Submitted to *Operations Research*
manuscript (Please, provide the mansucript number!)

Dynamic Capacity Management with General Upgrading

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This paper studies a capacity management problem with upgrading. A firm needs to procure multiple classes of capacities and then allocate the capacities to satisfy multiple classes of customers that arrive over time. A general upgrading rule is considered, i.e., unmet demand can be satisfied using multi-step upgrade. No replenishment is allowed and the firm has to make the allocation decisions without observing future demand. We first characterize the structure of the optimal allocation policy, which consists of parallel allocation and then sequential rationing. Specifically, the firm first uses capacity to satisfy the same-class demand as much as possible, then considers possible upgrading decisions in a sequential manner. We also propose a heuristic based on certainty equivalence control to solve the problem. Numerical analysis shows that the heuristic is fast and delivers close-to-optimal profit for the firm. Finally, we conduct extensive numerical studies to derive insights into the problem. It is found that under the proposed heuristic, the value of using sophisticated multi-step upgrading can be quite significant; however, using simple approximations for the initial capacity leads to negligible profit loss, which suggests that the firm's profit is not sensitive to the initial capacity decision if the optimal upgrading policy is used.

Key words: Capacity management, inventory, upgrading, dynamic programming, revenue management.

1. Introduction

Driven by intensified market competition and rapidly changing consumer trends, many firms have expanded their product lines to cater to different customer segments. On one hand, by offering products with a wide range of quality, design, and characteristics, firms can reach more consumers, generate additional sales, and extract higher profit margins. On the other hand, this has caused significant difficulties in matching supply with demand because the demand is less predictable at the individual segment level than at the aggregate level. Various operational strategies (e.g., postponement, component commonality, modular design) have been proposed for firms to enjoy the benefit of product differentiation while mitigating the risk of mismatches between supply and demand. This paper studies the influential practice of upgrading, where higher-quality products can be used to satisfy demand for a lower-quality product that is sold out. Such a practice takes advantage of risk pooling (product substitution essentially allows product/demand pooling), which results in several immediate benefits: First, it increases revenue by serving more demand; second, it enhances customer service by reducing lost sales; third, it may lead to lower inventory investment by hedging against demand uncertainty.

The practice of upgrading or substitution has been widely adopted in the business world. In the automobile industry, firms may shift demand for a dedicated capacity to a flexible capacity when the dedicated capacity is constrained (Wall 2003). In the semiconductor industry, faster memory chips can substitute for slower chips when the latter are no longer available (Leachman 1987). More examples in production/inventory control settings can be found in Bassok et al. (1999) and Shumsky and Zhang (2009). Similar practice is ubiquitous in the service industries as well. For instance, airlines may assign business-class seats to economy-class passengers, car rental companies may upgrade customers to more expensive cars, and hotels may use luxury rooms to satisfy demand for standard rooms.

Both practitioners and academics surely understand the importance of the upgrading practice. Substantial research has been conducted on how to manage upgrading in a variety of problem settings. Here we contribute to this large body of literature by studying a dynamic capacity management problem under general upgrading structure. For convenience, we use the terms "product" and "capacity" exchangeably throughout the paper, and similarly for "upgrading" and "substitution" (strictly speaking, upgrading is one-way substitution). A brief description of our problem is as follows. Consider a firm selling N products with differentiated qualities in a fixed horizon consisting of T periods. There are N classes of customers who arrive randomly in each period. Each customer requests one unit of the product; in the case of a stock-out, the customer can be satisfied with a higher-quality product at no extra charge. Unsatisfied demand is backlogged and the firm incurs a goodwill cost. The firm needs to first determine the procurement quantity of each product at the beginning of the horizon, and then decide how to distribute the products among incoming customers. Due to long ordering lead time, the firm cannot replenish inventory before the end of the horizon; as a result, the firm must dynamically allocate the products over time, before observing future demand.

This paper represents an extension of the recent work by Shumsky and Zhang (2009, referred to as SZ hereafter). As one of the first studies that incorporate dynamic allocation into substitution models, SZ make a simplifying assumption to maintain tractability. Specifically, they consider single-step upgrading, i.e., a demand can only be upgraded by the adjacent product. Clearly, this is a restrictive assumption because in many practical situations firms may have incentives to use multi-step upgrading to satisfy demand. Thus there is a need for a theoretical model that captures the realistic upgrading structure. The purpose of this paper is to fill this gap in the literature. While relaxing the single-step upgrading assumption, we attempt to address the following questions as in SZ: What is the optimal initial capacity? How should the products be allocated among customers over time? Are there any effective and efficient heuristics for solving the capacity management problem? The main findings from this paper are summarized as follows.

We start with the dynamic capacity allocation problem. In each period, the firm needs to use the available products to satisfy the realized demand. When a product is depleted while there is still demand for that product, the firm may use upgrading to satisfy customers. How to make such upgrading decisions is a key in substitution models. With the general upgrading structure, the optimal allocation policy is complicated by the fact that the upgrading decisions within a period are interdependent. Under the backlog assumption, we show that a Parallel and Sequential Rationing (PSR) policy is optimal among all possible policies. The PSR policy consists of two stages: In stage 1, the firm uses parallel allocation to satisfy demand as much as possible (i.e., demand is satisfied by the same-class capacity). Then in stage 2, the firm sequentially upgrades leftover demand, starting from the highest demand class; when upgrading a given demand class, the firm starts with the lowest capacity class. The optimality of such a sequential rationing scheme depends on an important property. That is, when using a particular class of capacity to upgrade demand, the upgrading decision does not depend on the status of the portion of the system below that class. The PSR can greatly reduce the computational complexity because the upgrading decisions do not have to be solved jointly. As an extension, we also consider the multi-horizon model with capacity replenishment and show that the PSR policy remains optimal in each horizon. Our theoretical results, though intuitive, turn out to be very challenging to prove. Indeed, our proofs rely on intricate arguments and fully exploit the special structure of the upgrading problem.

Despite the simplified policy structure given by the PSR, solving the problem is still quite involved due to the curse of dimensionality. So there is a need to search for fast heuristics that perform well for the firm. We present a heuristic that adapts certainty equivalence control (CEC) to exploit the sequential rationing property in the PSR policy. Such a heuristic is more appealing than the commonly used CEC heuristic, and we call it refined certainty equivalence control (RCEC) heuristic. Through extensive numerical experiments, we find that the RCEC heuristic delivers close-to-optimal profit for the firm.

The RCEC heuristic enables us to solve large problems effectively. Thus we can use numerical studies based on such a heuristic to derive more insights into the dynamic capacity management problem. First, compared to single-step upgrading, general upgrading (multi-step upgrading) can be highly valuable, especially when the capacities are severely unbalanced. Second, our numerical studies indicate that the firm's profit is not sensitive to the initial capacity decision, given that the optimal upgrading policy is used. For instance, either the newsvendor capacities (calculated assuming no upgrading) or the static capacities (calculated assuming complete demand information) provide nearly optimal profit for the firm. However, the negative impact of using suboptimal allocation policies could be quite significant. These findings suggest that from the practical perspective, deriving the optimal allocation policy should receive a higher priority than calculating the optimal initial capacity.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 describes the model setting. The optimal allocation policy is characterized by Sections 4 and 5. Section 6 extends the base model to multiple horizons with capacity replenishment. Section 7 proposes the RCEC heuristic and Section 8 presents the findings from numerical studies. The paper concludes with Section 9. All proofs are given in the appendix.

2. Literature Review

This paper falls in the vast literature on how to match supply with demand when there are multiple classes of uncertain demand. To facilitate the review, we may divide this literature into four major categories using the following criteria: (1) whether there are multiple capacity types or a single capacity type; and (2) whether the nature of capacity allocation is static or dynamic. A problem is called static if capacity allocation can be made after observing full demand information. The category that involves the single capacity and static allocation essentially reduces to the newsvendor model that is less relevant. Thus, our review below focuses on the representative studies from the other three categories.

The first category of studies involves multiple capacity types and static capacity allocation. In these studies, firms invest in capacities before demand is realized and then allocate capacities to customers after observing all demand. Due to the existence of multiple capacity types, the issue of substitution naturally arises. Van Mieghem (2003) and Yao and Zheng (2003) provide comprehensive surveys of this category of studies, which can be further divided into two groups. One group of papers study the optimal capacity investment and/or allocation decisions under substitution. Parlar and Goyal (1984) and Pasternack and Drezner (1991) are among the first to consider the simplest substitution structure with two products. Bassok et al. (1999) extend the problem to the general multi-product case. Hsu and Bassok (1999) introduce random yield into the substitution problem. By assuming single-level substitution, Netessine et al. (2002) study the impact of demand correlation on the optimal capacity levels. Van Mieghem and Rudi (2002) propose the notion of newsvendor networks that consist of multiple newsvendors and multiple periods of demand. Similar settings can be found in the studies on multi-period inventory models with transshipment, including Robinson (1990), Archibald et al. (1997), and Axsäter (2003). Although these studies involve multiple periods, replenishment is allowed and capacity allocation in each period is made with full demand information. The other group of studies focuses on the value of capacity flexibility. Fine and Freund (1990) and Van Mieghem (1998) consider two types of capacities (dedicated and flexible) and study the optimal investment in flexibility. Bish and Wang (2004) and Chod and Rudi (2005) incorporate pricing decisions when studying the value of resource flexibility. Jordan and Graves (1995) investigate a manufacturing flexibility design problem and discover the well-known chaining rule: Limited capacity flexibility, configured to connect all production facilities and products in a complete chain, can almost deliver the benefit of full flexibility. Their classic work on the design of flexibility has inspired numerous follow-up studies. For example, recently, Chou et al. (2010, 2011) have provided analytical evaluations of the chaining structure for both symmetric and asymmetric problem settings with large scales.

The second category of related literature studies the allocation of a single capacity to multi-class demand in a dynamic environment. This category dates back to the early work by Topkis (1968), who characterizes the optimal rationing policy that assigns capacity to different customer classes over time. Since then similar rationing policies have been applied to various industry settings. For instance, many revenue management studies focus on how to maximize firms' revenue through capacity rationing when there are multiple fare classes for a single seat type; see Talluri and van Ryzin (2004b) for a review of this literature. A stream of studies on production and inventory control has also derived threshold policies when serving multiple customer classes; see Ha (1997, 2000), de Véricourt et al. (2001, 2002), Deshpande et al. (2003), Savin et al. (2005), Ding et al. (2006) and the references therein.

The third category of studies involves multiple capacity types and dynamic capacity allocation. It differs from the first category mainly in that firms need to allocate capacities to customers without full demand information. There are relatively few papers in this category. Shumsky and Zhang (2009) consider a dynamic capacity management problem with single-step upgrading. They characterize the optimal upgrading policy and provide easy-to-compute bounds for the optimal protection limits that can help solve large problems. Xu et al. (2011) consider a two-product dynamic substitution problem where customers may or may not accept the substitution choice offered by seller. Our paper extends Shumsky and Zhang (2009) to allow general upgrading. We show that a sequential upgrading policy is optimal for such a problem and provide a fast heuristic that can effectively solve the optimal capacity investment and allocation decisions. Our problem can be framed as a network revenue management model with full upgrading, where the fares are fixed and each demand requests one unit of the corresponding resource (see Gallego and van Ryzin

1997). Gallego and Stefanescu (2009) introduce two continuous optimal control formulations for capacity allocation but concentrate on the analysis of deterministic cases. Steinhardt and Gönsch (2012) study a similar network revenue management problem but allow at most one customer in each period, which means there is at most one upgrading decision in each period. In contrast, we consider multiple interdependent upgrading decisions in each period that involve a much more challenging structure. Our work is also related to the studies on airline revenue management that involve multiple fare products. Talluri and van Ryzin (2004a) study revenue management under a general customer choice model. Zhang and Cooper (2005) consider the selling of parallel flights with dynamic customer choice among the flights. More recent developments include Liu and van Ryzin (2008) and Zhang (2011). In these studies, firms need to decide the subset of products from which a customer can choose; while in our paper, firms decide how to allocate capacities to realized demand. Therefore, both the model settings and results are quite different between these studies and our paper.

3. Model Setting

Consider a firm managing N types of products to satisfy customer demand. The products are indexed in decreasing quality so that product 1 has the highest quality while product N has the lowest quality. There are N corresponding classes of customer demand, i.e., a customer is called class j if she requests product j $(1 \leq j \leq N)$. The sales horizon consists of T discrete periods. The initial capacities of the products must be determined prior to the first period and no capacity replenishment is allowed during the sales horizon. (In Section 6, we extend the model to consider multiple horizons and allow for replenishment.) Customers arrive over time and the demand in each period is random. Let $\mathbf{D}^t = (d_1^t, d_2^t, \cdots, d_N^t)^\intercal \in \Re_+^N$ denote the demand vector for period t $(1 \leq t \leq T)$, where superscript $\intercal$ stands for the transpose operation. Throughout the paper we use bold letters for vectors and matrices, and use $(\mathbf{Z})_i$ for the i-th component of vector $\mathbf{Z}$ (or $(\mathbf{Z})_{ij}$ for the corresponding element in matrix $\mathbf{Z}$). For instance, $(\mathbf{D}^t)_i = d_i^t$ is the demand for product i in period t. We assume demand is independent across periods; however, demands for different products within a period can be correlated.

Let r_j be the revenue the firm collects from satisfying a class j customer. If product j is out of stock, then a class j customer could be upgraded at no extra charge by any product i as long as $i < j$. If a class j demand cannot be satisfied in period t, then it will be backlogged to the next period and the firm has to incur a goodwill cost g_j[1]. Define $\mathbf{G} = (g_1, \cdots, g_N) \in \Re_+^N$. To incorporate service settings like the car rental industry, we include a usage cost denoted by u_i for product i. We make the following assumptions:

Assumption 1 **(A1)**. $r_1 > r_2 > \cdots > r_N$.

ASSUMPTION 2 **(A2).** $g_1 > g_2 > \cdots > g_N$.

ASSUMPTION 3 **(A3).** $u_1 > u_2 > \cdots > u_N$.

We may define $\alpha_{ij} = r_j + g_j - u_i$ $(i \le j)$ as the profit margin for satisfying a class j customer using product i. Based on the above assumptions, we know $\alpha_{ij} > \alpha_{ik}$ and $\alpha_{jk} > \alpha_{ik}$ $(i < j < k)$. In other words, for a given capacity, it is more profitable to satisfy a higher class of demand; for a given demand, it is more profitable to use a lower class of capacity. These assumptions are similar to but more general than those made in SZ: We have relaxed the single step upgrading assumption in SZ ($\alpha_{ij} > 0$ only if $j = i+1$) and added Assumption (A2) about the backorder costs. Note that the above assumptions do not require all α_{ij} to be positive. Specifically, if $\alpha_{ij} < 0$ for some i and j, then the assumptions imply that $\alpha_{1j} < \cdots < \alpha_{ij} < 0$ and $\alpha_{iN} < \cdots < \alpha_{ij} < 0$, which are reasonable in practice.

The firm's objective is to maximize the expected profit over the sales horizon. There are two major decisions for the firm. First, the firm needs to determine the initial capacity before the start of the selling season; second, the firm needs to allocate the available capacities to satisfy demands in each period. Let $\mathbf{C} = (c_1, \cdots, c_N) \in \Re_+^N$ denote the capacity cost vector, $\mathbf{X}^t = (x_1^t, x_2^t, \cdots, x_N^t)^\intercal \in \Re_+^N$ the starting capacities in period t, and $\tilde{\mathbf{D}}^t = (\tilde{d}_1^t, \tilde{d}_2^t, \cdots, \tilde{d}_N^t)^\intercal \in \Re_+^N$ the backordered demand at the beginning of period t. We use $\mathbf{Y}^t$ for the capacity allocation matrix in period t, i.e., $(\mathbf{Y}^t)_{ij} = y_{ij}^t$ is the amount of product i offered to satisfy class j demand ($y_{ij} = 0$ if $i > j$). Define $\Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ as the optimal revenue-to-go function in period t given the state variable $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$. Then the buyer's problem can be formulated as follows:

$$\max_{\mathbf{X}^1 \in \Re_+^N} \Pi(\mathbf{X}^1) = \max_{\mathbf{X}^1 \in \Re_+^N} \left\{ \Theta^1(\mathbf{X}^1, \mathbf{0}) - \mathbf{C} \cdot \mathbf{X}^1 \right\}, \tag{1}$$

and for each period t $(1 \le t \le T)$:

$$\begin{aligned} \Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t) &= \mathop{\mathbb{E}}_{\mathbf{D}^t} \left\{ \Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t | \mathbf{D}^t) \right\} \\ &= \mathop{\mathbb{E}}_{\mathbf{D}^t} \left\{ \max_{\mathbf{Y}^t} \left[H(\mathbf{Y}^t | \tilde{\mathbf{D}}^t; \mathbf{D}^t) + \Theta^{t+1}(\mathbf{X}^{t+1}, \tilde{\mathbf{D}}^{t+1}) \right] \right\}, \end{aligned} \tag{2}$$

where

$$H(\mathbf{Y}^t | \tilde{\mathbf{D}}^t; \mathbf{D}^t) = \sum_{1 \le i \le j \le N} \alpha_{ij} y_{ij}^t - \mathbf{G} \cdot (\tilde{\mathbf{D}}^t + \mathbf{D}^t), \tag{3}$$

$$\mathbf{X}^{t+1} = \mathbf{X}^t - \mathbf{Y}^t \cdot \mathbf{1} \ge \mathbf{0}, \tag{4}$$

$$\tilde{\mathbf{D}}^{t+1} = \tilde{\mathbf{D}}^t + \mathbf{D}^t - (\mathbf{Y}^t)^\intercal \cdot \mathbf{1} \ge \mathbf{0}, \tag{5}$$

$$\mathbf{Y}^t \ge 0, \qquad \mathbf{1} = (1, 1, \cdots, 1)^\intercal.$$

We assume leftover products have zero value and unmet demand is lost at the end of the selling season, so $\Theta^{T+1} \equiv 0$. Note that the optimal revenue-to-go function $\Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ is recursively defined in (2). Given the allocation decision $\mathbf{Y}^t$, $H(\mathbf{Y}^t|\tilde{\mathbf{D}}^t; \mathbf{D}^t)$ in (3) denotes the single period revenue, which is the difference between the upgrading revenue and the goodwill cost. The state transition between two consecutive periods is governed by (4) and (5), which represent two constraints for the allocation decision $\mathbf{Y}^t$ in period t.

4. Parallel and Sequential Rationing (PSR)

This section starts analyzing the upgrading problem given in (1). Here we introduce several useful definitions and qualitatively characterize the optimal allocation policy. The formal optimality proof will be presented in the next section. As the first step, since

$$
\begin{aligned}
\Pi(\mathbf{0}) &= -\mathbf{G} \cdot \sum_{t=1}^{T} (T+1-t)\, \mathbb{E}[\mathbf{D}^t] > -\infty, \\
\lim_{\mathbf{X}^1 \to \infty} \Pi(\mathbf{X}^1) &= \sum_{t=1}^{T} \sum_{i=1}^{N} (r_i - u_i)\, \mathbb{E}[d_i^t] - \lim_{\mathbf{X}^1 \to \infty} \mathbf{C} \cdot \mathbf{X}^1 = -\infty,
\end{aligned}
\tag{6}
$$

and the fact that $\Pi(\mathbf{X}^1)$ is continuous in $\mathbf{X}^1 \in \Re_+^N$, we know there exists a finite $\mathbf{X}^* \in \Re_+^N$ that solves the optimization problem in (1).

From Murty (1983) and Rockafellar (1996), for any demand realization $\mathbf{D}^T$ in period T, it is straightforward to see $\Theta^T(\mathbf{X}^T, \tilde{\mathbf{D}}^T|\mathbf{D}^T)$ is concave in the state variable $(\mathbf{X}^T, \tilde{\mathbf{D}}^T)$, which are the right-hand side variables in the linear program defined by (2). Since concavity is preserved under the expectation operation on $\mathbf{D}^t$ $(1 \le t \le T)$ and the maximization operation with respect to the allocation decision $\mathbf{Y}^t$ (see, for example, Simchi-Levi et al. 2014, Proposition 2.1.3 and 2.1.15), Θ^t is again concave in $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ in each period t. Clearly, the revenue function

$$
\hat{\Theta}^t(\mathbf{Y}^t|\mathbf{X}^t, \tilde{\mathbf{D}}^t; \mathbf{D}^t) = H(\mathbf{Y}^t|\tilde{\mathbf{D}}^t; \mathbf{D}^t) + \Theta^{t+1}(\mathbf{X}^{t+1}, \tilde{\mathbf{D}}^{t+1}), \tag{7}
$$

in period t given state $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ and demand realization $\mathbf{D}^t$, is also concave in the allocation decision $\mathbf{Y}^t$. The concavity property is summarized in the following proposition whose formal proof is omitted.

PROPOSITION 1. *In period t, $\Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ is concave in $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$, and $\hat{\Theta}^t(\mathbf{Y}^t|\mathbf{X}^t, \tilde{\mathbf{D}}^t; \mathbf{D}^t)$ is concave in $\mathbf{Y}^t$.*

Notice that the allocation decision $\mathbf{Y}^t$ is constrained by a bounded polyhedron defined by (4)-(5) and $\hat{\Theta}^t$ in (7) is continuous in $\mathbf{Y}^t$. Thus, there always exists an optimal allocation to the general upgrading problem in each period t. For a given state $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ and demand realization $\mathbf{D}^t$, there are two types of decisions: parallel allocations y_{ii}^t for all i $(1 \le i \le N)$ and upgrading decisions

y_{ij}^t for classes i and j $(1 \le i < j \le N)$. These are dynamic decisions because they will not only determine revenue H in the current period but also affect future revenue $\Theta^{t+1}(\mathbf{X}^{t+1}, \tilde{\mathbf{D}}^{t+1})$.

It is straightforward to solve the parallel allocation problem. In our model, the maximum revenue we can get from a unit of capacity i is α_{ii} through the parallel allocation, i.e., capacity i is used to fulfill demand class i. It is suboptimal to satisfy demand from lower classes using capacity i when there is still unmet demand i. Further, the expected value of carrying over capacity i to the next period will not exceed α_{ii}, either. Hence the optimal strategy is to use the parallel allocation as much as possible, which implies $y_{ii}^t = \min(d_i^t + \tilde{d}_i^t,\ x_i^t)$. Another implication is that in the state variable $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$, class i $(1 \le i \le N)$ cannot be positive in both $\mathbf{X}^t$ and $\tilde{\mathbf{D}}^t$. Thus, we can use a single variable $\mathbf{M}^t = (\mathbf{X}^t - \tilde{\mathbf{D}}^t) = (m_1^t, m_2^t, \cdots, m_N^t)^\intercal$ to represent the state at the beginning of period t (before the parallel allocation): $m_i^t > 0$ means there is positive capacity for i while $m_i^t < 0$ means there is backordered demand for i. In the rest of the paper we will use $\mathbf{M}^t$ and $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ exchangeably.

The more challenging question is how to make the upgrading decisions after the parallel allocation. The state after the parallel allocation in period t is $(m_1^t - d_1^t, m_2^t - d_2^t, \cdots, m_N^t - d_N^t)^\intercal$. Note that $m_i^t - d_i^t > 0$ means that there is leftover capacity i, while $m_i^t - d_i^t < 0$ implies that there is unsatisfied demand i and capacity i must have been depleted. The firm needs to decide how much demand should be upgraded using higher class capacities. This is equivalent to a rationing problem, i.e., how much capacity should be protected to satisfy future demand. The upgrading problem in our model is different from the one studied in SZ. Particularly, with the single-step assumption in SZ, when capacity i is depleted, classes above i and those below i become independent of each other in future periods, and thus the upgrading problem is greatly simplified because all the upgrading decisions can be solved separately. However, with the general upgrading structure in our model, the upgrading decisions after parallel allocation are no longer isolated. In this case, we may have to solve all decisions jointly, which could be computationally intensive. Fortunately, close scrutiny shows that the following two observations can greatly reduce the complexity of the upgrading problem. The intuition and formal proofs of these observations will be presented in Section 5.1.

