently small. This is intuitive because if the firm's highest possible price is already low, then there is not much room for the firm to set different discount levels. Readers are referred to Hu (2015) for general conditions that guarantee that the period of the optimal cyclic pricing strategy is at most n.

When Assumption 1 fails, depending on the magnitude of violation, complicated cyclic behaviors can emerge, and we refer readers to Hu (2015) for a numerical study that illustrates the possible behaviors.

4. Conclusion

In this note we analyzed a dynamic pricing problem in a market with gain-seeking demands. In this model, demand depends on both the current selling price and the reference price, where the latter evolves according to an exponentially smoothing process of past prices.

We demonstrated the connection between the myopic pricing strategy and discontinuous maps in the dynamic system literature, and we identified conditions that lead to simple pricing dynamics: cyclic skimming pricing or cyclic penetrating pricing. Realizing the complexity of the problem, we restricted ourselves to a special case and proved that a cyclic skimming pricing strategy is optimal. We further provided conditions on the optimality of a high-low pricing strategy. Although our characterization of the optimal pricing strategy was built on a piecewise linear demand model, Proposition 3 can be extended to general nonlinear demand functions proposed in Popescu and Wu (2007) by imposing an assumption similar to Assumption 1. We relegate the extension to the online appendix (available as supplemental material at http://dx.doi.org/10.1287/opre.2015.1445).

Although our note stands from a prescriptive point of view and takes gain-seeking demands for granted, it would be worthwhile and interesting for future research to further study the causes of gain-seeking demands, to develop more sophisticated models by considering model endogeneity and consumer heterogeneity, and to statistically disentangle the effects from multiple possible consumer behaviors. On the other hand, a simple aggregate demand model has its own merits, such as minimum data requirements and ease in estimation. Therefore, to facilitate the use of such a model in practice, it would be valuable to develop efficient computational algorithms for simple heuristics that perform well under gain-seeking demands.

Finally, it would be interesting to study the impact of gain-seeking reference price effects on joint pricing and inventory decisions. Pricing and inventory integration has received much attention in the past few years (see, for example, Chen and Simchi-Levi 2004a, b, 2006, 2012). Recently, Chen et al. (2015) have incorporated the reference price effect into coordinated pricing and inventory models. However, their model focuses only on loss-averse demands.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2015.1445.

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Endnote

1. We point out two immediate extensions to our setting. First, demands are random with additive, mean 0 and independent random shocks. Second, there are several consumer/demand segments that are heterogeneous in demand parameters. It is easy to show that these two extensions can still be modeled and solved as problem (2).

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