First, the upgrading decision y_{ij}^t of using capacity i to upgrade demand j is independent of the demands and the capacities below class j. Second, for demand class j with $m_j^t - d_j^t < 0$, the upgrading decisions y_{ij}^t, $i = 1, \cdots, j-1$ can be solved sequentially in i starting from the lowest class i $(i < j)$ with positive capacity. Based on these observations, the upgrading problem can be sequentially solved as follows:

Step 1. Identify the smallest j $(1 \le j \le N)$ with $m_j^t - d_j^t < 0$ (the highest class with unmet demand);

Step 2. For the largest i (the lowest capacity class) less than j with $m_i^t - d_i^t > 0$, determine the upgrading quantity y_{ij}^t in period t (or equivalently, the quantity of capacity i to be protected for the next period). When solving y_{ij}^t, we can ignore the classes lower than j;

Step 3. Repeat Step 2 until all capacity classes available for upgrading demand j have been considered;

Step 4. Repeat Step 1 until all unmet demand classes have been considered.

To summarize, the firm may allocate capacity using the so-called Parallel and Sequential Rationing (PSR) policy. Under such a policy, the firm first performs the parallel allocation on each class to satisfy new demands, and then sequentially decides upgrading quantities for classes with unmet demand. For easy reference, hereafter we refer to the aforementioned allocation policy as the PSR policy.

The most crucial decision in the sequential upgrading procedure is to determine y_{ij}^t in Step 2. Consider the decision about how much capacity i should be used to upgrade demand j. It is clear that as long as the current upgrade revenue α_{ij} is greater than the expected marginal value in the future, capacity i should be used to upgrade demand j. Such an upgrading or rationing decision essentially specifies the protection levels for the capacities. Let p_{ij} be the optimal protection level of capacity i with respect to demand j, i.e., the firm should stop upgrading demand j by capacity i when the capacity level of i drops to p_{ij}. Since $\Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ is concave in $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ by Proposition 1, the expected marginal value of capacity i is monotonically increasing as capacity i decreases. Hence, the protection level p_{ij} in period t is the unique lower bound above which using capacity i to upgrade demand j is profitable. Define $\frac{\partial}{\partial p}\Theta^t = \left[\frac{\partial}{\partial p^+}\Theta^t, \frac{\partial}{\partial p^-}\Theta^t\right]$ as the subdifferential of Θ^t with respect to some variable p, where $\frac{\partial}{\partial p^-}\Theta^t$ and $\frac{\partial}{\partial p^+}\Theta^t$ are the left and right derivatives, respectively. Let

$$\mathbf{N}^t = (n_1^t, n_2^t, \cdots, n_N^t)^\intercal$$

denote the state of the system immediately before the epoch of determining y_{ij}^t. Specifically, $\mathbf{N}^t$ is a generic system state and can be recursively defined, i.e., $\mathbf{N}^t$ could be the resulting state after the parallel allocation and possibly some upgrading allocations as well (recall the upgrading decisions are made sequentially starting from the highest demand class). The optimal protection levels can be defined as follows.

Definition 1. The optimal protection level $p_{ij} \geq 0$ under state $\mathbf{N}^t = (n_1^t, n_2^t, \cdots, n_N^t)^\intercal$ is defined as

$$p_{ij} = \begin{cases} p & \text{if } \alpha_{ij} \in \frac{\partial}{\partial p}\Theta^{t+1}(n_1^t, \cdots, n_{i-1}^t, p, 0, \cdots, 0, -p, n_{j+1}^t, \cdots, n_N^t), \\ 0 & \text{if } \alpha_{ij} > \frac{\partial}{\partial p^+}\Theta^{t+1}(n_1^t, \cdots, n_{i-1}^t, p, 0, \cdots, 0, -p, n_{j+1}^t, \cdots, n_N^t)|_{p=0}. \end{cases} \tag{8}$$

With the protection levels p_{ij} and $\mathbf{N}^t$, the optimal upgrading decision y_{ij}^t is simply given by $\min\left((n_i^t - p_{ij})^+,\ (-n_j^t)^+\right)$ where $(x)^+ = \max(x, 0)$. Notice that there are 0's between classes i and j since the PSR policy does not consider y_{ij}^t if there exists a class s $(i < s < j)$ with positive capacity or unmet demand. When class s has a positive capacity, it is more profitable to upgrade demand j

with capacity s instead of capacity i, and it is unnecessary for us to consider y_{ij}^t if there is capacity s remaining after solving y_{sj}^t. When there is unmet demand for class s, capacity i should upgrade demand s first, and it would be suboptimal to upgrade demand j if class s still has unmet demand after upgrading y_{is}^t.

Before presenting the main results, we wish to further reduce the computation in the general upgrading problem by exploring its structure. With the single-step upgrading rule, SZ shows that whenever a capacity (say, i) is depleted, the entire problem decouples into two independent subproblems, where the first subproblem consists of products above i and the second consists of products below i (see Lemma 4 in SZ). Under the general upgrading rule, such a property in SZ clearly does not hold. However, it can be shown that under a similar but stronger condition, our problem can also be separated into independent subproblems, as stated in the next lemma.

LEMMA 1. *Consider an N-class general upgrading problem with state $\mathbf{N}^t = (n_1^t, n_2^t, \cdots, n_N^t)^\intercal$ in period t. If $\sum_{s=k}^{i} n_s^t \leq 0$ for all class $k \leq i$, then the problem can be separated into two independent subproblems: an upper part consisting of classes $(1, \cdots, i)$, and a lower part consisting of classes $(i+1, \cdots, N)$.*

Lemma 1 represents a generalized version of the separation property in SZ. For convenience, we say class i is separable if it satisfies the condition stated in Lemma 1. Notice that unlike the property in SZ, $n_i^t \leq 0$ is not enough to split the N-class general upgrading problem since there may be class k ($k < i$), which can upgrade demands in classes $(i+1, \cdots, N)$. However, the condition in Lemma 1 ensures that none of classes $(1, \cdots, i)$ has enough capacity to upgrade the demand in $(i+1, \cdots, N)$ when optimal upgrading is performed. Specifically, there may exist class $k < i$ with positive capacity that can upgrade the demand in $(i+1, \cdots, N)$, but it is more profitable for capacity k to satisfy the demand in classes $(k+1, \cdots, i)$ first, which will consume all of class k's capacity. Therefore, Lemma 1 asserts that the entire upgrading problem can be simplified by decomposition under certain conditions. That is, the profit of the N-class problem can be written as the sum of the profits from independent subproblems $(1, \cdots, i)$ and $(i+1, \cdots, N)$ whenever class i is separable. This observation will allow us to present the optimality proof in Section 5.1 on each of the non-separable subproblems, significantly simplifying the exposition.

5. Optimality and Properties of PSR

5.1. Optimality

We now present the optimality proof of the PSR policy. The proof essentially shows that two results hold for each period by induction: First, the marginal value of Θ^t in (8) has certain properties; second, the PSR policy is optimal due to these properties. As a preparation for the optimality proof, we introduce the concepts of greedy upgrading and effective state. By greedy upgrading, we refer to a PSR step with zero protection levels. That is, after the parallel allocation, the unmet demand will be sequentially upgraded as much as possible. In addition, we define the effective state as follows.

Definition 2. Consider a state vector $\mathbf{N}^t = (n_1^t, n_2^t, \cdots, n_N^t)$ in period t $(1 \le t \le T)$. For class r $(1 \le r \le N)$, the effective state $\hat{\mathbf{N}}_r^t = (\hat{n}_1^t, \cdots, \hat{n}_r^t, n_{r+1}^t, \cdots, n_N^t)$ is defined as the resulting state after applying the greedy upgrading for classes $(1, \cdots, r)$.

In fact, given any state $\mathbf{N}^t$ and its effective state $\hat{\mathbf{N}}_r^t$ for some class r, let h $(1 \le h \le r)$ denote the highest class with $\hat{n}_h^t > 0$, then class $h-1$ is separable in $\mathbf{N}^t$. To see this, note that given $\hat{n}_h^t > 0$, there is no upgrade between classes $(1, \cdots, h-1)$ and $(h, \cdots, r)$ when applying the greedy upgrading. Thus, for all class $k < h$, we have $\sum_{s=k}^{h-1} n_s^t \le \sum_{s=k}^{h-1} \hat{n}_s^t \le 0$, where the first inequality holds because there may be upgrade between classes $(1, \cdots, k-1)$ and $(k, \cdots, h-1)$ when performing the greedy upgrading, and the second inequality follows from the definition of class h. Hence, according to Lemma 1, $h-1$ is separable, and classes $(1, \cdots, h-1)$ can be ignored in the subsequent allocation decisions.

Consider a state vector $\mathbf{N}^t = (n_1^t, \cdots, n_N^t)$ in period t. For $1 \le i < j \le N$, define

$$\Delta_{ij}^{+-}\Theta^t(\mathbf{N}^t) = \frac{\partial}{\partial n_i^+}\Theta^t(\mathbf{N}^t) - \frac{\partial}{\partial n_j^-}\Theta^t(\mathbf{N}^t), \qquad \Delta_{ij}^{-+}\Theta^t(\mathbf{N}^t) = \frac{\partial}{\partial n_i^-}\Theta^t(\mathbf{N}^t) - \frac{\partial}{\partial n_j^+}\Theta^t(\mathbf{N}^t).$$

Notice that the protection level p_{ij} in (8) can be equivalently defined as

$$\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t)\,|_{p=p_{ij}} \le \alpha_{ij} \le \Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t)\,|_{p=p_{ij}},$$

where $\mathbf{N}^t = (n_1^t, \cdots, n_{i-1}^t, p, 0, \cdots, 0, -p, n_{j+1}^t, \cdots, n_N^t)$. Thus, the properties of $\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t)$ and $\Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t)$ will have useful implications on the optimal protection levels.

Now we are in the position to prove the optimality of the PSR by induction.

Proposition 2. *Consider an N-class general upgrading problem in period t $(1 \le t \le T)$ with state vector $\mathbf{N}^t$, where $(n_{i+1}^t, \cdots, n_{j-1}^t) \le 0$ and $n_j^t < 0$.*

1. *We have*

$$\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{+-}\Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t), \qquad \Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{-+}\Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t), \tag{9}$$

both of which are independent of the values of $(n_j^t, \cdots, n_N^t)$.

2. *The PSR policy solves the general upgrading problem in period t.*

Proposition 2 deserves some discussion. The first part of Proposition 2 states that the optimal protection level p_{ij} in period t is independent of the values of $(n_j^t, \cdots, n_N^t)$, while it is affected by the classes above i through the effective state $(\hat{n}_1^t, \cdots, \hat{n}_{i-1}^t)$. These results provide the rationale behind the sequential rationing in the PSR policy. We offer the following intuitive explanation of these results. First, we explain why $\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t)$ and $\Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t)$ are independent of $(n_j^t, \cdots, n_N^t)$. Before deciding p_{ij} or y_{ij}^t, without loss of generality, suppose there are p units of capacity i lined up to upgrade unmet demand j. Meanwhile, the unmet demand in class j can be treated as a waiting line, which will be satisfied in the first-come first-served sequence. To determine p_{ij} we need to compare α_{ij} with the expected value of each unit of capacity i. Consider the capacity unit 1, i.e., the first unit in front of the capacity line. Given the backorder assumption, capacity unit 1 can only satisfy either a future demand in classes $(i, \cdots, j-1)$ or the first unit in the waiting line of demand j. Hence, the expected value of capacity unit 1 in class i is independent of states $(n_{j+1}^t, \cdots, n_N^t)$. Furthermore, the above argument only relies on the fact that there exists unmet demand j. Thus, the expected value of capacity unit 1 is also independent of n_j^t, the length of the waiting line of demand j. Such an argument applies to any other capacity units of class i.

Next, we explain the equalities in (9). Note that for any class k $(1 < k < i)$ with positive capacity, it would not upgrade demand from class i or below in any optimal policy if there exists class r $(k < r < i)$ with backordered demand or positive capacity. In fact, it would be more valuable to use capacity k to upgrade demand r in the former case and to use capacity r to upgrade demand from class i or below in the latter case. The remaining capacity of class k after upgrading all backordered demands in classes $(k+1, \cdots, i-1)$ equals $\hat{n}_k^t$ as defined in the effective state. Therefore, the expected future value of capacity i in period t should equivalently depend on the effective state $(\hat{n}_1^t, \cdots, \hat{n}_{i-1}^t)$.

Finally, for any given period t under the PSR policy, the effective states of all intermediate states for classes $(1, 2, \ldots, i-1)$ are the same before we exhaust the capacity of class i. Thus, Proposition 2 implies that when solving p_{ij}, it is sufficient to use the first $i-1$ components of $\mathbf{M}^t - \mathbf{D}^t$ (i.e., the state of the system in period t after the parallel allocation) instead of $\mathbf{N}^t$ (i.e., the state of the system prior to deciding y_{ij}^t) in the PSR policy. This is a unique and interesting property of the general upgrading problem, allowing us to simultaneously and independently solve all protection levels based on $\mathbf{M}^t - \mathbf{D}^t$. Specifically, for any classes i and j $(1 \le i < j \le N)$ with $n_i^t > 0$ and $n_j^t < 0$, the protection level p_{ij} can be immediately determined by $\frac{\partial}{\partial p}\Theta^{t+1}(m_1^t - d_1^t, \cdots, m_{i-1}^t - d_{i-1}^t, p, 0, \cdots, 0, -p, 0, \cdots, 0)$.

5.2. Properties of Protection Levels

After establishing the optimality of the PSR policy, we explore some important properties related to the optimal protection levels from the PSR policy.

First, if both the initial capacity $\mathbf{X}^1$ and all demands are integer valued, similar to SZ, we can prove that there exists an integer valued optimal policy generated by the PSR policy.

PROPOSITION 3. *If initial capacity $\mathbf{X}^1$ and demand $\mathbf{D}^1, \cdots, \mathbf{D}^T$ are integer valued, there exists an integer valued optimal policy $\mathbf{Y}^1, \cdots, \mathbf{Y}^T$ derived by the PSR policy.*

To further characterize the protection level p_{ij} defined in (8), we need to deal with the marginal value of Θ^t with respect to each capacity level and unmet demand level. Intuitively, one may think that the profit will be higher if there is an additional unit of capacity $i-1$ $(1 < i \le N)$ rather than capacity i. But this is not necessarily true. When making upgrading decisions, one more unit of capacity from the higher class $i-1$ always provides more flexibility, but such a flexibility does not necessarily mean a higher profit since $\alpha_{ij} > \alpha_{i-1,j}$ $(i < j)$ by our assumption. Similarly, one more unit of demand in a lower class, which can be upgraded by more classes of capacities, has similar advantage but can not guarantee greater profit because $\alpha_{ij} < \alpha_{i,j+1}$ $(i \le j)$. However, we can show two different monotone properties of the protection levels. First, since lower demand has less value for any capacity, the protection level should increase in the class index of demand.

PROPOSITION 4. *For the same $(n_1^t, \cdots, n_{i-1}^t)$ in period t $(1 \le t \le T)$, $p_{ij} \le p_{i,j+1}$ when $i < j$.*

Because the general upgrading problem in period T is a transportation problem, $\Theta^T(\mathbf{X}^T, \tilde{\mathbf{D}}^T)$ is submodular in $(\mathbf{X}^T, -\tilde{\mathbf{D}}^T)$ (see Topkis 1998). This implies the protection level p_{ij} in period $T-1$ under state $\mathbf{N}^{T-1}$ is decreasing in $(n_1^{T-1}, \cdots, n_{i-1}^{T-1})$. In fact, the same monotonicity holds in earlier periods.

PROPOSITION 5. *The optimal protection level p_{ij} $(1 \le i < j \le N)$ in period t $(1 \le t \le T)$ are component-wise decreasing in $(n_1^t, \cdots, n_{i-1}^t)$.*

For any class i $(1 \le i \le N)$, this result assures that the more capacities (or less back-ordered demands) in classes higher than i, the more upgrades can be offered by class i. Note that larger $(n_1^t, \cdots, n_{i-1}^t)$ means higher probability of demand i being upgraded in remaining periods, which decreases the expected marginal value of capacity i and gives class i a greater incentive to upgrade lower demands in the current period.

It is noteworthy that although the result for the last period can be proved using lattice programming in Topkis (1998), the commonly used preservation property of supermodularity under maximization operations, Theorem 2.7.6 in Topkis (1998), does not apply. In fact, the revenue-to-go function $\Theta^T(\mathbf{X}^T, \tilde{\mathbf{D}}^T)$ in period T, used in the dynamic program recursion for period $T-1$, is the optimal objective value of a transportation problem that is submodular in $(\mathbf{X}^T, -\tilde{\mathbf{D}}^T)$. Therefore, our proof relies heavily on the structure of the general upgrading problem and fully utilizes the optimality of the PSR policy.

One may ask whether the optimal protection levels are decreasing over time, i.e., the protection level would be lower if there are fewer periods to go. Interestingly, though this is true in SZ, it does not hold in our upgrading problem. This is mainly due to the existence of the backorder cost. Note that the purpose of the protection levels is to balance the goodwill loss of carrying backorders and the revenue loss of losing future demand from the same class. For early periods that are still far away from the end of the horizon, because a backorder causes the goodwill loss in each period until it is upgraded, the protection levels may be lower to avoid high backorder costs; in contrast, when it is close to the end of the horizon, the protection levels may come back up because carrying backorders will be less costly.

We may use a two-product three-period example to explain this counter-intuitive result. Let $(2,-2)$ be the state after the parallel allocation, $\mathbf{D}^2=(0,0)$ and $\mathbf{D}^3=(1,0)$ with probability 1. Working backward to solve the p_{12} in period 2, since

$$\Theta^3(2,-2)-\Theta^3(1,-1)=\alpha_{12}-g_2<\alpha_{12},\quad \Theta^3(1,-1)-\Theta^3(0,0)=\alpha_{11}-g_2,$$

we have $p_{12}=1$ in period 2 if $\alpha_{11}-g_2>\alpha_{12}$. Since $\mathbf{D}^2=(0,0)$, there is

$$\Theta^2(2,-2)-\Theta^2(1,-1)=\alpha_{12}-g_2<\alpha_{12},\quad \Theta^2(1,-1)-\Theta^2(0,0)=\alpha_{11}-2g_2.$$

Therefore, if $\alpha_{11}-g_2>\alpha_{12}>\alpha_{11}-2g_2$, the optimal protection level p_{12} increases from 0 in period 1 to 1 in period 2. That is, the protection level does not necessarily decrease over time in our general upgrading problem[2].

6. Multiple Horizons with Capacity Replenishment

Now we extend our model to multiple horizons with capacity replenishment. Specifically, there are K $(K\geq 1)$ horizons, each consisting of T periods. Demands across horizons are independent and identically distributed. At the beginning of each horizon k $(1\leq k\leq K)$, the firm observes the leftover capacity $\mathbf{X}$ and unmet demand $\tilde{\mathbf{D}}$ carried over from the previous horizon. There are two decisions for the firm in each horizon: First, the firm decides how much capacity to replenish; second, it allocates capacity to satisfy demand as formulated in (2). For completeness, we assume unmet demand after the K-th horizon can also be satisfied by purchasing additional capacity. There is a unit cost vector $\mathbf{C}=(c_1,\cdots,c_N)\in\Re^N_+$ for capacity replenishment. The remaining capacity at the end of each horizon incurs a holding cost $\mathbf{h}=(h_1,\cdots,h_N)\in\Re^N_+$. The leftover capacity after the K-th horizon can be sold at the initial capacity cost, i.e., it has salvage value $\mathbf{C}$. Revenues and costs are discounted at a rate γ $(0<\gamma\leq 1)$ for each horizon. We introduce a discount factor here because a horizon might be long enough so that the time value of money should be taken into account. The rest of the model setting remains the same as in Section 3.

We show the optimality of a myopic policy in which the firm only needs to solve a single-horizon problem. For this purpose, define

$$\Theta^{T+1}(\mathbf{X}^{T+1},\tilde{\mathbf{D}}^{T+1}) = (\gamma\mathbf{C}-\mathbf{h})\cdot\mathbf{X}^{T+1} + \gamma(\boldsymbol{\alpha}-\mathbf{C})\cdot\tilde{\mathbf{D}}^{T+1}, \tag{10}$$

where $\boldsymbol{\alpha} = (\alpha_{11},\cdots,\alpha_{NN})$ is the revenue from parallel allocation. Since we assume the leftover capacity after the K-th horizon can be sold at the initial capacity cost and unmet demand after the K-th horizon can also be satisfied by purchasing additional capacity, at the end of the K-th horizon, the unit salvage value of the remaining capacity $\mathbf{X}^{T+1}$ is $\gamma\mathbf{C}-\mathbf{h}$, and the value of the back-ordered demand $\mathbf{D}^{T+1}$ is $\gamma(\boldsymbol{\alpha}-\mathbf{C})$. That is, the terminal value at the end of the K-th horizon is exactly given by $\Theta^{T+1}(\mathbf{X}^{T+1},\tilde{\mathbf{D}}^{T+1})$ if the remaining capacity is $\mathbf{X}^{T+1}$ and the back-ordered demand is $\tilde{\mathbf{D}}^{T+1}$, which is essential for the optimality of the myopic policy (similar terminal value assumptions have been commonly used in the literature). Let $\Pi(\mathbf{X};\gamma\mathbf{C}-\mathbf{h};\gamma(\boldsymbol{\alpha}-\mathbf{C}))$ denote the optimal profit of a single-horizon model with initial capacity $\mathbf{X}$ and a terminal value specified by $\Theta^{T+1}(\mathbf{X}^{T+1},\tilde{\mathbf{D}}^{T+1})$. Note that although $\gamma\mathbf{C}-\mathbf{h}$ and $\gamma(\boldsymbol{\alpha}-\mathbf{C})$ are constants, they are used as arguments in Π to emphasize the terminal values of the remaining capacity and back-ordered demand. This is different from the single-horizon model with zero end values in Section 3.

From the proof of Proposition 1, $\Theta^t(\mathbf{X}^t,\tilde{\mathbf{D}}^t)$, which is similarly defined as (2) with $\Theta^{T+1}\equiv\mathbf{0}$ being replaced by Θ^{T+1} in (10), is still concave in $(\mathbf{X}^t,\tilde{\mathbf{D}}^t)$. In particular, $\Pi(\mathbf{X};\gamma\mathbf{C}-\mathbf{h};\gamma(\boldsymbol{\alpha}-\mathbf{C}))$ is concave in $\mathbf{X}$ from the concavity of $\Theta^1(\mathbf{X},\mathbf{0})$. Furthermore, similarly as (6), we can show that there exists an optimizer $\mathbf{X}^*$ for the concave function $\Pi(\mathbf{X};\gamma\mathbf{C}-\mathbf{h};\gamma(\boldsymbol{\alpha}-\mathbf{C}))$:

$$\mathbf{X}^* \in \arg\max_{\mathbf{X}\in\Re_+^N} \Pi(\mathbf{X};\gamma\mathbf{C}-\mathbf{h};\gamma(\boldsymbol{\alpha}-\mathbf{C}))\,. \tag{11}$$

Note that $\mathbf{X}^*$ is the optimal capacity level for the replenishment model with $K=1$.

The next proposition characterizes the optimal capacity replenishment and allocation policies in the multi-horizon model, given that the firm starts with an initial capacity $\mathbf{X}\le\mathbf{X}^*$. It shows that the structural results from the base model in Section 3 remain valid in the multi-horizon model; thus we will focus on the base model in the rest of the paper.

Proposition 6. *Suppose the firm starts with an initial capacity* $\mathbf{X}\le\mathbf{X}^*$ *in* (11)*. The firm's optimal replenishment policy in the multi-horizon model is a base stock policy with the optimal base stock level* $\mathbf{X}^*$*. Furthermore, the PSR algorithm solves the optimal allocation decisions within each horizon.*

7. Heuristics and Benchmark Models

So far we have characterized the structure of the optimal allocation policy for our dynamic capacity management problem. In this section, we propose an effective heuristic for solving the optimal allocation policy. For future comparison, we also present two benchmark models that are simplified versions of the general upgrading problem.

7.1. Heuristics

We have shown that the PSR policy yields the optimal allocation decisions $\mathbf{Y}^t$ for the firm in period t, which essentially consist of the optimal protection levels for each capacity class. These optimal protection levels are defined by (8) and can be solved by backward induction. For instance, the optimal protection levels in period t depend on the revenue-to-go function Θ^{t+1}, which is determined by the protection levels used in period $t+1$. To evaluate Θ^{t+1}, one needs to derive the optimal protection levels for all possible states in period $t+1$ (note that these protection levels, though possessing the appealing properties established earlier, are still state-dependent). Due to the curse of dimensionality, solving the exact optimal upgrading decisions is quite difficult for large problems[3]. Therefore, we need to search for heuristics that can solve the problem effectively.

Since solving the allocation decision is equivalent to solving the Bellman equation (2) in period t, in order to develop efficient heuristics, we focus on the one-step lookahead policy that hinges upon reasonable approximations to Θ^{t+1}. The basic idea is as follows. Suppose $\bar{\Theta}^{t+1}_{\text{approx}}$ is an easy-to-compute and acceptable approximation to Θ^{t+1}. Given the initial state $(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ and the realized demand $\mathbf{D}^t$ in period t, we solve the following optimization program

$$\max_{\mathbf{Y}^t} \left[H(\mathbf{Y}^t|\tilde{\mathbf{D}}^t;\mathbf{D}^t) + \bar{\Theta}^{t+1}_{\text{approx}}(\mathbf{X}^{t+1}, \tilde{\mathbf{D}}^{t+1}) \right], \tag{12}$$

and obtain the corresponding allocation decision $\mathbf{Y}^t_{\text{approx}}(\mathbf{X}^t, \tilde{\mathbf{D}}^t|\mathbf{D}^t)$ in period t. Let Θ^t_{approx} be the revenue collected by applying the policy $(\mathbf{Y}^t_{\text{approx}}, \cdots, \mathbf{Y}^T_{\text{approx}})$ from period t to T. For simplicity, we do not distinguish between the policy and the decision (e.g., $\mathbf{Y}^t_{\text{approx}}$ and $\mathbf{Y}^t_{\text{approx}}(\mathbf{X}^t, \tilde{\mathbf{D}}^t|\mathbf{D}^t)$), since the proper interpretation is usually clear from the context. Note that $\mathbf{Y}^t_{\text{approx}}$ is a suboptimal policy in the general upgrading problem and $\Theta^t_{\text{approx}} \neq \bar{\Theta}^t_{\text{approx}}$ in general. Moreover, $\Theta^t_{\text{approx}}(\mathbf{N}^t) \leq \Theta^t(\mathbf{N}^t)$ for any state $\mathbf{N}^t$ in period t since $\Theta^t(\mathbf{N}^t)$ adopts the optimal policy from period t to T.

As pointed out by Bertsekas (2005b), even with readily available revenue-to-go approximations, computing Θ^t_{approx} may still involve substantial computational effort. A number of simplifications of the optimization in (12), including different $\bar{\Theta}^{t+1}_{\text{approx}}$ functions, have been considered. Here we present two of them that stand out both in terms of computational time and revenue performance. Because of the linearity in the upgrading problem, the first natural candidate is the traditional certainty equivalence control (CEC) heuristic in the literature (see Bertsekas 2005a, for example). The CEC is a suboptimal control that treats the uncertain quantities as fixed typical values in the stochastic dynamic program. In our case, we use demand means as typical values in evaluating the function $\bar{\Theta}^{t+1}_{\text{approx}}$. Thus, under the CEC, expectation calculations are no longer relevant, which can alleviate the computational burden in our problem. Specifically, the optimal allocation policy in period t is solved together with all future periods where the mean demand is used as approximation.

That is, the optimal allocation decision $\mathbf{Y}^t_{\text{CEC}}$ in the CEC heuristic will be obtained by solving the following linear program:

$$\begin{aligned}
\max_{(\mathbf{Y}^t_{\text{CEC}},\bar{\mathbf{Y}}^{t+1},\cdots,\bar{\mathbf{Y}}^T)\geq 0} \quad & \left\{ H(\mathbf{Y}^t_{\text{CEC}}|\tilde{\mathbf{D}}^t;\mathbf{D}^t) + \sum_{l=t+1}^{T} H(\bar{\mathbf{Y}}^l|\tilde{\mathbf{D}}^l;\boldsymbol{\mu}^l) \right\} \\
\text{s.t.} \quad & \tilde{\mathbf{D}}^{t+1} = \tilde{\mathbf{D}}^t + \mathbf{D}^t - (\mathbf{Y}^t_{\text{CEC}})^\intercal \cdot \mathbf{1}, \\
& \tilde{\mathbf{D}}^{l+1} = \tilde{\mathbf{D}}^l + \boldsymbol{\mu}^l - (\bar{\mathbf{Y}}^l)^\intercal \cdot \mathbf{1}, \qquad l = t+1,\cdots,T, \\
& \left(\mathbf{Y}^t_{\text{CEC}} + \sum_{l=t+1}^{T} \bar{\mathbf{Y}}^l \right) \mathbf{1} \leq \mathbf{X}^t, \\
& (\mathbf{Y}^t_{\text{CEC}})^\intercal \cdot \mathbf{1} \leq \tilde{\mathbf{D}}^t + \mathbf{D}^t, \\
& \left(\mathbf{Y}^t_{\text{CEC}} + \sum_{l=t+1}^{k} \bar{\mathbf{Y}}^l \right)^\intercal \mathbf{1} \leq \tilde{\mathbf{D}}^t + \mathbf{D}^t + \sum_{l=t+1}^{k} \boldsymbol{\mu}^l, \qquad k = t+1,\cdots,T,
\end{aligned} \tag{13}$$

where $\mathbf{X}^t$, $\tilde{\mathbf{D}}^t$, and $\mathbf{D}^t$ are the capacities, backorders, and realized demand in period t, respectively, and $(\boldsymbol{\mu}^1,\boldsymbol{\mu}^2,\cdots,\boldsymbol{\mu}^T)$ denote the mean demand vectors.

The solution to (13) yields the allocation decisions $(\mathbf{Y}^t_{\text{CEC}},\bar{\mathbf{Y}}^{t+1},\cdots,\bar{\mathbf{Y}}^T)$ for periods from t to T, where $(\bar{\mathbf{Y}}^{t+1},\cdots,\bar{\mathbf{Y}}^T)$ are discarded in the subsequent periods. We implement $\mathbf{Y}^t_{\text{CEC}}$ as the allocation decision for period t and then move on to solve problem (13) in period $t+1$. Let Θ^t_{CEC} be the revenue collected by applying the policy $(\mathbf{Y}^t_{\text{CEC}},\cdots,\mathbf{Y}^T_{\text{CEC}})$ in periods from t to T. Define $\Pi_{\text{CEC}}(\mathbf{X}) = \Theta^1_{\text{CEC}}(\mathbf{X},\mathbf{0})$ as the firm's total revenue given initial capacity $\mathbf{X}$ under the CEC heuristic.

Although the above CEC heuristic can simplify our problem, its computational time is still quite long. Consider an N-product general upgrading problem with t periods remaining, the CEC heuristic solves the allocation decisions in the current period as a transportation problem with N classes of capacities and tN classes of demands, whose running time is $O(tN^3(\log(tN)+N\log N))$ (see Brenner 2008). In addition, the optimal allocation is derived from the linear program in (13), which does not use the PSR procedure and the marginal analysis in (8). This means that the CEC might be further improved by exploiting the special properties inherited in our upgrading problem.

To this end, we further simplify the revenue-to-go function by applying greedy upgrading. So the approximation to Θ^{t+1} consists of two components: certainty equivalence control (CEC) and greedy upgrading. Under the CEC, again the mean demand is used as an approximation in all future periods. At the same time, $\bar{\Theta}^{t+1}_{\text{approx}}$ is simplified by adopting greedy upgrading from periods $t+1$ to T rather than solving the linear program as in the CEC heuristic. Such simplification, though suboptimal, is much easier to compute than the linear program[4]. Given these characteristics of the approximation, we call it refined certainty equivalence control (RCEC) and write $\bar{\Theta}^{t+1}_{\text{approx}}$ as $\bar{\Theta}^{t+1}_{\text{RCEC}}$. In addition to the above approximation, the RCEC heuristic then calculates the protection levels in (8) by replacing Θ^{t+1} with $\bar{\Theta}^{t+1}_{\text{RCEC}}$, and determines the allocation decision $\mathbf{Y}^t_{\text{RCEC}}$ in period t by applying the PSR policy to solve the following program

$$\max_{\mathbf{Y}^t}\left[H(\mathbf{Y}^t|\tilde{\mathbf{D}}^t;\mathbf{D}^t)+\bar{\Theta}^{t+1}_{\text{RCEC}}(\mathbf{X}^{t+1},\tilde{\mathbf{D}}^{t+1})\right].$$

Note that $\bar{\Theta}^s_{\text{RCEC}}$ $(s\ge t+1)$ can be defined recursively as follows:

$$\bar{\Theta}^s_{\text{RCEC}}(\mathbf{X}^s,\tilde{\mathbf{D}}^s)=H(\mathbf{Y}^s_\mu|\tilde{\mathbf{D}}^s;\boldsymbol{\mu}^s)+\bar{\Theta}^{s+1}_{\text{RCEC}}(\mathbf{X}^{s+1};\tilde{\mathbf{D}}^{s+1}),\tag{14}$$

where $\mathbf{X}^{s+1}=\mathbf{X}^s-\mathbf{Y}^s_\mu\cdot\mathbf{1}$, $\tilde{\mathbf{D}}^{s+1}=\tilde{\mathbf{D}}^s+\boldsymbol{\mu}^s-(\mathbf{Y}^s_\mu)^\intercal\cdot\mathbf{1}$, $\bar{\Theta}^{T+1}_{\text{RCEC}}\equiv 0$, and $\mathbf{Y}^s_\mu=\left(y^s_{ij}(\mu)\right)_{N\times N}$ is the solution to the following linear program:

$$\max_{\mathbf{Y}^s_\mu\ge 0}\left\{\sum_{1\le i\le j\le N}\alpha_{ij}y^s_{ij}(\mu)\middle|(\mathbf{Y}^s_\mu)^\intercal\cdot\mathbf{1}\le\boldsymbol{\mu}^s+\tilde{\mathbf{D}}^s,\ \mathbf{Y}^s_\mu\cdot\mathbf{1}\le\mathbf{X}^s\right\}.$$

Given the protection levels derived from $\bar{\Theta}^{t+1}_{\text{RCEC}}$, $\mathbf{Y}^t_{\text{RCEC}}$ is the allocation policy in period t solved by the PSR policy, and Θ^t_{RCEC} is the revenue collected by applying policy $(\mathbf{Y}^t_{\text{RCEC}},\cdots,\mathbf{Y}^T_{\text{RCEC}})$ in period t to T. Define $\Pi_{\text{RCEC}}(\mathbf{X})=\Theta^1_{\text{RCEC}}(\mathbf{X},\mathbf{0})$ as the firm's total revenue given initial capacity $\mathbf{X}$ under the RCEC heuristic, and $\mathbf{X}_{\text{RCEC}}$ as the optimal capacity that maximizes $\Pi_{\text{RCEC}}(\mathbf{X})$.

We now illustrate how the PSR policy can be used to significantly simplify the computation of the allocation policy $\mathbf{Y}^t_{\text{RCEC}}$ (through the protection levels) without the explicit function form of $\bar{\Theta}^{t+1}_{\text{RCEC}}$. Although greedy upgrading (rather the optimal allocation) is used in $\bar{\Theta}^{t+1}_{\text{RCEC}}$, it can be shown that for any state $\mathbf{N}^t=(n^t_1,\cdots,n^t_N)$,

$$\tfrac{\partial}{\partial p}\bar{\Theta}^{t+1}_{\text{RCEC}}(n^t_1,\cdots,n^t_{i-1},p,0,\cdots,0,-p,n^t_{j+1},\cdots,n^t_N)\tag{15}$$

is decreasing in p.[5] Thus, the protection levels in the PSR policy can be solved by a binary search, and it suffices to examine whether the protection level p_{ij} is between $\max(n^t_i+n^t_j,0)$ and n^t_i. The binary search starts with the two initial points $\max(n^t_i+n^t_j,0)$ and n^t_i. If it proceeds to evaluate more points in the middle (i.e., p_{ij} is strictly between $\max(n^t_i+n^t_j,0)$ and n^t_i), then there must remain both surplus capacity i and unmet demand j after performing the y^t_{ij} allocation. In this case, there will be no upgrade between classes $(1,\cdots,i-1)$ and $(j,\cdots,N)$, and it is unnecessary to compute the protection levels between these two sets. In addition, it indicates that when determining the upgrading between classes r and s for $i\le s<r\le j,(s,r)\ne(i,j)$, either surplus capacity s or unmet demand r must be zero and the binary search only needs to check the two boundary points. Consequently, the N classes can be partitioned into a number of blocks (say K blocks), and upgrading takes place only within each block. Moreover, in each block there is at most one pair of i and j such that the binary search evaluates more than two points to determine p_{ij}. For block k $(1\le k\le K)$ with size n_k $(2\le n_k\le N)$, the total number of calls to compute the derivative in

(15) is no more than $O(n_k^2 + \log|X|)$, where $|X|$ is the upper bound of the initial capacity in each class. Since there is no upgrade between blocks, to solve the allocation decision in each period, the total number of calls to evaluate $\bar{\Theta}^{t+1}_{\text{RCEC}}$ would be bounded by $O(N^2 + N\log|X|)$. It is noteworthy that the advantage of using protection levels is that they can fully characterize the upgrading policy; these protection levels can be computed via binary search that makes a small number of calls to $\bar{\Theta}^{t+1}_{\text{RCEC}}$, so there is no need to undertake the more demanding task of deriving the explicit expression of $\bar{\Theta}^{t+1}_{\text{RCEC}}$.

Consider an N-product general upgrading problem with t periods remaining. Since greedy upgrading can be solved in the running time of $O(tN^2)$, from the above analysis, the RCEC has a running time of $O(tN^3(N + \log|X|))$ in the worst scenario, which is significantly shorter than the CEC when $|X|$ is moderate. More appealingly, the PSR policy can further reduce the computational complexity in practice. Recall the discussion after Proposition 2, the protection level p_{ij} $(1 \le i < j \le N)$ in period t only depends on the effective state above i, which is decided by $\mathbf{M}^t - \mathbf{D}^t$. Thus, we can use parallel computing technique and solve all protection levels independently based on $\mathbf{M}^t - \mathbf{D}^t$.

A common feature of the RCEC and CEC heuristics is that both use mean demand in future periods as an approximation. There is a critical difference between these two heuristics. In the RCEC, the PSR procedure is used; in particular, the optimal protection level is determined by (8) (i.e., by comparing the upgrading value to the future marginal value). By contrast, in the CEC, the optimal allocation is derived from the linear program in (13), which utilizes neither the PSR procedure nor (8). It seems the adoption of the PSR procedure in the RCEC plays an important role in both reducing the computational complexity and improving the approximation performance, which will be further discussed in Section 8.1.

7.2. Benchmark Models

For future comparison, we introduce two benchmark models in this subsection. The first one is called the crystal ball (CB) model. In this model, the firm has perfect demand forecast when allocating the capacities in each period. Such a benchmark has been widely adopted in the literature because it offers the "perfect hindsight" upper bound of the firm's optimal profit. For instance, it has been used in SZ but is called static model because the firm essentially faces a static capacity allocation problem given complete demand information. Let ω represent a sample path of demand

$(\mathbf{D}^1,\cdots,\mathbf{D}^T)$ over the sales horizon, and $\mathbf{D}^t(\omega)$ the demand in period t on sample path ω. Then, the firm's expected profit from period t to T is defined as $\mathbb{E}_\omega[\Theta^t(\mathbf{X}^t,\tilde{\mathbf{D}}^t;\omega)]$, where

$$\begin{aligned}
\Theta^t(\mathbf{X}^t,\tilde{\mathbf{D}}^t;\omega) = \max_{\mathbf{Y}^t,\cdots,\mathbf{Y}^T} & \sum_{l=t}^{T} H(\mathbf{Y}^l|\tilde{\mathbf{D}}^l;\mathbf{D}^l(\omega)) \\
\text{s.t.} \quad & \tilde{\mathbf{D}}^{l+1} = \tilde{\mathbf{D}}^l + \mathbf{D}^l(\omega) - (\mathbf{Y}^l)^\intercal \cdot \mathbf{1} \qquad l=t,\cdots,T \\
& \sum_{l=t}^{T} \mathbf{Y}^l \cdot \mathbf{1} \le \mathbf{X}^t, \\
& \sum_{l=t}^{k} (\mathbf{Y}^l)^\intercal \cdot \mathbf{1} \le \tilde{\mathbf{D}}^t + \sum_{l=t}^{k} \mathbf{D}^l(\omega), \qquad k=t,\cdots,T, \\
& \mathbf{Y}^l \ge 0, \qquad l=t,\cdots,T.
\end{aligned}$$

The firm's optimal profit in the crystal ball model is given by

$$\max_{\mathbf{X}^1\in\Re_+^N} \Pi_{\text{CB}}(\mathbf{X}^1) = \max_{\mathbf{X}^1\in\Re_+^N} \left\{ \mathbb{E}_\omega[\Theta^1(\mathbf{X}^1,\mathbf{0};\omega)] - \mathbf{C}\cdot\mathbf{X}^1 \right\}, \tag{16}$$

which can be used to benchmark the performance of our heuristic in the dynamic upgrading problem.

The second benchmark is the model without product upgrading. In this case, the firm's problem reduces to N independent newsvendors (NV) with backorders. The firm's expected profit can be written as

$$\begin{aligned}
\max_{\mathbf{X}^1\in\Re_+^N} \Pi_{\text{NV}}(\mathbf{X}^1) = \max_{\mathbf{X}^1\in\Re_+^N} & \left\{ \mathbb{E}_{\{\mathbf{D}^1,\cdots,\mathbf{D}^T\}} \sum_{s=1}^{N}\sum_{t=1}^{T} \left[\alpha_{ss}\min(x_s^t,d_s^t) - g_s(\tilde{d}_s^t + d_s^t)\right] - \mathbf{C}\cdot\mathbf{X}^1 \right\} \\
\text{s.t.} \quad & x_s^{t+1} = (x_s^t - d_s^t)^+,\ \tilde{d}_s^{t+1} = \tilde{d}_s^t + (d_s^t - x_s^t)^+, \\
& x_s^1 = (\mathbf{X}^1)_s,\ d_s^t = (\mathbf{D}^t)_s, \qquad s=1,\cdots,N,\ t=1,\cdots,T.
\end{aligned} \tag{17}$$

Note that although the two benchmark models (CB and NV) are similar to the static and independent newsvendor models used in SZ, due to the backlogging assumption, the firm has to allocate capacity in each period in our model, rather than accumulate the demand for the entire selling season and then allocate the capacity as in SZ. In Section 8, we use Monte Carlo simulation to generate demand sample paths and use the sample average approximation method to solve the two benchmark models.

8. Numerical Studies

In this section, we conduct numerical studies to derive insights into the capacity management problem. First, we test the performance of the RCEC heuristic proposed in the previous section. After that, by using the heuristic and benchmark models, we investigate the importance of the allocation mechanism and the capacity sizing decision. For simplicity, we focus on integral demands.

8.1. Performance of RCEC

Due to the complexity of the problem, we use extensive numerical experiments to test the performance of the heuristics. These experiments are conducted using MATLAB R2013a on an Intel Core i7-2600 desktop with 12Gb RAM. We focus on the RCEC heuristic because it will be used later for further numerical investigation.

The first set of experiments has $N = 4$ and $T = 3$. For such a problem size, we can use backward induction to evaluate the firm's optimal profit $\Pi(\mathbf{X})$ given in (1). Later we will also discuss the performance of the RCEC for larger problem sizes where it is difficult to evaluate $\Pi(\mathbf{X})$ directly. Given an initial capacity $\mathbf{X} \in \Re_+^N$, define the performance measure

$$\Delta_{\text{opt}} = \left| \frac{\Pi_{\text{RCEC}}(\mathbf{X}) - \Pi(\mathbf{X})}{\Pi(\mathbf{X})} \right| * 100\%, \tag{18}$$

i.e., the percentage of profit loss by using $\Pi_{\text{RCEC}}(\mathbf{X})$ rather than $\Pi(\mathbf{X})$.

To calculate $\Pi(\mathbf{X}) = \Theta^1(\mathbf{X}, \mathbf{0})$, we use the Monte Carlo method and consider a comprehensive range of scenarios, which capture different fluctuation patterns of demand means along the selling horizon (i.e., variation of $\mathbb{E}[\mathbf{D}^t]$ from $t = 1$ to T), different correlations between classes of demands in each period (i.e., $\text{Corr}(d_i^t, d_j^t)$ for all $1 \le i \le j \le N$), different demand distributions (i.e., Normal distribution with integer rounding and Poisson distribution), and various economic parameters (i.e., revenue $(r_1, \cdots, r_N)$, goodwill cost $(g_1, \cdots, g_N)$, usage cost $(u_1, \cdots, u_N)$, and capacity cost $(c_1, \cdots, c_N)$). Furthermore, to ensure the robustness of the results, we also test a number of different initial capacities $\mathbf{X}$ used in (18), which consist of both realistic and extreme scenarios. In total there are 4212 experiments in this numerical study. A full description of the setup of the numerical study is lengthy and thus given in the appendix.

The statistics for the Δ_{opt} value are reported in Table 1. It can be seen that the RCEC performs

Mean	Std.	Median	90%-percentile	95%-percentile	99%-percentile	Max.
0.40	1.13	0.15	0.77	1.27	5.58	17.99

Table 1 The percentage profit loss (Δ_{opt}) of RCEC relative to the optimal solution.

very well in this numerical study. Among all the experiments tested, the 90^{th} percentile of the profit loss is 0.77%, and the average is 0.40%.

Next we test the performance of the RCEC in larger problems. Specifically, we consider problems with $N = 5$ products and up to $T = 60$ periods. Given such sizes, it is extremely time-consuming to evaluate the optimal revenue function $\Pi(\mathbf{X})$. Instead, we use $\Pi_{\text{CB}}(\mathbf{X})$ from the crystal ball (CB) model defined in (16) as the benchmark for comparison. Recall that $\Pi_{\text{CB}}(\mathbf{X})$ is an upper bound of the optimal revenue $\Pi(\mathbf{X})$ for any $\mathbf{X}$, and the following relationship holds: $\Pi_{\text{CB}}(\mathbf{X}) \geq \Pi(\mathbf{X}) \geq \Pi_{\text{RCEC}}(\mathbf{X})$. Define

$$\Delta_{\text{CB}} = \left| \frac{\Pi_{\text{RCEC}}(\mathbf{X}) - \Pi_{\text{CB}}(\mathbf{X})}{\Pi_{\text{CB}}(\mathbf{X})} \right| * 100\%.$$

Then Δ_{CB} is an upper bound of Δ_{opt}, the percentage profit loss of the RCEC (i.e., $\Pi_{\text{RCEC}}(\mathbf{X})$) relative to the optimal revenue (i.e., $\Pi(\mathbf{X})$).

Similar experiment design has been used as Table 1 except that now we consider 5 products with several different T values. This allows us to examine up to 4 levels of upgrading. Also by varying T we can study the impact of the number of periods (or the frequency of upgrading decisions) on the problem. Specifically, T takes values from a set $\{3, 15, 30, 60\}$. For each T, there are 13260 experiments in total in this numerical study. To save space, we provide a detailed description in the appendix.

We summarize the statistics of Δ_{CB} for different T's in Table 2. The table shows that the value of Δ_{CB} is increasing in the number of periods, T. The RCEC ignores the randomness of the demand in future periods (recall that the mean demand is used). Thus, compared to $\Pi_{\text{CB}}(\mathbf{X})$, more demand information is lost as T increases. Table 2 also indicates that the value of Δ_{CB} is small in general: Even for $T = 60$, Δ_{CB} is 8.21% at the 90^{th} percentile, and the average is about 3.52%. This observation has two implications. First, since Δ_{CB} is the upper bound of Δ_{opt}, we know that Δ_{opt} is generally small in the tested examples. This means that for the 5-product numerical experiments, the RCEC also performs well. Second, the observation implies that the difference between $\Pi_{\text{CB}}(\mathbf{X})$ and $\Pi(\mathbf{X})$ is small. In other words, the value of advance demand information is not significant for many cases. Such a result is in line with some of the findings reported in the literature. For instance, SZ finds from numerical study that when the optimal upgrading policy is used, the firm's expected revenue is consistently within 1% of the revenue in a static model (i.e., the crystal ball model). Similarly, Acimovic and Graves (2015) find in a dynamic order fulfillment setting that the crystal ball model improves the performance of the proposed heuristic by 2%, i.e., the performance difference between the crystal ball model and the true optimum is smaller than 2%.

T	Mean	Std.	Median	90%-percentile	Max.
3	0.14	0.38	0.00	0.34	6.74
15	1.52	2.51	0.23	4.83	12.06
30	2.37	3.36	0.42	5.37	23.37
60	3.52	3.82	0.59	8.21	39.36

Table 2 The percentage profit loss (Δ_{CB}) of RCEC relative to the CB solution.

We now compare the performances of the RCEC and the CEC. Define the ratio

$$\eta = \frac{\Pi_{\text{RCEC}}(\mathbf{X})}{\Pi_{\text{CEC}}(\mathbf{X})}$$

to measure the relative performances of the two heuristics. So a ratio higher (lower) than 1 implies that the RCEC outperforms (underperforms) the CEC. We calculate the ratio for the problem instances used in the numerical study underlying Table 2 (i.e., $N = 5$ and $T = \{3, 15, 30, 60\}$). The statistics of the ratio values are summarized in Table 3 since the results are consistent across different T's. Meanwhile, as we mentioned earlier, we also compare the actual computation times of the CEC and the RCEC heuristics in these instances, i.e., the total computation times for deriving the capacity allocations and obtaining the values of Π_{CEC} and Π_{RCEC} for any given $\mathbf{X}$. Specifically, we use MOSEK toolbox for MATLAB version 7 to solve the linear program in (13) in the CEC heuristic, and we apply the binary search to solve the protection levels in (8) while replacing Θ^{t+1} by $\bar{\Theta}^{t+1}_{\text{RCEC}}$ in (14). Similarly, we define

$$\eta_{\text{time}} = \frac{\text{Time for solving } \Pi_{\text{RCEC}}(\mathbf{X})}{\text{Time for solving } \Pi_{\text{CEC}}(\mathbf{X})},$$

whose statistics are also reported in Table 3.

We observe that the CEC may outperform the RCEC in some instances (e.g., the ratio can be as low as 2%)[6]; however, for the majority of the examples, the RCEC performs better than the CEC . More importantly, the reduction of computation time from CEC to RCEC is substantial: all else being equal, the average time for solving a test instance using the RCEC is only 9% of that using the CEC.

	Mean	Std.	Min.	25%-percentile	Median	Max.
η	1.00	0.04	0.02	1.00	1.00	4.81
η_{time}	0.09	0.05	0.00	0.05	0.08	1.13

Table 3 Comparison of RCEC and CEC.

Why does the RCEC exhibit a better overall performance? We offer the following plausible explanation. In both the CEC and RCEC heuristics, we replace the future random demands by their means in each period. Such an approximation clearly will change our original problem and result in suboptimal solutions. In the RCEC, the optimal protection level is determined by comparing two values: The first is the upgrading value from using the product in the current period; the second is the expected marginal value of the product if it is saved to the next period. For illustration, consider the upgrading of demand j using capacity i in period t. The latter value is defined as $\bar{\Theta}^{t+1}_{\text{RCEC}}(\mathbf{X}^{t+1} + \mathbf{e}_i, \tilde{\mathbf{D}}^{t+1} + \mathbf{e}_j | \mu^{t+1}, \cdots, \mu^T) - \bar{\Theta}^{t+1}_{\text{RCEC}}(\mathbf{X}^{t+1}, \tilde{\mathbf{D}}^{t+1} | \mu^{t+1}, \cdots, \mu^T)$, where $\mathbf{e}_s$ $(s = i, j)$ is the unit vector with 1 in position s. The mean demand approximation may introduce biases into the two revenue functions. However, since the expected marginal revenue is defined as the difference between the two revenue functions, these biases may be canceled out to some degree. In other words, the inaccuracies introduced by certainty equivalence control might be reduced in the RCEC heuristic. Note that such a cancellation effect does not exist in the traditional CEC heuristic. Therefore, the RCEC generally outperforms the CEC. In addition, the RCEC is more attractive than the CEC in terms of computational time in our numerical study.

One may also use the deflected linear decision rule (DLDR) method proposed in Chen et al. (2008) to approximate Θ^t in the PSR algorithm. Let Θ^t_{DLDR} be the revenue collected by using $\mathbf{Y}^t_{\text{DLDR}}$'s in the remaining sales horizon, and denote $\Pi_{\text{DLDR}}(\mathbf{X}) = \Theta^1_{\text{DLDR}}(\mathbf{X}, \mathbf{0})$ as the expected revenue under the DLDR heuristic. We evaluate $\Pi_{\text{DLDR}}(\mathbf{X})$ in the numerical study described above and find that $\Pi_{\text{DLDR}}(\mathbf{X})$ and $\Pi_{\text{RCEC}}(\mathbf{X})$ are almost identical in all the problem instances.

Based on the results in Tables 1 and 2, we conclude that the RCEC performs very well in a wide range of problem situations. In addition, the RCEC greatly reduces the computational complexity of the original problem. Therefore, in the rest of the paper, we will use the RCEC to solve the dynamic capacity management problem. Although the RCEC only provides an approximation of the optimal policy, we expect the insights to carry over to the optimal policy because the performance of the RCEC is close to optimal in most of our numerical studies.

8.2. Value of Optimal Upgrading

Given the efficiency and effectiveness of the RCEC heuristic, we are ready to derive more insights into the problem using numerical studies. There are a couple of natural questions we would like to address. First, what is the value of using multi-step upgrading? Second, what is the value of using the optimal capacity? Both questions are important from a practical standpoint because managers need to know how complex an upgrading structure should be used and how to determine the initial capacity. This subsection focuses on the first question, and the second will be addressed in the next subsection.

Let $\Pi^k_{\text{RCEC}}(\mathbf{X})$ be the revenue function given initial capacity $\mathbf{X}$ and k-level upgrading (i.e., product i can be used to satisfy class j demand only if $i \le j \le i+k$). Note that when $k=0$, no upgrading is allowed, and $\Pi^0_{\text{RCEC}}(\mathbf{X}) = \Pi_{\text{NV}}(\mathbf{X})$, where $\Pi_{\text{NV}}(\mathbf{X})$ is the optimal revenue in the newsvendor model in (17). Define

$$\Delta^k_{\text{RCEC}} = \frac{\Pi^k_{\text{RCEC}}(\mathbf{X}) - \Pi^{k-1}_{\text{RCEC}}(\mathbf{X})}{\Pi^{k-1}_{\text{RCEC}}(\mathbf{X})} * 100\%, \qquad k = 1, 2, 3, 4,$$

which measures the percentage profit gain from one additional level of upgrading under the RCEC [7].

We evaluate the values of Δ^k_{RCEC} using the same parameters as those for Table 2 except the initial capacities. Intuitively, upgrade is more valuable when the capacity is unbalanced, i.e., there is excess capacity for some products while there is shortage for the others. Such unbalance may occur even if the initial capacities are optimally set, because demand may fluctuate due to seasonality and trend while capacities are determined for the long term. Thus, when choosing the initial capacity we use the following procedure. Start with the optimal capacity under the RCEC, denoted as $\mathbf{X}_{\text{RCEC}}$ (the computation of $\mathbf{X}_{\text{RCEC}}$ will be discussed in the next subsection); then set the capacity for one product (say, product j) to 0 while adding capacity $(\mathbf{X}_{\text{RCEC}})_j$ to a higher-quality product; finally, scale the entire capacity vector by different multipliers. Mathematically, for $1 \le i < j \le 5$, we consider all initial capacity $\mathbf{X}$, whose components are given by

$$(\mathbf{X})_i = \lambda((\mathbf{X}_{\text{RCEC}})_i + (\mathbf{X}_{\text{RCEC}})_j),\ (\mathbf{X})_j = 0,\ (\mathbf{X})_s = \lambda(\mathbf{X}_{\text{RCEC}})_s,\ \forall\ s \in \{1,2,3,4,5\} \setminus \{i,j\},$$

where $\lambda \in \{0.9,\ 1,\ 1.1\}$. There are 10 combinations of the initial capacities for each λ and parameter set; one example is $\mathbf{X} = ((\mathbf{X}_{\text{RCEC}})_1 + (\mathbf{X}_{\text{RCEC}})_2, 0, (\mathbf{X}_{\text{RCEC}})_3, (\mathbf{X}_{\text{RCEC}})_4, (\mathbf{X}_{\text{RCEC}})_5)$. A full list of the initial capacities are given in the appendix. We believe such a design captures the possible capacity scenarios that may happen over time as the firm allocates products to satisfy realized demand, especially those with unbalanced capacities. Moreover, the mean of total demand over the selling horizon remains the same for different $T \in \{3, 15, 30, 60\}$, which implies that less demand information is available within each period for larger T. The numerical results for different T values are given in Table 4.

T	Upgrading Level k	Mean	Median	90%-percentile
3	1	29.75	20.64	51.63
	2	5.71	2.09	15.21
	3	1.45	0.11	4.99
	4	0.25	0.01	0.28
15	1	25.96	20.25	47.44
	2	4.86	2.99	12.88
	3	0.79	0.04	2.70
	4	0.07	0	0.09
30	1	20.38	19.88	45.63
	2	3.89	1.63	11.70
	3	0.67	0.02	2.02
	4	0.05	0	0.07
60	1	16.77	12.57	32.70
	2	2.88	1.07	10.23
	3	0.40	0	2.1
	4	0.03	0	0.04

Table 4 The value of using multi-step upgrading (Δ^k_{RCEC}).

There are several observations from Table 4. First, Table 4 suggests that the firm's profit increases in the number of upgrading levels k and the marginal value decreases in k. In particular, we can see that most of the gains are achieved with a single level of upgrading, though more can be achieved with multi-step upgrading. In some cases, the value from two or more levels of upgrading can be quite high. For instance, with $T = 3$, the benefit of moving from one-step upgrading to two-step upgrading can be as high as 15.21% at the 90^{th} percentile (i.e., for at least 10% of the scenarios, the value is more than 15.21%). This result implies that single-step upgrading may not capture the full benefit of upgrading, and multi-step upgrading is needed in certain cases.

Second, Table 4 indicates that the value of multi-step upgrading decreases in T. That is, using more upgrading levels will be less beneficial for the firm when there are more time periods in the selling horizon. Close scrutiny reveals that there is a key contributing factor to this interesting observation. A large T value means there are more time periods, which allows "chain allocation" to be more likely to happen. To see this, first consider $T = 1$. In this case, under single-step upgrading, product 1 cannot be used to satisfy demand 3. However, with $T = 2$, it is possible that product 2 is used to satisfy demand 3 in period 1; then, in the second period, product 1 is used to satisfy demand 2. These two allocations essentially mean that product 1 is used to satisfy demand 3. Evidence of such a chain allocation has been observed in our numerical study. For instance, we examine the upgrading patterns for scenarios with $T = 3$ and $T = 15$ under the full upgrading structure. With $T = 3$, the average numbers of upgrades from product 1 to demand 2 (i.e., $\sum_{t=1}^{T} y_{12}^t$), product 2 to demand 3 (i.e., $\sum_{t=1}^{T} y_{23}^t$), and product 1 to demand 3 (i.e., $\sum_{t=1}^{T} y_{13}^t$) are 10.6, 13.3, and 5.1, respectively, while the corresponding numbers with $T = 15$ are 12.9, 15.0, and 4.0. Note that there are more upgrades from product 1 to demand 3 with $T = 3$ than with $T = 15$ even though there are fewer chances for upgrading with $T = 3$. The chain allocation is analogous to multi-step upgrading; the only difference is that it can be better executed when there are more time periods. Therefore, multi-step upgrading is less valuable since it can be implemented even under single-step upgrading, but in a different way.

Finally, the numerical experiments suggest that multi-step upgrading is most valuable when the initial capacity is unbalanced. For example, for $T = 3$, when the optimal initial capacity $\mathbf{X}_{\text{RCEC}}$ is used, the incremental value of moving from 2-level to 3-level upgrading is 0.04% on average; however, for initial capacity $\mathbf{X} = ((\mathbf{X}_{\text{RCEC}})_1, (\mathbf{X}_{\text{RCEC}})_2 + (\mathbf{X}_{\text{RCEC}})_5, (\mathbf{X}_{\text{RCEC}})_3, (\mathbf{X}_{\text{RCEC}})_4, 0)$, the counterpart value is 5.10%. This indicates that the multi-step upgrading is quite important because unbalanced capacity may arise over time, even if the problem starts with the optimal initial capacity.

What is the benefit of using more upgrading levels if the optimal initial capacities are used? To answer this question, let $\mathbf{X}_{\text{RCEC}}(k)$ $(k = 0, 1, \cdots, 4)$ be the optimal initial capacities obtained from the RCEC heuristic with k-level upgrading, and redefine

$$\Delta_{\text{RCEC}}^k = \frac{\Pi_{\text{RCEC}}^k(\mathbf{X}_{\text{RCEC}}(k)) - \Pi_{\text{RCEC}}^{k-1}(\mathbf{X}_{\text{RCEC}}(k-1))}{\Pi_{\text{RCEC}}^{k-1}(\mathbf{X}_{\text{RCEC}}(k-1))} * 100\%, \qquad k = 1, 2, 3, 4$$

which is the percentage profit gain from one additional level of upgrading under the RCEC if the corresponding optimal initial capacities are used. Using the same set of parameters as in Table 4, we obtain the numerical results given in Table 5.

Upgrading Level k	Mean	Median	90%-percentile
1	1.34	0.87	3.61
2	0.40	0.21	1.14
3	0.14	0.05	0.42
4	0.05	0.01	0.17

Table 5 The value of using multi-step upgrading (Δ^k_{RCEC}) under optimal initial capacity.

As one may expect, the values of using multi-step upgrading are much smaller in Table 5 because the initial capacities have been accordingly adjusted, and this lowers the benefit of using more levels of upgrading. However, the value of multi-step upgrading should not be overlooked either: the additional profit by moving from one-step to two-step upgrading captures about 30% of that from no upgrading to one-step upgrading.[8] Our numerical analysis also indicates that the more variable the demand, the higher the value of upgrading. We omit the details because similar observations have already been made in Bassok et al. (1999) and SZ.

8.3. Capacity Sizing Decision vs. Allocation Mechanism

The profit of the upgrading problem hinges upon both the initial capacity and the allocation mechanism. This raises an interesting question: Which decision is more important, capacity sizing or allocation mechanism? This is a practical question because the firm may wish to focus limited resources on improving the decision that has a bigger impact on profit. To shed some light on this question, we measure the importance of each decision using the profit loss when a suboptimal decision is applied rather than the optimal one. Next, we describe the suboptimal decisions that will be used.

In our problem, it is time-consuming to derive the optimal initial capacity even if we can efficiently solve the optimal allocation decision by the RCEC heuristic. So we consider two simple alternatives. The first alternative is to use the optimal capacity $\mathbf{X}_{\text{CB}}$ in the crystal ball model. The crystal ball model is called static model in SZ, who find that $\mathbf{X}_{\text{CB}}$ yields nearly optimal revenue for the firm in their single-step upgrading model. To check whether the result carries over to our general upgrading model, define

$$\Delta_{\mathbf{X}_{\text{CB}}} = \left| \frac{\Pi_{\text{RCEC}}(\mathbf{X}_{\text{CB}}) - \Pi_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}})}{\Pi_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}})} \right| * 100\%$$

to measure the performance of the crystal ball capacity $\mathbf{X}_{\text{CB}}$. With the same parameters used for Tables 2, 3, and 4, we evaluate $\Delta_{\mathbf{X}_{\text{CB}}}$ for 624 examples and summarize the results in Table 6 (the first row). It can be seen that $\Delta_{\mathbf{X}_{\mathrm{CB}}}$ is generally negligible in the numerical study: The average revenue difference is 0.02% and the maximum is 1.10%.

Note that since the true optimal capacity is unknown, we use $\mathbf{X}_{\mathrm{RCEC}}$, the optimal initial capacity under RCEC, as the benchmark for the comparison. It is worth mentioning that despite its excellent performance, the firm's total revenue function $\Pi_{\mathrm{RCEC}}(\mathbf{X})$ is not necessarily concave in $\mathbf{X}$ under the RCEC heuristic. A non-concave example is provided in the appendix. Thus if needed, we resort to the following approach to determine $\mathbf{X}_{\mathrm{RCEC}}$. We start with the optimal capacity from the crystal ball model and apply the grid search to find the optimal initial capacity: For each search step, we move from $\mathbf{X} = (x_1, x_2, \cdots, x_N)^\intercal$ to the point in set $\{\mathbf{X}' = (x_1', x_2', \cdots, x_N')^\intercal : \max_{i\in\{1,2,\dots,N\}} |x_i' - x_i| = 1\}$, which gives the largest function value of Π_{RCEC}. That is, we conduct a local search along the grid; such an approach has also been used in SZ.

	Mean	Std.	Median	90%-percentile	Max.
$\Delta_{\mathbf{X}_{\mathrm{CB}}}$	0.02	$5.32 * 10^{-2}$	0	0.04	1.10
$\Delta_{\mathbf{X}_{\mathrm{NV}}}$	0.30	$2.89 * 10^{-1}$	0.26	0.68	1.62
Δ_{greedy}	5.60	5.68	8.03	12.81	13.75

Table 6 Capacity decision vs. allocation mechanism.

An even simpler alternative is to use the newsvendor capacity $\mathbf{X}_{\mathrm{NV}}$, i.e., the optimal capacity under no upgrading. Similarly, in the same numerical study, we define

$$\Delta_{\mathbf{X}_{\mathrm{NV}}} = \left| \frac{\Pi_{\mathrm{RCEC}}(\mathbf{X}_{\mathrm{NV}}) - \Pi_{\mathrm{RCEC}}(\mathbf{X}_{\mathrm{RCEC}})}{\Pi_{\mathrm{RCEC}}(\mathbf{X}_{\mathrm{RCEC}})} \right| * 100\%$$

and present the statistics of $\Delta_{\mathbf{X}_{\mathrm{NV}}}$ in Table 6 (the second row). We can see that $\Delta_{\mathbf{X}_{\mathrm{NV}}}$ is greater than $\Delta_{\mathbf{X}_{\mathrm{CB}}}$ in general, but it offers reasonably good performance as well. The average and maximum revenue differences are 0.30% and 1.62%, respectively. In particular, the number at the 90^{th} percentile is 0.68%, which means that the newsvendor capacity performs quite well for the majority of the scenarios. From the above observations, one can see that these simple alternatives to the optimal capacity perform reasonably well. Therefore, as long as the optimal upgrading policy is used, the value of using the optimal capacity seems to be very small in our problem setting.

Next, we consider the impact of using a suboptimal allocation policy. We first use greedy upgrading as the suboptimal policy. Under such a policy, unmet demands are upgraded in the order characterized by the PSR policy; however, there is no rationing and surplus capacities are used for upgrading as much as possible. It serves as a reasonable suboptimal policy because it is intuitive and straightforward to implement in practice. Furthermore, the RCEC heuristic incorporates greedy upgrading to simplify its computation. Specifically, let $\Pi_{\text{greedy}}(\mathbf{X})$ be the expected profit using greedy upgrading given initial capacity $\mathbf{X}$. We define

$$\Delta_{\text{greedy}} = \left| \frac{\Pi_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}}) - \Pi_{\text{greedy}}(\mathbf{X}_{\text{RCEC}})}{\Pi_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}})} \right| * 100\%$$

as the profit loss due to greedy upgrading. The same parameters for $\Delta_{\mathbf{X}_{\text{CB}}}$ and $\Delta_{\mathbf{X}_{\text{NV}}}$ have been used, and the statistics of Δ_{greedy} are presented in Table 6 (the third row). The average profit loss due to greedy upgrading is 5.598%, which is much larger than those for $\Delta_{\mathbf{X}_{\text{CB}}}$ and $\Delta_{\mathbf{X}_{\text{NV}}}$. In addition to greedy upgrading, we also test suboptimal allocation policies that involve only k-step ($k = 0, \cdots, N-2$) upgrading. The magnitudes of profit losses are still generally much larger than those for $\Delta_{\mathbf{X}_{\text{CB}}}$ and $\Delta_{\mathbf{X}_{\text{NV}}}$. To save space, the detailed results are presented in the appendix.

The above numerical results based on the RCEC heuristic suggest that the benefit of choosing an effective allocation mechanism outweighs that of choosing an accurate initial capacity. Based on these observations, in practice, the firm may decide the initial capacity by using simple approximations (e.g., either the NV or CB model) and focus on optimally allocating the capacity during the sales horizon.

9. Conclusion

This paper studies a firm's capacity investment and allocation problem in a dynamic setting with stochastic demand. There are multiple demand classes, which can be satisfied by multiple classes of capacities. Demand arrives in discrete time periods, and the firm needs to make capacity allocation decisions in each period before observing future demand. A general upgrading structure is considered, which is broad enough to cover a wide range of practical upgrading situations. One may also view this as an inventory management problem with one-way dynamic substitution.

We first show that for any given initial capacity, a Parallel and Sequential Rationing (PSR) policy is optimal for the firm. Under the PSR policy, the firm can make upgrading decisions in each period sequentially rather than jointly, which greatly reduces the complexity of the capacity allocation problem. Despite the well-structured PSR policy, the dynamic allocation problem is still subject to the curse of dimensionality. Thus we propose a Refined Certainty Equivalence Control (RCEC) heuristic that improves over the traditional CEC methodology by exploiting the property of the PSR policy. Through extensive numerical experiments, we find that the RCEC heuristic is highly efficient and yields nearly optimal revenue for the firm. With the help of the RCEC heuristic, we conduct numerical studies to derive managerial insights about the dynamic capacity management problem. Our numerical studies indicate that the multi-step upgrading could be significantly valuable, especially when the capacities are not balanced (either due to suboptimal initial investment or unexpected demand realizations over time). We find that using simple approximations (e.g., the newsvendor model without substitution and the crystal ball model with perfect demand forecast) for the initial capacities leads to negligible profit loss, while the negative impact of using a suboptimal allocation (e.g., greedy upgrading) could be quite significant. In this sense, the allocation mechanism plays a more important role in our problem than the capacity sizing decision.

There are several interesting directions for future research. First, it is worthwhile exploring models with general non-stationary model parameters. The PSR policy remains optimal as long as the profit margin is monotonically decreasing over time. However, with general non-stationary model parameters, the optimal policy is still unknown. It is possible that the parallel allocation is not used to the maximum extent, and the upgrading decision may depend on the demands and capacities below the class being upgraded if a higher profit margin is expected in some later period. Second, it is a challenge to analyze models with lost sales. The backorder assumption used in this paper is critical for the optimal PSR allocation policy. It is not clear how the optimal policy looks under the lost-sales assumption. Third, it would be interesting to take pricing decisions into account, i.e., the firm may adjust prices over time depending on the evolution of demand and remaining capacity levels. Finally, recently there has been a growing interest in studying opportunistic consumer behavior in operations problems. In our upgrading setting, a consumer may intentionally choose the product that is sold out, hoping to receive a free upgrade later. It would be interesting to investigate how such a behavior may affect the firms' operational strategies (e.g., upgrading policies).

Endnotes

1. The backorder assumption is used mainly for tractability. Notice that an unmet demand could be upgraded in any subsequent period, so it is reasonable to assume that the customers are willing to wait for potential upgrades, i.e., unsatisfied demands can be backlogged.
2. This counter-intuitive example remains valid for any goodwill cost g_2 if the length T satisfies $\alpha_{11} - (T-2)g_2 > \alpha_{12} > \alpha_{11} - (T-1)g_2$ and $\mathbf{D}^2 = \cdots = \mathbf{D}^{T-1} = (0,0)$ and $\mathbf{D}^T = (1,0)$.
3. To deal with the dimensionality issue, SZ propose a series of bounds to approximate the optimal protection levels. For instance, when computing the protection level for product i, one may consider only the capacity for $i-1$, while assuming the products above $i-1$ to be either ∞ (this gives a lower bound of the protection level) or 0 (this gives an upper bound). It has been found that under the single-step upgrading assumption, these bounds are very tight and yield nearly optimal revenue for the firm. However, such bounds do not work well in our model, where general upgrading is allowed.

4. We have tested the heuristic without the greedy upgrading and found that the performance is almost identical. That is, the use of greedy upgrading in this heuristic can significantly reduce the computational complexity but has a negligible impact on the revenue performance.

5. Since future demands are known, there exists a period s $(t+1 \le s \le T)$ in which capacity i will be depleted. From the expression in (15), a marginal change of p only affects the greedy upgrading in period s because both capacity i and backorder demand j change simultaneously in p. In particular, capacity i is used to sequentially satisfy demands from class i to j in period s. As p increases, the additional units of capacity i will be used to satisfy demands from lower classes that have smaller profit margins. Thus, the partial derivative is a decreasing step function of p.

6. The differences between RCEC and CEC seem to be quite large in extreme cases. The reason is that the optimal expected profits in these cases are very close to zero and thus a small perturbation in Π_{RCEC} and Π_{CEC} may drastically change the ratio η.

7. Note that Δ^k_{RCEC} may be negative since Π^k_{RCEC} may be different from the optimal value. However, we only observed 15 cases out of 74880 (i.e., 0.02%) in our numerical study.

8. In our numerical study, upgrade constitutes 1.92% of the total satisfied demands on average when the optimal initial capacity is used, and 26.59% when the suboptimal initial capacities are adopted. If the firm uses frequent upgrading to satisfy customer demand (e.g., the initial capacity is poorly decided), customers may learn about the upgrading pattern and become opportunistic. That is, a class i customer may intentionally ask for product j $(i<j)$, hoping that she will be upgraded when product j is out of stock. Incorporating such a behavior is out of the scope of this paper and therefore left for future research.

Acknowledgments

We thank the department editor Chung-Piaw Teo, the anonymous associate editor and reviewers for insightful comments and suggestions that have significantly improved the paper. Specifically, Section 6 was suggested by the review team and Section 8.3 was suggested by Chung-Piaw Teo. This research of Xin Chen is partly supported by NSF grants CMMI-1030923, CMMI-1363261 and CMMI-1538451, and China NSFC Grants 71228203 and 71520107001.

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Appendices to "Dynamic Capacity Management with General Upgrading"

The appendices contain the supplemental materials to the main paper. Appendices A and B are devoted to the proofs of the main results in this paper. Appendix C provides the analysis of the multi-horizon model with replenishment in Section 6. A detailed description of the numerical studies can be found in Appendix D. Finally, Appendix E offers more details of the proofs that have been omitted in Appendix B for concision.

Appendix A: Preliminary

A.1. Notations

The following notations are used in this appendix to simplify our exposition. Consider a vector $\mathbf{Z}=(z_1,\cdots,z_N)\in\Re^N$, for $1\le i<j\le N$, we define

$$(\mathbf{Z})_i = z_i$$

$$(\mathbf{Z})_{i,\cdots,j} = (z_i, z_{i+1},\cdots,z_j)$$

$$\mathbf{Z}_{ij} = (z_1,\cdots,z_{i-1},z_i+1,z_{i+1},\cdots,z_{j-1},z_j-1,z_{j+1},\cdots,z_N).$$

Notice that the above notations are still valid for $\mathbf{Z}=(z_r,\cdots,z_k)$ $(1<r\le i\le j\le k<N)$ if we artificially set $\mathbf{Z}=(0,\cdots,0,z_r,\cdots,z_k,0,\cdots,0)\in\Re^N$.

For state vector $\mathbf{N}^t$, recall the effective state $\hat{\mathbf{N}}_r^t$ of classes $(1,\cdots,r)$ defined in Definition 2. If $r=N$, we use $\hat{\mathbf{N}}^t$ instead of $\hat{\mathbf{N}}_N^t$ to simplify our notation.

Moreover, for class i $(1\le i\le N)$ in period t $(1\le t\le T)$, we define

$$\partial_i^-\Theta^t(\mathbf{Z}) = \frac{\partial}{\partial z_i^-}\Theta^t(\mathbf{Z}), \qquad \partial_i^+\Theta^t(\mathbf{Z}) = \frac{\partial}{\partial z_i^+}\Theta^t(\mathbf{Z}).$$

Recall Δ_{ij}^{-+} and Δ_{ij}^{+-} $(1\le i<j\le N)$, we have

$$\Delta_{ij}^{-+}\Theta^t(\mathbf{Z}) = \frac{\partial}{\partial z_i^-}\Theta^t(\mathbf{Z}) - \frac{\partial}{\partial z_j^+}\Theta^t(\mathbf{Z}), \qquad \Delta_{ij}^{+-}\Theta^t(\mathbf{Z}) = \frac{\partial}{\partial z_i^+}\Theta^t(\mathbf{Z}) - \frac{\partial}{\partial z_j^-}\Theta^t(\mathbf{Z}).$$

Using the notations above, the protection level $p_{ij}=p$ in period t if and only if $\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t)\le \alpha_{ij}\le\Delta_{ij}^{-+}\Theta^t(\mathbf{N}^t)$ from (8), where $\mathbf{N}^t=(n_1^t,\cdots,n_{i-1}^t,p,0,\cdots,0,-p,n_{j+1}^t,\cdots,n_N^t)$.

In the essence of Δ_{ij}^{-+} and Δ_{ij}^{+-}, we define the marginal perturbation of class i and j (referred to as MP_{ij} hereafter) as $\Theta^t(\mathbf{Z}+\epsilon(\mathbf{e}_i-\mathbf{e}_j))-\Theta^t(\mathbf{Z})$, where $\epsilon\in\Re$ is a small number and $\mathbf{e}_s$ $(s=i,j)$ is the unit vector with 1 in position s.

A.2. Independence Property

Consider a state vector $\mathbf{N}^t=(n_1^t,\cdots,n_N^t)$ and its effective state $\hat{\mathbf{N}}_{i-1}^t=(\hat{n}_1^t,\cdots,\hat{n}_{i-1}^t,n_i^t,\cdots,n_N^t)$ in period t. In Lemma EC.4 and EC.5, we will show Θ^t has the following independence property if $(n_{i+1}^t,\cdots,n_{j-1}^t)\le 0$ and $n_j^t<0$:

1. In period t $(1 \leq t \leq T-1)$,

$$\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{+-}\Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t), \qquad \Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{-+}\Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t).$$

2. $\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t)$ and $\Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t)$ are independent of the values of $(n_j^t, \cdots, n_N^t)$.

Given the independence property of Θ^{t+1}, the protection levels in period t have a similar property. Specifically, consider two different state vectors $\mathbf{N} = (n_1, \cdots, n_N)^\intercal$ and $\mathbf{N}' = (n_1', \cdots, n_N')^\intercal$ with the same effective state for the first $i-1$ classes. If $(n_{i+1}, \cdots, n_{j-1}) = (n_{i+1}', \cdots, n_{j-1}') \leq 0$ and $n_j = n_j' < 0$, then the protection level p_{ij} under state $\mathbf{N}$ is the same as that under $\mathbf{N}'$. Furthermore, the protection level p_{ij} under state $\mathbf{N}$ is independent of the values of $(n_j, \cdots, n_N)$. Hereafter, when speaking of the independence property, we do not distinguish between Θ^{t+1} and the protection levels in period t, since the proper interpretation is usually clear from the context.

Remark EC.1. Note that the independence property holds under the conditions $(n_{i+1}^t, \cdots, n_{j-1}^t) \leq 0$ and $n_j^t < 0$. However, in the proofs of Lemma EC.4 and EC.5, we only need $n_j^t \leq 0$ to prove the results of $\Delta_{ij}^{+-}\Theta^{t+1}$.

A.3. Foundation Results

Lemma 1 gives the condition of splitting the N-class general upgrading problem into subproblems, which reduces the complexity of the analysis.

Lemma 1 *Consider an N-class general upgrading problem with state $\mathbf{N}^t = (n_1^t, n_2^t, \cdots, n_N^t)^\intercal$ in period t. If $\sum_{s=k}^{i} n_s^t \leq 0$ for all class $k \leq i$, then the problem can be separated into two independent subproblems: an upper part consisting of classes $(1, \cdots, i)$, and a lower part consisting of classes $(i+1, \cdots, N)$.*

Proof. This result holds if none of the optimal policies would upgrade demand j by capacity k when there remains unmet demand i $(k < i < j)$ in the same period. For simplicity, we only prove the latter claim in the integer case. For any demand sample path $(\mathbf{D}^t, \cdots, \mathbf{D}^T)$, let $(\mathbf{Y}^t, \cdots, \mathbf{Y}^T)$ be the optimal decisions. We assume without loss of generality that $y_{i-1,j}^t = (\mathbf{Y}^t)_{i-1,j} \geq 1$ $(i < j)$ while there remains unmet demand i after period t.

We construct decisions $\bar{\mathbf{Y}}^s$ $(s = t, \cdots, T)$ that yield higher profit than the optimal decisions, which will be a contradiction. Let $\bar{\mathbf{Y}}^t$ be the same as $\mathbf{Y}^t$ except that $\bar{y}_{i-1,i}^t = y_{i-1,i}^t + 1$ and $\bar{y}_{i-1,j}^t = y_{i-1,j}^t - 1$. In the remaining periods s $(s = t+1, \cdots, T)$, we apply allocation decision $\bar{\mathbf{Y}}^s = \mathbf{Y}^s$ whenever $\mathbf{Y}^s$ is feasible. If the optimal decisions are feasible in periods $t+1$ to T, the profit increase by using $\bar{\mathbf{Y}}^s$ $(t \leq s \leq T)$ instead of the optimal decisions is $\alpha_{i-1,i} - \alpha_{i-1,j} + (T - t + 1)(g_i - g_j) > 0$, which is a contradiction.

Otherwise, let l $(t+1 \leq l \leq T)$ be the first period that $\mathbf{Y}^l$ is not feasible. From our construction, it is clear that there exists $y_{ki}^l \geq 1$ $(k < i)$ in $\mathbf{Y}^l$ that is not feasible after applying $\bar{\mathbf{Y}}^s$ $(s = t, \cdots, l-1)$.

Let $\bar{\mathbf{Y}}^l$ be the same as $\mathbf{Y}^l$ except that $\bar{y}^l_{ki} = y^l_{ki} - 1$ and $\bar{y}^l_{kj} = y^l_{kj} + 1$. Since the states after applying $\bar{\mathbf{Y}}^s$ $(s = t, \cdots, l)$ are the same as that for $\mathbf{Y}^s$ $(s = t, \cdots, l)$, $\bar{\mathbf{Y}}^s = \mathbf{Y}^s$ $(s = l+1, \cdots, T)$ are feasible in the remaining periods. Thus, the profit increase by using $\bar{\mathbf{Y}}^s$ $(t \le s \le T)$ instead of the optimal ones is $(l - t)(g_i - g_j) > 0$, which contradicts the optimality assumption.

This concludes our proof. □

Lemmas EC.1 and EC.2 illustrate the bounds of the profit differences under different states.

LEMMA EC.1. *Consider a state vector* $\mathbf{N} = (n_1, \cdots, n_N)$ *with* $n_i \ge 0$ *and* $n_j \ge 0$ $(1 \le i < j \le N)$. *Then,*

$$\partial_i^+ \Theta^t(\mathbf{N}) - \partial_j^+ \Theta^t(\mathbf{N}) \ge u_j - u_i \tag{EC.1}$$

and

$$\partial_i^- \Theta^t(\mathbf{N}) - \partial_j^- \Theta^t(\mathbf{N}) \ge u_j - u_i \qquad \text{if } n_i > 0 \text{ and } n_j > 0. \tag{EC.2}$$

Proof. We use the sample path argument to prove (EC.1). For each demand sample path, it is sufficient to prove

$$\Theta^t(\mathbf{N} + \epsilon \mathbf{e}_i) - \Theta^t(\mathbf{N} + \epsilon \mathbf{e}_j) \ge \epsilon (u_j - u_i), \tag{EC.3}$$

where $\epsilon > 0$, $\mathbf{e}_s$ $(s = i, j)$ is the unit vector with 1 in position s. The same argument can be applied to (EC.2).

Given a demand sample path $(\mathbf{D}^t, \cdots, \mathbf{D}^T)$, let $(\mathbf{Y}^t, \cdots, \mathbf{Y}^T)$ be the corresponding optimal solutions in period t to T under initial state $\mathbf{N} + \epsilon \mathbf{e}_j$ in period t. For initial state $\mathbf{N} + \epsilon \mathbf{e}_i$, we sequentially construct solutions $(\bar{\mathbf{Y}}^t, \cdots, \bar{\mathbf{Y}}^T)$ based on $(\mathbf{Y}^t, \cdots, \mathbf{Y}^T)$ from period t to T. Specifically, $\bar{\mathbf{Y}}^l = \mathbf{Y}^l$ in period l $(t \le l \le T)$ if $\mathbf{Y}^l$ is feasible, and we write $\epsilon_l = 0$. Otherwise, if $\mathbf{Y}^l$ is not feasible, from the assumption of the initial states, the total demands which are satisfied by capacity j in $\mathbf{Y}^l$ is greater than the existing capacity j with initial state $\mathbf{N} + \epsilon \mathbf{e}_i$, and we denote the difference as ϵ_l $(0 < \epsilon_l \le \epsilon_1)$. To construct a feasible solution $\bar{\mathbf{Y}}^l$, we use capacity i to satisfy demands which cannot be fulfilled by capacity j. By applying such $(\bar{\mathbf{Y}}^t, \cdots, \bar{\mathbf{Y}}^T)$, the unmet demands in periods t to T are the same for both initial states, and $\sum_{l=t}^T \epsilon_l \le \epsilon$.

Note that $\alpha_{si} - \alpha_{sj} = u_j - u_i < 0$ for any class s $(s \ge j)$, and unmet demand vectors in period t to T are the same for both initial states. Since $(\bar{\mathbf{Y}}^t, \cdots, \bar{\mathbf{Y}}^T)$ are feasible solutions to the general upgrading problem with initial state $\mathbf{N} + \epsilon \mathbf{e}_i$, we have

$$\Theta^t(\mathbf{N} + \epsilon \mathbf{e}_i) - \Theta^t(\mathbf{N} + \epsilon \mathbf{e}_j) \ge (u_j - u_i) \sum_{l=t}^{T} \epsilon_l \ge \epsilon (u_j - u_i),$$

which completes the proof. □

Lemma EC.2. *Consider a state vector* $\mathbf{N} = (n_1, \cdots, n_N)$ *with* $n_i \le 0$ *and* $n_j \le 0$ $(1 \le i < j \le N)$. *Then,*

$$\partial_i^+ \Theta^t(\mathbf{N}) - \partial_j^+ \Theta^t(\mathbf{N}) \ge r_j - r_i \qquad \text{if } n_i < 0 \text{ and } n_j < 0$$

and

$$\partial_i^- \Theta^t(\mathbf{N}) - \partial_j^- \Theta^t(\mathbf{N}) \ge r_j - r_i.$$

Proof. It is similar to the proof of Lemma EC.1. □

Appendix B: Proofs of the Main Results

This section presents the proofs of the main results in the paper. The proofs of some intermediate results are lengthy and therefore presented in Appendix E, including Lemmas EC.6 to EC.10 and Propositions EC.1 to EC.3.

In §B.1, we prove the desired properties in period T. §B.2 considers a general period t by following the similar logic for period T. §B.3 completes the optimality proof. §B.4 proves two properties of the protection levels.

B.1. Final Period T

Lemma EC.3. *The PSR policy solves the general upgrading problem* (2) *in period* T *with all protection levels being* 0.

Proof. Note that $\Theta^{T+1} \equiv 0$ and the solution $\mathbf{Y}^T$ generated by the PSR is a Monge sequence which solves the general upgrading problem in period T (see Bassok et al. 1999). □

We follow the notations in the main paper. Recall the state vector $\mathbf{N}^t = (n_1^t, \cdots, , n_N^t)$ in period t, and $\hat{\mathbf{N}}_{i-1}^t = (\hat{n}_1^t, \cdots, \hat{n}_{i-1}^t, n_i^t, \cdots, n_N^t)$, where $(\hat{n}_1^t, \cdots, \hat{n}_{i-1}^t)$ is the effective state of $(n_1^t, \cdots, n_{i-1}^t)$. Then, Lemma EC.4 shows the independence property of Θ^T.

Lemma EC.4. *Consider an* N*-class general upgrading problem in period* $T-1$ *with state vector* $\mathbf{N}^{T-1}$, *where* $(n_{i+1}^{T-1}, \cdots, n_{j-1}^{T-1}) \le 0$ *and* $n_j^{T-1} < 0$. *Then,*

$$\Delta_{ij}^{+-} \Theta^T(\mathbf{N}^{T-1}) = \Delta_{ij}^{+-} \Theta^T(\hat{\mathbf{N}}_{i-1}^{T-1}), \qquad \Delta_{ij}^{-+} \Theta^T(\mathbf{N}^{T-1}) = \Delta_{ij}^{-+} \Theta^T(\hat{\mathbf{N}}_{i-1}^{T-1}). \tag{EC.4}$$

In addition, they are independent of the values of $(n_j^{T-1}, \cdots, n_N^{T-1})$.

Proof. For any $t = 1, \cdots, T$, given $\mathbf{D}^t = (d_1, \cdots, d_N)$ as realized demand in period t, we have

$$\Delta_{ij}^{+-} \Theta^t(\mathbf{N}^{t-1}) = \Delta_{ij}^{+-} \mathbb{E}\left\{\Theta^t(\mathbf{N}^{t-1}|\mathbf{D}^t)\right\} = \mathbb{E}\left\{\Delta_{ij}^{+-} \Theta^t(\mathbf{N}^{t-1}|\mathbf{D}^t)\right\} \tag{EC.5}$$

and

$$\Delta_{ij}^{-+} \Theta^t(\mathbf{N}^{t-1}) = \Delta_{ij}^{-+} \mathbb{E}\left\{\Theta^t(\mathbf{N}^{t-1}|\mathbf{D}^t)\right\} = \mathbb{E}\left\{\Delta_{ij}^{-+} \Theta^t(\mathbf{N}^{t-1}|\mathbf{D}^t)\right\}. \tag{EC.6}$$

Both the continuity of $\Theta^t(\mathbf{N}^{t-1}|\mathbf{D}^t)$ and the existence of its left and right derivatives (see Rockafellar 1996) assure the last equality in (EC.5-EC.6) (see Zorich 2004, P.409).

We focus on Δ_{ij}^{+-} in (EC.4) since the same method applies to Δ_{ij}^{-+}. For any demand realization $\mathbf{D}^T = (d_1, \cdots, d_N)$ in period T, we next show

$$\Delta_{ij}^{+-}\Theta^T(\mathbf{N}^{T-1}|\mathbf{D}^T) = \Delta_{ij}^{+-}\Theta^T(\hat{\mathbf{N}}_{i-1}^{T-1}|\mathbf{D}^T), \tag{EC.7}$$

and it is independent of the values of $(n_j^{T-1}, \cdots, n_N^{T-1})$.

For any $\mathbf{D}^T$, without loss of generality, we assume classes $(1, \cdots, N)$ can not be separated based on $\mathbf{N}^{T-1} - \mathbf{D}^T$. Otherwise, from Lemma 1, we can consider independent subproblems instead. With this assumption, classes $(1, \cdots, N)$ are also not separable based on $\hat{\mathbf{N}}_{i-1}^{T-1} - \mathbf{D}^T$ by Proposition EC.1 in Appendix E.

To solve the N-class general upgrading problem in period T, we first solve subproblems $(1, \cdots, i-1)$ with initial state $(\mathbf{N}^{T-1})_{1,\cdots,i-1}$ and $(\hat{\mathbf{N}}_{i-1}^{T-1})_{1,\cdots,i-1}$ by the PSR. Then, we use the PSR to solve the subproblem $(1, \cdots, N)$, where the initial states of classes $(1, \cdots, i-1)$ are the states after solving the subproblem $(1, \cdots, i-1)$ by the PSR.

Since the upgrading problem in period T is a transportation problem, given the special cost structure, the optimal allocation decisions in subproblem $(1, \cdots, i-1)$ are independent from classes $(i, \cdots, N)$. Particularly, the optimal decisions within classes $(1, \cdots, i-1)$ remain unchanged with respect to MP_{ij}. Moreover, from Proposition EC.2, the result of applying the PSR to subproblem $(1, \cdots, i-1)$ with initial state $(\mathbf{N}^{T-1})_{1,\cdots,i-1}$ is the same as that with initial state $(\hat{\mathbf{N}}_{i-1}^{T-1})_{1,\cdots,i-1}$. In other words, the initial states in subproblem $(1, \cdots, N)$ are the same for both initial states $(\mathbf{N}^{T-1}, \mathbf{D}^T)$ and $(\hat{\mathbf{N}}_{i-1}^{T-1}, \mathbf{D}^T)$. Thus, (EC.7) is true. In addition, $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}^{T-1}|\mathbf{D}^T)$ is independent of the values of $(n_j^{T-1}, \cdots, n_N^{T-1})$ from Lemma EC.7. This completes the proof. □

B.2. Earlier Periods

Lemma EC.5 proves the independence property of Θ^{t+1} by backward induction.

LEMMA EC.5. *Consider an N-class general upgrading problem in period t with state vector $\mathbf{N}^t$, where $(n_{i+1}^t, \cdots, n_{j-1}^t) \le 0$ and $n_j^t < 0$. If the PSR policy solves the general upgrading problem in period $t+1$ and the independence property holds for Θ^{t+2}, then,*

$$\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{+-}\Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t), \qquad \Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{-+}\Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t). \tag{EC.8}$$

In addition, $\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t)$ and $\Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t)$ are independent of the values of $(n_j^t, \cdots, n_N^t)$.

Proof. As discussed in the proof of Lemma EC.4, Δ_{ij}^{+-} and Δ_{ij}^{-+} in (EC.8) are well-defined. We prove the equality regarding Δ_{ij}^{+-} in (EC.8) and the corresponding independence property for any demand realization $\mathbf{D}^{t+1} = (d_1, \cdots, d_N)$ in period $t+1$. From Lemma 1, we can assume classes $(1, \cdots, N)$ are not separable under $\mathbf{N}^t - \mathbf{D}^{t+1}$, which is also true under $\hat{\mathbf{N}}^t_{i-1} - \mathbf{D}^{t+1}$ by Proposition EC.1.

Splitting the N-class general upgrading problem into subproblems: $(1, \cdots, i-1)$, $(1, \cdots, j)$ and $(1, \cdots, N)$, we start with the subproblem $(1, \cdots, i-1)$.

1. Because the protection levels within classes $(1, \cdots, i-1)$ in period $t+1$ are defined by Θ^{t+2}, which satisfies the independence property by assumption, the allocation decisions within classes $(1, \cdots, i-1)$ in period $t+1$ remain unchanged with respect to MP_{ij}. Let $\mathbf{N}'_{i-1}$ be the outcome of applying the PSR policy to subproblem $(1, \cdots, i-1)$ with states $((\mathbf{N}^t)_{1,\cdots,i-1}, (\mathbf{D}^{t+1})_{1,\cdots,i-1})$. Denote k $(1 \le k \le i-1)$ as the highest class such that $(\mathbf{N}'_{i-1})_{k,\cdots,i-1} \ge 0$ and $(\mathbf{N}'_{i-1})_k > 0$. Since the PSR is optimal in period $t+1$ by assumption, we only need to consider upgrading decisions among classes $(k, \cdots, N)$ in the rest of the subproblems. Similarly, we can define $\hat{\mathbf{N}}'_{i-1}$ and $\hat{k}$ for subproblem $(1, \cdots, i-1)$ with states $((\hat{\mathbf{N}}^t_{i-1})_{1,\cdots,i-1}, (\mathbf{D}^{t+1})_{1,\cdots,i-1})$. From Proposition EC.3, we know that $\hat{k} = k$ and $(\hat{\mathbf{N}}'_{i-1})_{k,\cdots,i-1} = (\mathbf{N}'_{i-1})_{k,\cdots,i-1}$. In other words, after solving subproblem $(1, \cdots, i-1)$, the initial state of classes $(k, \cdots, N)$ are the same for both $\mathbf{N}^t$ and $\hat{\mathbf{N}}^t_{i-1}$. Notice that we assume both k and $\hat{k}$ exist; otherwise, both k and $\hat{k}$ do not exist from Proposition EC.3, which means that considering upgrading decisions in classes $(i, \cdots, N)$ is sufficient, which is a simpler case.
2. From the definition of the protection levels, although there is no upgrade between classes $(1, \cdots, k-1)$ and $(k, \cdots, N)$, the states of classes $(1, \cdots, k-1)$ can still affect the protection levels within classes $(k, \cdots, N)$ in period $t+1$. Fortunately, the effective state of $(\hat{\mathbf{N}}'_{i-1})_{1,\cdots,k-1}$ is the same as that of $(\mathbf{N}'_{i-1})_{1,\cdots,k-1}$ by Proposition EC.2. From the independence property assumption of Θ^{t+2}, the protection levels within classes $(k, \cdots, N)$ are the same for both initial states.

To summarize, for initial states $\mathbf{N}^t$ and $\hat{\mathbf{N}}^t_{i-1}$, the capacities of classes $(k, \cdots, i-1)$ after solving subproblem $(1, \cdots, i-1)$, which can upgrade the demands in classes $(i, \cdots, N)$, are the same. Moreover, the protection levels within classes $(k, \cdots, N)$ are also the same. Therefore, we only analyze the allocation decisions within classes $(k, \cdots, N)$ under initial state $\mathbf{N}^t$, which can again be split into subproblems $(k, \cdots, j)$ and $(k, \cdots, N)$.

Apply the PSR to subproblem $(k, \cdots, j)$ with state $(\mathbf{N}_j, (0, \cdots, 0, (\mathbf{D}^{t+1})_{i,\cdots,j}))$, where $\mathbf{N}_j = ((\mathbf{N}'_{i-1})_{k,\cdots,i-1}, (\mathbf{N}^t)_{i,\cdots,j})$, and let $\mathbf{N}'_j$ be the resulting states of classes $(k, \cdots, j)$ after applying $\mathbf{Y}_j$, which are the optimal allocation decisions within classes $(k, \cdots, j)$. Since $(\mathbf{N}^t)_{i+1,\cdots,j} \le 0$, the protection levels used in subproblem $(k, \cdots, j)$, which determine the upgrades from classes $(k, \cdots, i)$, only depend on $(\mathbf{N}'_{i-1})_{k,\cdots,i-1}$ by the independence property assumption of Θ^{t+2}. We consider two cases based on whether there is unmet demand j in $\mathbf{N}'_j$:

1. $(\mathbf{N}'_j)_j = 0$: Define h $(k \le h \le i)$ as the class which satisfies the last unit of demand j when the PSR solves subproblem $(k, \cdots, j)$. In fact,

$$h = \begin{cases} r, & \text{if } r < i \text{ and } \sum_{s=r+1}^{i-1} (\mathbf{N}'_{i-1})_s \le -\sum_{s=i}^{j} ((\mathbf{N}^t)_s - d_s) < \sum_{s=r}^{i-1} (\mathbf{N}'_{i-1})_s \\ i, & \text{if } \sum_{s=i}^{j} ((\mathbf{N}^t)_s - d_s) > 0. \end{cases}$$

In this case, $\mathbf{N}'_j$ is the same as the result of applying the greedy upgrading to subproblem $(k, \cdots, j)$, i.e., $\mathbf{N}'_j = \hat{\mathbf{N}}_j$, where $\hat{\mathbf{N}}_j$ is the effective state of $\mathbf{N}_j$. Specifically,

$$(\hat{\mathbf{N}}_j)_l = \begin{cases} (\mathbf{N}'_{i-1})_l, & \text{if } k \le l < h \\ \sum_{s=h}^{i-1} (\mathbf{N}'_{i-1})_s + \sum_{s=i}^{j} ((\mathbf{N}^t)_s - d_s), & \text{if } l = h < i \\ \sum_{s=i}^{j} ((\mathbf{N}^t)_s - d_s), & \text{if } l = h = i \\ 0, & \text{otherwise,} \end{cases} \tag{EC.9}$$

for class l $(k \le l \le j)$. Note that class h $(k \le h \le i)$ must exist since classes $(1, \cdots, N)$ are not separable, and h and $\hat{\mathbf{N}}_j$ remain the same with respect to MP_{ij}. Furthermore, from the discussion of $\mathbf{N}'_j$, we can see that $\mathbf{Y}_j$ is the same as optimal allocation decisions given initial state $(\mathbf{N}_j, (0, \cdots, 0, (\mathbf{D}^{t+1})_{i,\cdots,j}))$ in period T where the protection levels are zero. Hence,

$$\begin{aligned} \Theta^{t+1}(\mathbf{N}^t|\mathbf{D}^{t+1}) =& \Theta^T \left((\mathbf{N}^t)_{1,\cdots,i-1} - \mathbf{N}'_{i-1} | (\mathbf{D}^{t+1})_{1,\cdots,i-1} \right) + \Theta^T \left(\mathbf{N}_j | (0, \cdots, 0, (\mathbf{D}^{t+1})_{i,\cdots,j}) \right) \\ &+ \Theta^{t+1} \left(((\mathbf{N}'_{i-1})_{1,\cdots,k-1}, \hat{\mathbf{N}}_j, (\mathbf{N}^t)_{j+1,\cdots,N}) | (0, \cdots, 0, (\mathbf{D}^{t+1})_{j+1,\cdots,N}) \right), \end{aligned} \tag{EC.10}$$

where the first two terms are the corresponding revenues of subproblems $(1, \cdots, i-1)$ and $(k, \cdots, j)$, and the last term is the sum of the current revenue of subproblem $(k, \cdots, j)$ and the expected value in the remaining periods. Thus,

$$\Delta_{ij}^{+-} \Theta^{t+1}(\mathbf{N}^t|\mathbf{D}^{t+1}) = \Delta_{ij}^{+-} \Theta^T \left(\mathbf{N}_j | (0, \cdots, 0, (\mathbf{D}^{t+1})_{i,\cdots,j}) \right), \tag{EC.11}$$

which is clearly independent of $(n_{j+1}^t, \cdots, n_N^t)$. Also, (EC.11) is independent of n_j^t by Lemma EC.7. Note that the first term in (EC.10) has been omitted from (EC.11) since the allocation decisions in subproblem $(1, \cdots, i-1)$ remain unchanged with respect to MP_{ij}. Moreover, the last term in (EC.10) has also been dropped from (EC.11) because its initial states remain the same with respect to MP_{ij}.

Similarly, for initial state $\hat{\mathbf{N}}_{i-1}^t$, we have

$$\Delta_{ij}^{+-} \Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t|\mathbf{D}^{t+1}) = \Delta_{ij}^{+-} \Theta^T \left(\mathbf{N}_j | (0, \cdots, 0, (\mathbf{D}^{t+1})_{i,\cdots,j}) \right)$$

since the allocation decisions in subproblem $(k, \cdots, j)$ are the same for both initial state $\mathbf{N}^t$ and $\hat{\mathbf{N}}_{i-1}^t$. Therefore, we have

$$\Delta_{ij}^{+-} \Theta^{t+1}(\mathbf{N}^t|\mathbf{D}^{t+1}) = \Delta_{ij}^{+-} \Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t|\mathbf{D}^{t+1}),$$

which is independent of the values of $(n_j^t, \cdots, n_N^t)$ by (EC.11);

2. $(\mathbf{N}'_j)_j < 0$: Since the PSR is optimal in period $t+1$, there is no upgrade between classes $(k,\cdots,j)$ and $(j+1,\cdots,N)$. By the definition of the effective state, $\hat{\mathbf{N}}_j$ in (EC.9), which remains unchanged with respect to MP_{ij}, is the effective state of $\mathbf{N}'_j$. Thus, the allocation decisions within classes $(j+1,\cdots,N)$ stay the same with respect to MP_{ij} by the independence property assumption of Θ^{t+2}, and we denote $\mathbf{N}'_{j+}$ as the result of applying the PSR to classes $(j+1,\cdots,N)$. Therefore, we have

$$\begin{aligned}&\Theta^{t+1}(\mathbf{N}^t|\mathbf{D}^{t+1})\\=&\Theta^T\left((\mathbf{N}^t)_{1,\cdots,i-1}-\mathbf{N}'_{i-1}|(\mathbf{D}^{t+1})_{1,\cdots,i-1}\right)+\Theta^T\left(\mathbf{N}_j-\mathbf{N}'_j|(0,\cdots,0,(\mathbf{D}^{t+1})_{k,\cdots,j})\right)\\&+\Theta^T\left((\mathbf{N}^t)_{j+1,\cdots,N}-\mathbf{N}'_{j+}|(0,\cdots,0,(\mathbf{D}^{t+1})_{j+1,\cdots,N})\right)+\Theta^{t+2}\left((\mathbf{N}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right),\end{aligned}\tag{EC.12}$$

where the first three terms are the corresponding revenues of subproblems $(1,\cdots,i-1)$, $(k,\cdots,j)$, and $(j,\cdots,N)$, and the last term is the expected revenue-to-go function. As we discussed earlier, we have

$$\begin{aligned}&\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t|\mathbf{D}^{t+1})\\=&\Delta_{ij}^{+-}\Theta^T\left(\mathbf{N}_j-\mathbf{N}'_j|(0,\cdots,0,(\mathbf{D}^{t+1})_{k,\cdots,j})\right)+\frac{\partial}{\partial n_i^+}\Theta^{t+2}\left((\mathbf{N}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right)\\&-\frac{\partial}{\partial n_j^-}\Theta^{t+2}\left((\mathbf{N}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right),\end{aligned}\tag{EC.13}$$

where the first term is independent of $(n_{j+1}^t,\cdots,n_N^t)$ by construction. Moreover, recall that the protection levels used in subproblem $(k,\cdots,j)$ only depend on $(\mathbf{N}'_{i-1})_{k,\cdots,i-1}$, and demand j is not fully satisfied in this case, thus the allocation decisions $\mathbf{Y}_j$ as well as $\mathbf{N}_j-\mathbf{N}'_j$, which is the capacity used in subproblem $(k,\cdots,j)$, do not depend on n_j^t. Hence, the first term in (EC.13) is also independent of n_j^t. Similarly, for initial state $\hat{\mathbf{N}}_{i-1}^t$, we have

$$\begin{aligned}&\Delta_{ij}^{+-}\Theta^{t+1}(\hat{\mathbf{N}}_{i-1}^t|\mathbf{D}^{t+1})\\=&\Delta_{ij}^{+-}\Theta^T\left(\mathbf{N}_j-\mathbf{N}'_j|(0,\cdots,0,(\mathbf{D}^{t+1})_{k,\cdots,j})\right)+\frac{\partial}{\partial n_i^+}\Theta^{t+2}\left((\hat{\mathbf{N}}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right)\\&-\frac{\partial}{\partial n_j^-}\Theta^{t+2}\left((\hat{\mathbf{N}}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right).\end{aligned}\tag{EC.14}$$

To complete the proof, from (EC.13) and (EC.14), we use the induction assumption of Θ^{t+2} to show

$$\begin{aligned}&\frac{\partial}{\partial n_i^+}\Theta^{t+2}\left((\mathbf{N}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right)-\frac{\partial}{\partial n_j^-}\Theta^{t+2}\left((\mathbf{N}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right)\\=&\frac{\partial}{\partial n_i^+}\Theta^{t+2}\left((\hat{\mathbf{N}}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right)-\frac{\partial}{\partial n_j^-}\Theta^{t+2}\left((\hat{\mathbf{N}}'_{i-1})_{1,\cdots,k-1},\mathbf{N}'_j,\mathbf{N}'_{j+}\right),\end{aligned}\tag{EC.15}$$

which is independent of $(n_j^t,\cdots,n_N^t)$. First of all, since there is no upgrade between classes $(1,\cdots,k-1)$ and $(k,\cdots,N)$ in period $t+1$, and the PSR sequentially satisfies demands in each class, the marginal change of n_i^t only affects the state of a single class in $\mathbf{N}'_j$, which is the same for both initial states $\mathbf{N}^t$ and $\hat{\mathbf{N}}^t_{i-1}$. Denote such a class as r, then $k \le r \le j$. Given $(\mathbf{N}'_{i-1})_{1,\cdots,k-1}$ and $(\hat{\mathbf{N}}'_{i-1})_{1,\cdots,k-1}$ have the same effective state from the previous argument, to apply the induction assumption, we only need to show $(\mathbf{N}'_j)_{r+1,\cdots,j} \le 0$ where $(\mathbf{N}'_j)_j < 0$ by assumption. Suppose to the contrary that $(\mathbf{N}'_j)_l > 0$ for class l $(r < l < j)$. Note that initial states $(\mathbf{N}^t)_{i+1,\cdots,j} \le 0$, thus class $l \le i$. Since the demands in classes $(i,\cdots,j)$ should be satisfied by class l prior to class r by the PSR, given $(\mathbf{N}'_j)_l > 0$, there is no upgrade between classes $(k,\cdots,l-1)$ and $(l,\cdots,j)$, i.e., the marginal change of n_i^t should not affect the state of class r, which is a contradiction. Hence, by applying the induction assumption to (EC.15), we have

$$\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t|\mathbf{D}^{t+1}) = \Delta_{ij}^{+-}\Theta^{t+1}(\hat{\mathbf{N}}^t_{i-1}|\mathbf{D}^{t+1}),$$

which is independent of the values of $(n_j^t,\cdots,n_N^t)$. This concludes the proof.

□

B.3. Optimality

Proposition 2 *Consider an N-class general upgrading problem in period t $(1 \le t \le T)$ with state vector $\mathbf{N}^t$, where $(n_{i+1}^t,\cdots,n_{j-1}^t) \le 0$ and $n_j^t < 0$.*

1. *We have*

$$\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{+-}\Theta^{t+1}(\hat{\mathbf{N}}^t_{i-1}), \qquad \Delta_{ij}^{-+}\Theta^{t+1}(\mathbf{N}^t) = \Delta_{ij}^{-+}\Theta^{t+1}(\hat{\mathbf{N}}^t_{i-1}),$$

both of which are independent of the values of $(n_j^t,\cdots,n_N^t)$.

2. *The PSR policy solves the general upgrading problem in period t.*

Proof. In the proof, we show the two properties in Proposition 2 can be preserved under backward induction. The proof of period T is given in the end of this proof.

Suppose they are true for Θ^{t+1}, we verify the two properties for Θ^t.

2. Optimality of the PSR policy

Consider initial state $\mathbf{X}^t = (x_1,\cdots,x_N)$ and $\tilde{\mathbf{D}}^t = (\tilde{d}_1,\cdots,\tilde{d}_N)$, for any demand realization $\mathbf{D}^t = (d_1,\cdots,d_N)$ in period t, we next verify $\mathbf{Y}^t = (y_{ij})_{N\times N}$ derived by the PSR are optimal in period t. First, from the discussion in the main paper, $y_{kk} = \min(d_k + \tilde{d}_k, x_k)$ $(1 \le k \le N)$ in the PSR is optimal.

For upgrading decisions y_{ij} $(i>j)$ in $\mathbf{Y}^t$, we consider an equivalent representation of the general upgrading problem in (2). Let $\mathbf{Z}=(z_1,\cdots,z_N)^\top=\mathbf{Y}^t\mathbf{1}$, the optimal solution $\mathbf{W}=(w_{ij})_{N\times N}$ in the following linear program is the same as $\mathbf{Y}^t=(y_{ij})_{N\times N}$ in period t:

$$\begin{aligned}\max_{\mathbf{W}\geq 0}\quad & \sum_{1\leq i\leq j\leq N}\alpha_{ij}w_{ij}\\ \text{s.t.}\quad & \sum_j w_{ij}\leq z_i,\quad i=1,2,\cdots,N,\\ & \sum_i w_{ij}\leq d_j+\tilde{d}_j,\quad j=1,2,\cdots,N.\end{aligned}\tag{EC.16}$$

Since the parallel allocation is optimal, $z_i=x_i$ $(1\leq i\leq N)$ in $\mathbf{Z}$ is optimal if $x_i\leq d_i+\tilde{d}_i$. Furthermore, we need to show the optimality of z_i for all classes i's with $x_i>d_i+\tilde{d}_i$, i.e., the classes with surplus capacities after the parallel allocation. Since the general upgrading problem is concave, we only need to examine $\frac{\partial}{\partial z_i^+}\Theta^{t+1}(\mathbf{N})$ and $\frac{\partial}{\partial z_i^-}\Theta^{t+1}(\mathbf{N})$, where

$$\mathbf{N}=\mathbf{X}^t-\tilde{\mathbf{D}}^t-\mathbf{D}^t-\mathbf{Y}^t\mathbf{1}+(\mathbf{Y}^t)^\intercal\mathbf{1}=\mathbf{X}^t-\tilde{\mathbf{D}}^t-\mathbf{D}^t-\mathbf{Z}+(\mathbf{W})^\intercal\mathbf{1}$$

is the state at the beginning of period $t+1$.

Without loss of generality, we assume class 1 is the highest class with $x_1>d_1+\tilde{d}_1$ and analyze the optimality of z_1 by cases.

1. $z_1=d_1+\tilde{d}_1$: We only need to prove that increasing z_1 is suboptimal since $z_1\geq y_{11}=d_1+\tilde{d}_1$. Let k $(k>1)$ be the highest class with $(\mathbf{N})_k<0$. Note that z_1 is clearly optimal if class k does not exist, i.e., there is no backlogged demand in classes $(1,\cdots,N)$ in $\mathbf{N}$.
   (a) $(\mathbf{N})_{2,\cdots,k-1}=0$: When solving the protection level p_{1k} and the allocation decision y_{1k} by the PSR, $(\mathbf{N})_{1,\cdots,k}$ are the states of classes $(1,\cdots,k)$. Meanwhile, the upgrading decisions within classes $(k+1,\cdots,N)$ have not been considered, whose states are the states after the parallel allocation, i.e., $(\mathbf{X}^t-\tilde{\mathbf{D}}^t-\mathbf{D}^t)_{k+1,\cdots,N}$. Thus,

$$\begin{aligned}0\geq&\alpha_{1k}-\Delta_{1k}^{-+}\Theta^{t+1}\left((\mathbf{N})_{1,\cdots,k},(\mathbf{X}^t-\tilde{\mathbf{D}}^t-\mathbf{D}^t)_{k+1,\cdots,N}\right)\\ =&\alpha_{1k}-\Delta_{1k}^{-+}\Theta^{t+1}(\mathbf{N})=\alpha_{1k}+\frac{\partial}{\partial z_1^+}\Theta^{t+1}(\mathbf{N}),\end{aligned}\tag{EC.17}$$

   where the first equality is from the independence property assumption of Θ^{t+1}, and the second equality follows from the fact that $\mathbf{N}$ changes to $\mathbf{N}+\epsilon(-\mathbf{e}_1+\mathbf{e}_k)$ when z_1 marginally changes to $z_1+\epsilon$, where $\epsilon>0$. Hence, increasing z_1 is suboptimal.
   (b) There exists class i $(1<i<k)$ with $(\mathbf{N})_i>0$: Without loss of generality, we assume that i is the lowest class in $(2,\cdots,k-1)$ with $(\mathbf{N})_i>0$. In this case, the PSR considers protection level p_{ik} and ignores the potential upgrade from class 1 to k, and we will show it is indeed optimal to do so. Since $(\mathbf{N})_{1,\cdots,k}$ are the states of classes $(1,\cdots,k)$ when considering the

protection level p_{ik} by the PSR, and $\mathbf{N}$ changes to $\mathbf{N}+\epsilon(-\mathbf{e}_i+\mathbf{e}_k)$ when z_i marginally changes to $z_i+\epsilon$, where $\epsilon>0$. We have

$$0 \geq \alpha_{ik} - \Delta_{ik}^{-+}\Theta^{t+1}\left((\mathbf{N})_{1,\cdots,k},(\mathbf{X}^t-\tilde{\mathbf{D}}^t-\mathbf{D}^t)_{k+1,\cdots,N}\right) = \alpha_{ik} + \frac{\partial}{\partial z_i^+}\Theta^{t+1}(\mathbf{N}).$$

Moreover, because $(\mathbf{N})_1>0$ and $(\mathbf{N})_i>0$,

$$\partial_1^-\Theta^{t+1}(\mathbf{N}) - \partial_i^-\Theta^{t+1}(\mathbf{N}) \geq u_i - u_1$$

from Lemma EC.1.

Notice that $\mathbf{N}$ changes to $\mathbf{N}+\epsilon(-\mathbf{e}_1+\mathbf{e}_k)$ when z_1 marginally changes to $z_1+\epsilon$, then

$$\frac{\partial}{\partial z_1^+}\Theta^{t+1}(\mathbf{N}) - \frac{\partial}{\partial z_i^+}\Theta^{t+1}(\mathbf{N}) = \partial_i^-\Theta^{t+1}(\mathbf{N}) - \partial_1^-\Theta^{t+1}(\mathbf{N}). \tag{EC.18}$$

Thus, from $\alpha_{ik}-\alpha_{1k}=u_1-u_i$, we have

$$\frac{\partial}{\partial z_1^+}\Theta^{t+1}(\mathbf{N}) + \alpha_{1k} \leq \frac{\partial}{\partial z_i^+}\Theta^{t+1}(\mathbf{N}) + \alpha_{1k} + u_1 - u_i = \frac{\partial}{\partial z_i^+}\Theta^{t+1}(\mathbf{N}) + \alpha_{ik} \leq 0, \tag{EC.19}$$

which means increasing z_1 is not optimal.

2. $z_1>d_1+\tilde{d}_1$: Let j $(j>1)$ be the lowest class with $y_{1j}>0$ in $\mathbf{Y}^t$. Similar to the previous case, from the PSR, $(\mathbf{N})_{1,\cdots,j}$ are the states after performing the last unit of upgrade y_{1j}. In this case, $\mathbf{N}$ changes to $\mathbf{N}+\epsilon(\mathbf{e}_1-\mathbf{e}_j)$ when z_1 marginally changes to $z_1-\epsilon$, where $\epsilon>0$, then

$$0 \leq \alpha_{1j} - \Delta_{1j}^{+-}\Theta^{t+1}\left((\mathbf{N})_{1,\cdots,j},(\mathbf{X}^t-\tilde{\mathbf{D}}^t-\mathbf{D}^t)_{j+1,\cdots,N}\right) = \alpha_{1j} + \frac{\partial}{\partial z_1^-}\Theta^{t+1}(\mathbf{N}). \tag{EC.20}$$

Thus, decreasing current z_1 is costly.

Furthermore, for all class i $(1<i<j)$ with $x_i>d_i+\tilde{d}_i$, $z_i=x_i$ by the PSR policy. First, we only need to show decreasing these z_i's is not optimal.

When z_i marginally changes to $z_i-\epsilon$, there is a *chain reaction*. From (EC.16), decreasing z_i by ϵ is equivalent to reducing the upgrade y_{ik_i} by ϵ, where k_i is the lowest class upgraded by capacity i. Then, unmet demand k_i increases by ϵ unit, and demand k_i will be upgraded by capacity s $(1\leq s<i)$, which is the lowest class with $x_s>\tilde{d}_s+d_s$. Meanwhile, k_s, the lowest class of demands upgraded by capacity s prior to changing z_i, has an additional ϵ unit unmet demand, which can be similarly analyzed as class k_i. The *chain reaction* continues, and $\mathbf{N}$ changes to $\mathbf{N}+\epsilon(\mathbf{e}_i-\mathbf{e}_j)$, i.e., the unmet demand j is increased by ϵ unit.

When z_i marginally changes to $z_i-\epsilon$, only unmet demand j and capacity i changed in the aforementioned *chain reaction*, then the objective function in (EC.16) decreases by $\epsilon\alpha_{ij}$. Meanwhile, given $(\mathbf{N})_1\geq 0$ and $(\mathbf{N})_i=0$, similar to (EC.18), we have

$$\frac{\partial}{\partial z_1^-}\Theta^{t+1}(\mathbf{N}) - \frac{\partial}{\partial z_i^-}\Theta^{t+1}(\mathbf{N}) \leq u_1 - u_i$$

by Lemma EC.1. Thus, from $\alpha_{ij}-\alpha_{1j}=u_1-u_i$,

$$\frac{\partial}{\partial z_i^-}\Theta^{t+1}(\mathbf{N})+\alpha_{ij}\geq\frac{\partial}{\partial z_1^-}\Theta^{t+1}(\mathbf{N})+u_i-u_1+\alpha_{ij}=\frac{\partial}{\partial z_1^-}\Theta^{t+1}(\mathbf{N})+\alpha_{1j}\geq 0.$$

Hence, $z_i=x_i$ is optimal for all class i $(1<i<j)$ with $x_i>d_i+\tilde{d}_i$.

Next, we have to prove that increasing z_1 itself is also suboptimal.

(a) $(\mathbf{N})_j<0$: In this case, the protection level p_{1j} is binding in the PSR, i.e., the upgrade between classes 1 and j stops when the quantity of capacity 1 reaches p_{1j}. From the definition of p_{1j}, and the fact that $\mathbf{N}$ changes to $\mathbf{N}+\epsilon(-\mathbf{e}_1+\mathbf{e}_j)$ when z_1 marginally changes to $z_1+\epsilon$, we have

$$\begin{aligned}0\geq&\alpha_{1j}-\Delta_{1j}^{-+}\Theta^{t+1}\left((\mathbf{N})_{1,\cdots,j},(\mathbf{X}^t-\tilde{\mathbf{D}}^t-\mathbf{D}^t)_{j+1,\cdots,N}\right)\\=&\alpha_{1j}+\frac{\partial}{\partial z_1^+}\Theta^{t+1}(\mathbf{N}).\end{aligned}\tag{EC.21}$$

From (EC.20) and (EC.21), we know the optimality of z_1.

(b) $(\mathbf{N})_j=0$: The upgrading decision y_{1j} is bounded because there is no unmet demand j remaining, and we do not have (EC.21) directly from solving p_{1j}. However, similar to the case when $z_1=d_1+\tilde{d}_1$, increasing z_1 is still suboptimal. Particularly, if there exists k $(k>j)$ as the highest class with $(\mathbf{N})_k<0$, and $(\mathbf{N})_s=0$ for all class s $(j<s<k)$, then we have (EC.17) that affirms the optimality of z_1. On the other hand, if there exists class i $(j<i<k)$ with $(\mathbf{N})_i>0$, then (EC.19) is valid, which also proves the optimality of z_1.

To summarize, we have proved that z_1 is optimal. In addition, if $z_1>d_1+\tilde{d}_1$ and class j is the lowest class with $y_{1j}>0$ in $\mathbf{Y}^t$, we have also shown the optimality of z_i $(1<i<j)$ with $x_i>d_i+\tilde{d}_i$. The same argument can be sequentially applied to the rest of z_s's since $(\mathbf{N})_{1,\cdots,s-1}$ are the states of classes $(1,\cdots,s-1)$ when solving the protection levels within classes $(s,\cdots,N)$ in the PSR policy.

Therefore, the PSR policy solves the general upgrading problem in period t.

1. Independence property of Θ^t

As the PSR solves the general upgrading problem in period t, and the independence property of Θ^{t+1} holds by Property 1 of the induction assumption, all requirements of Lemma EC.5 are satisfied, thus the independence property of Θ^t also holds.

To conclude the proof, we now consider period T. The PSR solves the general upgrading problem in period T by Lemma EC.3. And Lemma EC.4 asserts the independence property of Θ^T. Therefore, we can use the backward induction and complete the proof. □

B.4. Properties of the Protection Levels

PROPOSITION 3 *If initial capacity $\mathbf{X}^1$ and demand $\mathbf{D}^1,\cdots,\mathbf{D}^T$ are integer valued, there exists an integer valued optimal policy $\mathbf{Y}^1,\cdots,\mathbf{Y}^T$ derived by the PSR policy.*

Proof. The proof is similar to the proof of Proposition 3 in Shumsky and Zhang (2009). □

PROPOSITION 4 *For the same* $(n_1^t, \cdots, n_{i-1}^t)$ *in period* t $(1 \le t \le T)$, $p_{ij} \le p_{i,j+1}$ *when* $i < j$.

Proof. Suppose to the contrary that $p_{ij} > p_{i,j+1}$ in period t. Let $\bar{p} = \frac{p_{ij}+p_{i,j+1}}{2}$, and denote $\mathbf{N} = (n_1^t, \cdots, n_{i-1}^t, \bar{p}, 0, \cdots, 0, n_{j+2}^t, \cdots, n_N^t)$. From the concavity in Proposition 1 and the independence property in Proposition 2, we have $\Delta_{i,j+1}^{+-}\Theta^{t+1}(\mathbf{N}) \le \alpha_{i,j+1}$ given $\bar{p} > p_{i,j+1}$. Similarly, we have $\Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}) \ge \alpha_{ij}$ since $\bar{p} < p_{ij}$.

However, since $\alpha_{ij} - \alpha_{i,j+1} = r_j + g_j - r_{j+1} - g_{j+1}$ and $g_j > g_{j+1}$,

$$\begin{aligned}
&\Delta_{i,j+1}^{+-}\Theta^{t+1}(\mathbf{N}) = \partial_i^{+}\Theta^{t+1}(\mathbf{N}) - \partial_{j+1}^{-}\Theta^{t+1}(\mathbf{N})\\
\ge&\partial_i^{+}\Theta^{t+1}(\mathbf{N}) - \partial_j^{-}\Theta^{t+1}(\mathbf{N}) + r_{j+1} - r_j = \Delta_{ij}^{+-}\Theta^{t+1}(\mathbf{N}) + r_{j+1} - r_j\\
\ge&\alpha_{ij} + r_{j+1} - r_j > \alpha_{i,j+1},
\end{aligned}$$

where the first inequality follows from Lemma EC.2. This is a contradiction. Hence, $p_{ij} \le p_{i,j+1}$ when $i < j$. □

Appendix C: Multi-Horizon Model with Replenishment

PROPOSITION 6 *Suppose the firm starts with an initial capacity* $\mathbf{X} \le \mathbf{X}^*$ *in* (11). *The firm's optimal replenishment policy in the multi-horizon model is a base stock policy with the optimal base stock level* $\mathbf{X}^*$. *Furthermore, the PSR policy solves the optimal allocation decisions within each horizon.*

Proof. We prove this proposition by induction. Let $V_k(\mathbf{X}, \tilde{\mathbf{D}})$ $(1 \le k \le K)$ denote the expected revenue-to-go function at the beginning of the k-th horizon with capacity $\mathbf{X}$ and backlogged demand $\tilde{\mathbf{D}}$. Where possible, the index of periods in each horizon is denoted by superscripts while subscripts denote the index of horizons. We inductively prove the following three properties.

1. The PSR policy optimally solves the capacity allocation decisions in the k-th horizon;
2. The optimal replenishment policy in the k-th horizon is a base stock policy with the optimal base stock level $\mathbf{X}^*$;
3. If $\mathbf{X} \le \mathbf{X}^*$, $V_k(\mathbf{X}, \tilde{\mathbf{D}})$ is affine in $\mathbf{X}$ with slope $\mathbf{C}$ and $\tilde{\mathbf{D}}$ with slope $\boldsymbol{\alpha} - \mathbf{C}$.

Suppose all properties hold in the $(k+1)$-th horizon. It suffices to consider capacity $\mathbf{X} \leq \mathbf{X}^*$. Since $V_{k+1}(\mathbf{X}, \tilde{\mathbf{D}})$ is affine in $(\mathbf{X}, \tilde{\mathbf{D}})$, in horizon k we have

$$\begin{aligned}\Theta_k^T(\mathbf{X}, \tilde{\mathbf{D}}) &= \mathop{\mathbb{E}}_{\mathbf{D}_k^T}\left\{\max_{\mathbf{Y}_k^T}\left[H(\mathbf{Y}_k^T|\tilde{\mathbf{D}};\mathbf{D}_k^T) - \mathbf{h}\mathbf{X}_k^{T+1} + \gamma V_{k+1}(\mathbf{X}_k^{T+1}, \tilde{\mathbf{D}}_k^{T+1})\right]\right\}\\ &= \mathop{\mathbb{E}}_{\mathbf{D}_k^T}\left\{\max_{\mathbf{Y}_k^T}\left[H(\mathbf{Y}_k^T|\tilde{\mathbf{D}};\mathbf{D}_k^T) - \mathbf{h}\mathbf{X}_k^{T+1} + \gamma V_{k+1}(\mathbf{X}^*, 0) + \gamma\mathbf{C}(\mathbf{X}_k^{T+1} - \mathbf{X}^*) + \gamma(\boldsymbol{\alpha} - \mathbf{C})\tilde{\mathbf{D}}_k^{T+1}\right]\right\}\\ &= \mathop{\mathbb{E}}_{\mathbf{D}_k^T}\left\{\max_{\mathbf{Y}_k^T}\left[H(\mathbf{Y}_k^T|\tilde{\mathbf{D}};\mathbf{D}_k^T) + (\gamma\mathbf{C} - \mathbf{h})\mathbf{X}_k^{T+1} + \gamma(\boldsymbol{\alpha} - \mathbf{C})\tilde{\mathbf{D}}_k^{T+1} + \gamma\left(V_{k+1}(\mathbf{X}^*, 0) - \mathbf{C}\mathbf{X}^*\right)\right]\right\}\\ &= \mathop{\mathbb{E}}_{\mathbf{D}_k^T}\left\{\max_{\mathbf{Y}_k^T}\left[H(\mathbf{Y}_k^T|\tilde{\mathbf{D}};\mathbf{D}_k^T) + (\gamma\mathbf{C} - \mathbf{h})\mathbf{X}_k^{T+1} + \gamma(\boldsymbol{\alpha} - \mathbf{C})\tilde{\mathbf{D}}_k^{T+1}\right]\right\} + \gamma\left(V_{k+1}(\mathbf{X}^*, 0) - \mathbf{C}\mathbf{X}^*\right),\end{aligned} \tag{EC.22}$$

where $\gamma\left(V_{k+1}(\mathbf{X}^*, 0) - \mathbf{C}\mathbf{X}^*\right)$ is a constant. From the proof of Proposition 2, the PSR policy optimally solves the T-th period allocation decisions in the k-th horizon, where the protection levels are based on $\Theta^{T+1}(\mathbf{X}^{T+1}, \tilde{\mathbf{D}}^{T+1})$ in (10). Moreover, it is clear that $\Theta_k^T(\mathbf{X}, \tilde{\mathbf{D}})$ is also concave in $(\mathbf{X}, \tilde{\mathbf{D}})$ from the proof of Proposition 1. Inductively, for $t = T-1, \cdots, 1$, we know

$$\Theta_k^t(\mathbf{X}, \tilde{\mathbf{D}}) = \mathop{\mathbb{E}}_{\mathbf{D}_k^t}\left\{\max_{\mathbf{Y}_k^t}\left[H(\mathbf{Y}_k^t|\tilde{\mathbf{D}};\mathbf{D}_k^t) + \Theta_k^{t+1}(\mathbf{X}_k^{t+1}, \tilde{\mathbf{D}}_k^{t+1})\right]\right\}$$

is concave in $(\mathbf{X}, \tilde{\mathbf{D}})$, and we can show that the PSR policy solves the capacity allocation decisions for horizon k.

From the Bellman equation, we have

$$V_k(\mathbf{X}, \tilde{\mathbf{D}}) = \max_{\mathbf{Z} \geq \mathbf{X}} G(\mathbf{Z}),$$

where $G(\mathbf{Z}) = \Theta_k^1(\mathbf{Z}, \mathbf{0}) + \boldsymbol{\alpha}\tilde{\mathbf{D}} - \mathbf{C}(\mathbf{Z} - \mathbf{X} + \tilde{\mathbf{D}})$. From (EC.22), we have

$$G(\mathbf{Z}) = \Pi\left(\mathbf{Z}; \gamma\mathbf{C} - \mathbf{h}, \gamma(\boldsymbol{\alpha} - \mathbf{C})\right) + (\boldsymbol{\alpha} - \mathbf{C})\tilde{\mathbf{D}} + \mathbf{C}(\mathbf{X} - \gamma\mathbf{X}^*) + \gamma V_{k+1}(\mathbf{X}^*, 0).$$

By the definition of $\mathbf{X}^*$, the optimal replenishment policy in the k-th horizon is a base stock policy with optimal base stock level $\mathbf{X}^*$. Furthermore, for $\mathbf{X} \leq \mathbf{X}^*$,

$$V_k(\mathbf{X}, \tilde{\mathbf{D}}) = \Pi\left(\mathbf{X}^*; \gamma\mathbf{C} - \mathbf{h}, \gamma(\boldsymbol{\alpha} - \mathbf{C})\right) + (\boldsymbol{\alpha} - \mathbf{C})\tilde{\mathbf{D}} + \mathbf{C}(\mathbf{X} - \gamma\mathbf{X}^*) + \gamma V_{k+1}(\mathbf{X}^*, 0)$$

is affine in $\mathbf{X}$ with slope $\mathbf{C}$ and $\tilde{\mathbf{D}}$ with slope $\boldsymbol{\alpha} - \mathbf{C}$.

To conclude the proof, we consider the last horizon profit, $V_K(\mathbf{X}, \tilde{\mathbf{D}})$. Since

$$\Theta_K^T(\mathbf{X}, \tilde{\mathbf{D}}) = \mathop{\mathbb{E}}_{\mathbf{D}_K^T}\left\{\max_{\mathbf{Y}_K^T}\left[H(\mathbf{Y}_K^T|\tilde{\mathbf{D}};\mathbf{D}_K^T) - \mathbf{h}\mathbf{X}_K^{T+1} + \gamma\mathbf{C}\mathbf{X}_K^{T+1} + \gamma(\boldsymbol{\alpha} - \mathbf{C})\tilde{\mathbf{D}}_K^{T+1}\right]\right\}$$

by definition, the optimality of the PSR policy can be similarly proved. Meanwhile, if $\mathbf{X} \leq \mathbf{X}^*$,

$$V_K(\mathbf{X}, \tilde{\mathbf{D}}) = \max_{\mathbf{Z} \geq \mathbf{X}} \left[\Pi\left(\mathbf{Z}; \gamma\mathbf{C} - \mathbf{h}, \gamma(\boldsymbol{\alpha} - \mathbf{C})\right) + (\boldsymbol{\alpha} - \mathbf{C})\tilde{\mathbf{D}} + \mathbf{C}\mathbf{X} \right],$$

so the base stock policy is optimal and $V_K(\mathbf{X}, \tilde{\mathbf{D}})$ is affine in $\mathbf{X}$ with slope $\mathbf{C}$ and $\tilde{\mathbf{D}}$ with slope $\boldsymbol{\alpha} - \mathbf{C}$. Therefore, all properties hold for the K-th horizon, which completes the proof. □

Appendix D: Numerical Studies

D.1. Numerical study with $N = 4$ and $T = 3$

In Table 1, we consider problems with size $N = 4$ and $T = 3$. For such a problem size, we can use backward induction to calculate the firm's optimal revenue, which serves as a benchmark to evaluate the performance of the RCEC heuristics. Below we describe the design of the numerical study in detail. The description consists of three parts: demand patterns, economic parameters, and initial capacity.

D.1.1. Demand patterns To cover a wide range of demand scenarios, we consider 13 evolution patterns for product demand means in Table EC.1. For each evolution pattern, we define vectors $\boldsymbol{\mu}^t = (\mu_1^t, \cdots, \mu_N^t)^\intercal$ $(t = 1, \cdots, T)$, where μ_i^t is the demand mean of product i in period t. The demand mean patterns in Table EC.1 cover some typical realistic scenarios. For instance, in pattern 4, the expected demand for high-quality products are higher than that for low-quality products when the period is close to the end of horizon, which corresponds to revenue management situations.

Given an evolution pattern $\boldsymbol{\mu}^t$ $(t = 1, \cdots, T)$ for the demand means, we generate a sample of random demands for each product in each period. Specifically, given the demand mean $\boldsymbol{\mu}^t$ in period t, we generate demand $\mathbf{D}^t$ by using either Poisson distribution or multivariate normal distribution with covariance matrix

$$\begin{pmatrix} 0.5 & 0.15 & 0.075 & 0.0375 \\ 0.15 & 0.5 & 0.15 & 0.075 \\ 0.075 & 0.15 & 0.5 & 0.15 \\ 0.0375 & 0.075 & 0.15 & 0.5 \end{pmatrix} * \boldsymbol{\mu}^t,$$

and

$$\begin{pmatrix} 0.5 & -0.15 & -0.075 & -0.0375 \\ -0.15 & 0.5 & -0.15 & -0.075 \\ -0.075 & -0.15 & 0.5 & -0.15 \\ -0.0375 & -0.075 & -0.15 & 0.5 \end{pmatrix} * \boldsymbol{\mu}^t.$$

The first covariance matrix represents positive correlation between the products, while the second represents negative correlation between the products. For normal distribution, we truncate the demand realizations at zero and round them to the nearest integer values. By the above construction, there are total $39 = 3 * 13$ demand scenarios.

Pattern	Description	Example ($T=3$)
Linear	1. Product 1 demand increases, product 2 demand is flat, product 3 and 4 demands decrease with the same rate.	$\begin{pmatrix}2&4&6&6\\4&4&4&4\\6&4&2&2\end{pmatrix}$
	2. Product 1 demand increases, product 2 demand is flat, product 3 demand decreases, product 4 demand decreases in half of the rate of product 3.	$\begin{pmatrix}2&4&6&4\\4&4&4&3\\6&4&2&2\end{pmatrix}$
	3. Product 1, 2, 3 and 4 demands are flat.	$\begin{pmatrix}4&4&4&4\\4&4&4&4\\4&4&4&4\end{pmatrix}$
	4. Product 1 and 2 demands increase with the same rate, product 3 and 4 demands decrease with the same rate.	$\begin{pmatrix}2&2&6&6\\4&4&4&4\\6&6&2&2\end{pmatrix}$
	5. Product 1 demand increases, product 2 demand increases in half of the rate of product 1, product 3 demand decreases, product 4 demand decreases in half of the rate of product 3.	$\begin{pmatrix}2&2&6&4\\4&3&4&3\\6&4&2&2\end{pmatrix}$
	6. Product 1 and 2 demands increase with the same rate, product 3 demand is flat, product 4 demand decreases.	$\begin{pmatrix}2&2&4&6\\4&4&4&4\\6&6&4&2\end{pmatrix}$
	7. Product 1 demand increases, product 2 demand increases in half of the rate of product 1, product 3 demand is flat, product 4 demand decreases.	$\begin{pmatrix}2&2&4&6\\4&3&4&4\\6&4&4&2\end{pmatrix}$
	8. Product 1 and 2 demands increase with the same rate, product 3 and 4 demands are flat.	$\begin{pmatrix}2&2&4&4\\4&4&4&4\\6&6&4&4\end{pmatrix}$
	9. Product 1 demand increases, product 2 demand increases in half of the rate of product 1, product 3 and 4 demands are flat.	$\begin{pmatrix}2&2&4&4\\4&3&4&4\\6&4&4&4\end{pmatrix}$
Alternating	10. Products 1 and 3 start with positive demand, while products 2 and 4 start with zero demand.	$\begin{pmatrix}4&0&4&0\\0&4&0&4\\4&0&4&0\end{pmatrix}$
	11. Products 1 and 3 start with positive demand, where demand 3 is smaller than demand 1 in each period, and products 2 and 4 start with zero demand, where demand 4 is smaller than demand 2 in each period.	$\begin{pmatrix}6&0&2&0\\0&6&0&2\\6&0&2&0\end{pmatrix}$
	12. Products 2 and 4 start with positive demand, while products 1 and 3 start with zero demand.	$\begin{pmatrix}0&4&0&4\\4&0&4&0\\0&4&0&4\end{pmatrix}$
	13. Products 2 and 4 start with positive demand, where demand 4 is smaller than demand 2 in each period, and products 1 and 3 start with zero demand, where demand 3 is smaller than demand 1 in each period.	$\begin{pmatrix}0&6&0&2\\6&0&2&0\\0&6&0&2\end{pmatrix}$

Table EC.1 Demand patterns with 4 products.

D.1.2. Economic parameters We also consider a wide variety of values for the economic parameters while using the same backorder cost $(g_1, g_2, g_3, g_4) = (1.0, 0.9, 0.8, 0.7)$. Recall the upgrading revenue is given by $\alpha_{ij} = r_j + g_j - u_i$; instead of specifying r_j and u_i, we choose to specify α_{ij}, which is sufficient for the numerical study. Four different matrices of $\boldsymbol{\alpha} = (\alpha_{ij})_{4\times 4}$ have been considered in the numerical study. The capacity costs (c_1, c_2, c_3, c_4) are decided by $c_i = 0.3\alpha_{ii}$ $(i = 1, \cdots, 4)$ for each matrix.

1. Matrix 1: The parallel revenue decreases in product class, and the upgrading revenues are close to the parallel revenue.
$$\begin{pmatrix} 16 & 14 & 12 & 10 \\ 0 & 15 & 13 & 11 \\ 0 & 0 & 14 & 12 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$
2. Matrix 2: The parallel revenue decreases in product class, and the upgrading revenues are decreasing in the number of levels of upgrading (e.g., 1-step upgrading revenue is 11 and 2-step upgrading is either 7 or 8).
$$\begin{pmatrix} 16 & 11 & 7 & 4 \\ 0 & 15 & 11 & 8 \\ 0 & 0 & 14 & 11 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$
3. Matrix 3: The parallel revenue decreases in product class, and α_{12} and α_{34} are higher than α_{23}.
$$\begin{pmatrix} 16 & 14 & 5 & 3 \\ 0 & 15 & 6 & 4 \\ 0 & 0 & 14 & 12 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$
4. Matrix 4: The parallel revenue is constant across products, the 2-step upgrading revenue is constant, and α_{23} is higher than the other 1-step upgrading revenue.
$$\begin{pmatrix} 16 & 10 & 9 & 3 \\ 0 & 16 & 15 & 9 \\ 0 & 0 & 16 & 10 \\ 0 & 0 & 0 & 16 \end{pmatrix}$$

D.1.3. Initial capacity When choosing the initial capacity, we start with optimal capacity level $\mathbf{X}_{\text{RCEC}}$ using the RCEC heuristic. To ensure the robustness of the results, we also consider a number of variants of $\mathbf{X}_{\text{RCEC}}$, among which some are extreme capacity scenarios. In particular, we use $\mathbf{X}_{\text{RCEC}} = (x_1^{\text{RCEC}}, x_2^{\text{RCEC}}, x_3^{\text{RCEC}}, x_4^{\text{RCEC}})$ to construct the following five patterns of initial capacity:

1. $\mathbf{X} = \lambda \mathbf{X}_{\text{RCEC}}$
2. For each $i \in \{1,2,3\}$: $(\mathbf{X})_i = \lambda(x_i^{\text{RCEC}} + x_{i+1}^{\text{RCEC}})$, $(\mathbf{X})_{i+1} = 0$, $(\mathbf{X})_s = \lambda x_s^{\text{RCEC}}$, $\forall\ s \in \{1,2,3,4\} \setminus \{i, i+1\}$
3. For each $i \in \{2,3,4\}$: $(\mathbf{X})_i = 0$, $(\mathbf{X})_s = \lambda x_s^{\text{RCEC}}$, $\forall\ s \in \{1,2,3,4\} \setminus \{i\}$
4. $\mathbf{X} = \lambda(x_1^{\text{RCEC}} + x_2^{\text{RCEC}}, 0, x_3^{\text{RCEC}} + x_4^{\text{RCEC}}, 0)$

5. $\mathbf{X} = \lambda(x_1^{\text{RCEC}} + x_3^{\text{RCEC}}, x_2^{\text{RCEC}} + x_4^{\text{RCEC}}, 0, 0)$,

where $\lambda = \{0.75,\ 1,\ 1.25\}$. Each pattern corresponds to a realistic or extreme scenario. For instance, in Pattern 2, a certain product has extremely low inventory level while the adjacent high-quality product is abundant; in Pattern 5, the last two products have extremely low investment while there are plenty of higher level products. Note that in some of the patterns (e.g., Patterns 2-5), upgrading would be frequently performed. The parameter λ is used to adjust the capacity-demand ratio (e.g., $\lambda = 0.75$ implies that the aggregate capacity level is relatively low). For each λ, there are 9 initial capacity scenarios; so there are totally 27 capacity scenarios.

To summarize, we test $39 * 4 * 27 = 4212$ problem instances by the above construction. They cover a wide range of possible situations that may arise in practice.

D.2. Numerical study with $N = 5$ and $T \in \{3, 15, 30, 60\}$

This is the major numerical study in this paper; it serves several purposes. First, we test the performance of the RCEC heuristic for problems with larger sizes in Tables 2 and 3; second, we examine the value of multi-step upgrading in Tables 4 and 5; third, we investigate the importance of the allocation mechanism and the capacity sizing decision in Table 6. To make the results comparable across different T values, we make a couple of assumptions: (1) For each product i, the expected total demand throughout the sales horizon is the same for different T values; that is, the sum $\sum_{t=1}^{T} \mu_i^t$ in each demand evolution pattern $\boldsymbol{\mu}^t = (\mu_1^t, \cdots, \mu_N^t)$ $(t = 1, \cdots, N)$ is the same for different T values, which is set to be 60 for each i. (2) For each parameter combination, the capacity cost is the same for different T's. (3) The per-period goodwill cost is linearly decreasing in T (using a constant goodwill cost independent of T will not affect the performance result about RCEC). Below we describe the design of the numerical study in detail. Again the description consists of three parts: demand patterns, economic parameters, and initial capacity.

D.2.1. Demand patterns Similar to the first numerical study in Section D.1, we consider 13 demand evolution patterns in Table EC.2.

Again, given an evolution pattern $\boldsymbol{\mu}^t$ $(t = 1, \cdots, T)$ for the demand means, we generate a sample of random demands for each product in each period. Specifically, given the demand mean $\boldsymbol{\mu}^t$ in period t, we generate demand $\mathbf{D}^t$ by using either Poisson distribution or multivariate normal distribution with a positive covariance matrix

$$\begin{pmatrix} 0.5 & 0.15 & 0.12 & 0.09 & 0.06 \\ 0.15 & 0.5 & 0.15 & 0.12 & 0.09 \\ 0.12 & 0.15 & 0.5 & 0.15 & 0.12 \\ 0.09 & 0.12 & 0.15 & 0.5 & 0.15 \\ 0.06 & 0.09 & 0.12 & 0.15 & 0.5 \end{pmatrix} * \boldsymbol{\mu}^t,$$

Pattern	Description	Example ($T=3$)
Linear	1. Product 1 demand increases, product 2 and 3 demands are flat, product 4 and 5 demands decrease with the same rate.	$\begin{pmatrix} 4 & 8 & 8 & 12 & 12 \\ 8 & 8 & 8 & 8 & 8 \\ 12 & 8 & 8 & 4 & 4 \end{pmatrix}$
	2. Product 1 demand increases, product 2 and 3 demands are flat, product 4 demand decreases, product 5 demand decreases in half of the rate of product 4.	$\begin{pmatrix} 4 & 8 & 8 & 12 & 8 \\ 8 & 8 & 8 & 8 & 6 \\ 12 & 8 & 8 & 4 & 4 \end{pmatrix}$
	3. Product 1, 2, 3, 4 and 5 demands are flat.	$\begin{pmatrix} 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \end{pmatrix}$
	4. Product 1, 2 and 3 demands increase with the same rate, product 4 and 5 demands decrease with the same rate.	$\begin{pmatrix} 4 & 4 & 4 & 12 & 12 \\ 8 & 8 & 8 & 8 & 8 \\ 12 & 12 & 12 & 4 & 4 \end{pmatrix}$
	5. Product 1 demand increases, product i ($i=2,3$) demand increases in half of the rate of product $i-1$, product 4 demand decreases, product 5 demand decreases in half of the rate of product 4.	$\begin{pmatrix} 4 & 4 & 4 & 12 & 8 \\ 8 & 6 & 5 & 8 & 6 \\ 12 & 8 & 6 & 4 & 4 \end{pmatrix}$
	6. Product 1 and 2 demands increase with the same rate, product 3 demand is flat, product 4 and 5 demands decrease with the same rate.	$\begin{pmatrix} 4 & 4 & 8 & 12 & 12 \\ 8 & 8 & 8 & 8 & 8 \\ 12 & 12 & 8 & 4 & 4 \end{pmatrix}$
	7. Product 1 demand increases, product 2 demand increases in half of the rate of product 1, product 3 demand is flat, product 4 demand decreases, product 5 demand decreases in half of the rate of product 4.	$\begin{pmatrix} 4 & 4 & 8 & 12 & 8 \\ 8 & 6 & 8 & 8 & 6 \\ 12 & 8 & 8 & 4 & 4 \end{pmatrix}$
	8. Product 1 and 2 demands increase with the same rate, product 3, 4 and 5 demands are flat.	$\begin{pmatrix} 4 & 4 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 12 & 12 & 8 & 8 & 8 \end{pmatrix}$
	9. Product 1 demand increases, product 2 demand increases in half of the rate of product 1, product 3, 4 and 5 demands are flat.	$\begin{pmatrix} 4 & 4 & 8 & 8 & 8 \\ 8 & 6 & 8 & 8 & 8 \\ 12 & 8 & 8 & 8 & 8 \end{pmatrix}$
Alternating	10. Products 1, 3 and 5 start with positive demand, while products 2 and 4 start with zero demand.	$\begin{pmatrix} 8 & 0 & 8 & 0 & 8 \\ 0 & 8 & 0 & 8 & 0 \\ 8 & 0 & 8 & 0 & 8 \end{pmatrix}$
	11. Products 1, 3 and 5 start with positive demand, where demand i ($i=3,5$) is smaller than demand $i-2$ in each period, and products 2 and 4 start with zero demand, where demand 4 is smaller than demand 2 in each period.	$\begin{pmatrix} 16 & 0 & 8 & 0 & 4 \\ 0 & 16 & 0 & 8 & 0 \\ 16 & 0 & 8 & 0 & 4 \end{pmatrix}$
	12. Products 2 and 4 start with positive demand, while products 1, 3 and 5 start with zero demand.	$\begin{pmatrix} 0 & 8 & 0 & 8 & 0 \\ 8 & 0 & 8 & 0 & 8 \\ 0 & 8 & 0 & 8 & 0 \end{pmatrix}$
	13. Products 2 and 4 start with positive demand, where demand 4 is smaller than demand 2 in each period, and products 1, 3 and 5 start with zero demand, where demand i ($i=3,5$) is smaller than demand $i-2$ in each period.	$\begin{pmatrix} 0 & 16 & 0 & 8 & 0 \\ 16 & 0 & 8 & 0 & 4 \\ 0 & 16 & 0 & 8 & 0 \end{pmatrix}$

Table EC.2 Demand patterns with 5 products.

and a negative covariance matrix

$$\begin{pmatrix} 0.5 & -0.15 & -0.12 & -0.09 & -0.06 \\ -0.15 & 0.5 & -0.15 & -0.12 & -0.09 \\ -0.12 & -0.15 & 0.5 & -0.15 & -0.12 \\ -0.09 & -0.12 & -0.15 & 0.5 & -0.15 \\ -0.06 & -0.09 & -0.12 & -0.15 & 0.5 \end{pmatrix} * \boldsymbol{\mu}^t.$$

The rest of the details are the same as in the first numerical study and therefore omitted. There are totally $3*13=39$ demand scenarios.

D.2.2. Economic parameters For all problem instances, we use the same backorder cost vector $T*(g_1,\cdots,g_5)=(6.0,5.7,5.4,5.1,4.8)$. Then we use 4 sets of parameters for the revenue $(r_1,\cdots,r_5)$, usage cost $(u_1,\cdots,u_5)$ and capacity cost $(c_1,\cdots,c_5)$, which is explained next. Since the vector $(g_1,\cdots,g_5)$ decreases in T, each element of $\boldsymbol{\alpha}=(\alpha_{ij})_{N\times N}$ decreases in T as well, where α_{15} is the smallest entry with $T=15$. For illustration, we provide $\boldsymbol{\alpha}$ for $T=15$ in each of the parameter set for revenue $(r_1,\cdots,r_5)$, usage cost $(u_1,\cdots,u_5)$ and capacity cost $(c_1,\cdots,c_5)$. In particular, given α_{15} in $T=15$, the capacity cost $(c_1,\cdots,c_5)$ are determined by $c_1=0.7*\alpha_{15}$, and $c_{i+1}=c_i-0.05$ $(i>1)$ for parameter set 1 and $c_{i+1}=c_i-0.02$ $(i>1)$ for the other sets.

1. Parameter set 1: Upgrading revenue is close to the parallel revenue.

$$(r_1,\cdots,r_5)=(20,18,16,14,12)$$
$$(u_1,\cdots,u_5)=(5,4,3,2,1)$$
$$(c_1,\cdots,c_5)=(5.124,5.074,5.024,4.974,4.924).$$

   For $T=15$, there is

$$\boldsymbol{\alpha}=\begin{pmatrix} 15.4 & 13.38 & 11.36 & 9.34 & 7.32 \\ 0 & 14.38 & 12.36 & 10.34 & 8.32 \\ 0 & 0 & 13.36 & 11.34 & 9.32 \\ 0 & 0 & 0 & 12.34 & 10.32 \\ 0 & 0 & 0 & 0 & 11.32 \end{pmatrix}$$

2. Parameter set 2: Revenues of 1-step upgrading are almost identical for different classes.

$$(r_1,\cdots,r_5)=(26,21,17,14,12)$$
$$(u_1,\cdots,u_5)=(11,7,4,2,1)$$
$$(c_1,\cdots,c_5)=(0.924,0.904,0.884,0.864,0.844).$$

   For $T=15$, there is

$$\boldsymbol{\alpha}=\begin{pmatrix} 15.4 & 10.38 & 6.36 & 3.34 & 1.32 \\ 0 & 14.38 & 10.36 & 7.34 & 5.32 \\ 0 & 0 & 13.36 & 10.34 & 8.32 \\ 0 & 0 & 0 & 12.34 & 10.32 \\ 0 & 0 & 0 & 0 & 11.32 \end{pmatrix}$$

3. Parameter set 3: α_{12} is much smaller than parallel revenue α_{11}. However, α_{23}, α_{34} and α_{45} are close to α_{22}, α_{33} and α_{44}, respectively.

$$(r_1,\cdots,r_5)=(26,21,17,14,12)$$
$$(u_1,\cdots,u_5)=(11,7,4,2,1)$$
$$(c_1,\cdots,c_5)=(2.324,2.304,2.284,2.264,2.244).$$

For $T=15$, there is

$$\boldsymbol{\alpha}=\begin{pmatrix}15.4 & 9.38 & 7.36 & 5.34 & 3.32\\ 0 & 14.38 & 12.36 & 10.34 & 8.32\\ 0 & 0 & 13.36 & 11.34 & 9.32\\ 0 & 0 & 0 & 12.34 & 10.32\\ 0 & 0 & 0 & 0 & 12.32\end{pmatrix}$$

4. Parameter set 4: α_{12} and α_{45} are close to parallel revenues α_{11} and α_{44}, respectively. However, α_{23} and α_{34} are much smaller than α_{22} and α_{33}.

$$(r_1,\cdots,r_5)=(26,21,17,14,12)$$
$$(u_1,\cdots,u_5)=(11,7,4,2,1)$$
$$(c_1,\cdots,c_5)=(1.624,1.604,1.584,1.564,1.544).$$

For $T=15$,

$$\boldsymbol{\alpha}=\begin{pmatrix}15.4 & 13.38 & 8.36 & 4.34 & 2.32\\ 0 & 14.38 & 9.36 & 5.34 & 3.32\\ 0 & 0 & 13.36 & 9.34 & 7.32\\ 0 & 0 & 0 & 12.34 & 10.32\\ 0 & 0 & 0 & 0 & 11.32\end{pmatrix}$$

There are totally 4 parameter combinations.

D.2.3. Initial capacity Similar to the first numerical study, we use $\mathbf{X}_{\text{RCEC}}$ to construct the following five patterns of initial capacity.

1. $\mathbf{X}=\lambda\mathbf{X}_{\text{RCEC}}$
2. For each $i,j\in\{1,2,3,4,5\}$ with $i<j$: $(\mathbf{X})_i=\lambda\left((\mathbf{X}_{\text{RCEC}})_i+(\mathbf{X}_{\text{RCEC}})_j\right)$, $(\mathbf{X})_j=0$, $(\mathbf{X})_s=\lambda(\mathbf{X}_{\text{RCEC}})_s$, $\forall\ s\in\{1,2,3,4,5\}\setminus\{i,j\}$
3. For each $i\in\{2,3,4,5\}$: $(\mathbf{X})_i=0$, $(\mathbf{X})_s=\lambda(\mathbf{X}_{\text{RCEC}})_s$, $\forall\ s\in\{1,2,3,4,5\}\setminus\{i\}$
4. $\mathbf{X}=\lambda\left((\mathbf{X}_{\text{RCEC}})_1+(\mathbf{X}_{\text{RCEC}})_2,0,(\mathbf{X}_{\text{RCEC}})_3+(\mathbf{X}_{\text{RCEC}})_4,0,(\mathbf{X}_{\text{RCEC}})_5\right)$
5. $\mathbf{X}=\lambda\left((\mathbf{X}_{\text{RCEC}})_1,(\mathbf{X}_{\text{RCEC}})_2+(\mathbf{X}_{\text{RCEC}})_4,0,(\mathbf{X}_{\text{RCEC}})_3+(\mathbf{X}_{\text{RCEC}})_5,0\right)$,

where $\lambda=\{0.7,0.9,1.0,1.1,1.3\}$. Again the parameter λ is used to adjust the capacity-demand ratio (e.g., $\lambda=0.7$ implies that the aggregate capacity level is relatively low). For each λ, there are 17 initial capacity scenarios; so there are totally 85 capacity scenarios.

To summarize, we test $39*4*85=13260$ problem instances by the above construction. They cover a wide range of possible situations that may arise in practice.

D.3. Impact of Allocation Mechanism: Suboptimal k-Step Upgrading

We examine the profit loss of adopting suboptimal, k-step upgrading ($k = 0, \cdots, N-2$). Given $\mathbf{X}_{\text{RCEC}}$ as the optimal initial capacity under full-step upgrading, define the profit loss of using only k-step upgrading as

$$\Delta_{k-step} = \left| \frac{\Pi_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}}) - \Pi^k_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}})}{\Pi_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}})} \right| * 100\%, \qquad k = 0, 1, 2, 3,$$

where $\Pi^k_{\text{RCEC}}(\mathbf{X}_{\text{RCEC}})$ is the revenue from the k-step upgrading. The statistics are presented in Table EC.3. We can see that the magnitudes of profit losses are still generally much larger than those for $\Delta_{\mathbf{X}_{\text{CB}}}$ and $\Delta_{\mathbf{X}_{\text{NV}}}$ (given in Table 6).

	Mean	Std.	Median	90%-percentile	Max.
Δ_{0-step}	4.29	4.28	2.85	9.36	31.80
Δ_{1-step}	1.08	1.39	0.45	3.37	7.72
Δ_{2-step}	0.33	0.56	0.091	1.04	3.54
Δ_{3-step}	0.10	0.25	0.01	0.29	1.98

Table EC.3 Profit loss of suboptimal allocation with k-step upgrading.

Appendix E: Additional Details

The following lemma shows the relation between $\mathbf{N}$ and its effective state $\hat{\mathbf{N}}$.

LEMMA EC.6. *Suppose $\hat{\mathbf{N}} = (\hat{n}_1, \cdots, \hat{n}_N)$ is the effective state of $\mathbf{N} = (n_1, \cdots, n_N)$, then $\sum_{s=i}^j \hat{n}_s \le \sum_{s=i}^j n_s$ if $\hat{n}_i > 0$, and $\sum_{s=i}^j \hat{n}_s \ge \sum_{s=i}^j n_s$ if $\hat{n}_{j+1} > 0$. Especially, $\sum_{s=i}^N \hat{n}_s \ge \sum_{s=i}^N n_s$.*

Proof. The proof follows from the definition of the effective state. For any class k ($1 \le k \le N$), when applying the greedy upgrading to $\mathbf{N}$, there is no upgrade between classes $(1, \cdots, k-1)$ and $(k, \cdots, N)$ if $\hat{n}_k > 0$, and such an upgrade may exist if $\hat{n}_k \le 0$. Hence, $\sum_{s=i}^j \hat{n}_s \le \sum_{s=i}^j n_s$ if $\hat{n}_i > 0$, and $\sum_{s=i}^j \hat{n}_s \ge \sum_{s=i}^j n_s$ if $\hat{n}_{j+1} > 0$. The same argument shows that $\sum_{s=i}^N \hat{n}_s \ge \sum_{s=i}^N n_s$. $\square$

The next proposition shows that separation can be preserved under the effective state operation.

PROPOSITION EC.1. *Suppose $\hat{\mathbf{N}} = (\hat{n}_1, \cdots, \hat{n}_N)$ is the effective state of $\mathbf{N} = (n_1, \cdots, n_N)$. For any demand realization $\mathbf{D}$, class i ($i < N$) is the lowest class which is separable in $\mathbf{N} - \mathbf{D}$ if and only if i is the lowest class which is separable in $\hat{\mathbf{N}} - \mathbf{D}$.*

Proof. Suppose class i is the lowest separable class in $\mathbf{N} - \mathbf{D}$ but is not separable in $\hat{\mathbf{N}} - \mathbf{D}$. Then, there exists class a $k \le i$ such that $\sum_{s=k}^i (\hat{n}_s - d_s) > 0$. First, we must have $k < i$; otherwise, $n_i \ge \hat{n}_i > d_i \ge 0$, which means class i is not separable in $\mathbf{N} - \mathbf{D}$ and is a contradiction. Given $k < i$, without loss of generality, we assume k is the lowest class with $\sum_{s=k}^i (\hat{n}_s - d_s) > 0$, which also implies $\hat{n}_k > d_k \ge 0$ since $\sum_{s=k+1}^i (\hat{n}_s - d_s) \le 0$. Thus, $\sum_{s=k}^i n_s \ge \sum_{s=k}^i \hat{n}_s$ by Lemma EC.6, and

$\sum_{s=k}^{i}(n_s - d_s) \geq \sum_{s=k}^{i}(\hat{n}_s - d_s) > 0$ which contradicts the assumption of class i being separable in $\mathbf{N}-\mathbf{D}$. Hence, class i must be separable in $\hat{\mathbf{N}}-\mathbf{D}$ as well.

Next, we prove that i is the lowest separable class in $\hat{\mathbf{N}}-\mathbf{D}$. Suppose to the contrary that class $i' > i$ is the lowest separable class in $\hat{\mathbf{N}}-\mathbf{D}$, i.e., $\sum_{s=k}^{i'}(\hat{n}_s - d_s) \leq 0$ for all classes k $(1 \leq k \leq i')$. Then, $\hat{n}_{i'+1} - d_{i'+1} > 0$; otherwise, $i'+1$ will be the lowest separable class. Because class i is the lowest separable class in $\mathbf{N}-\mathbf{D}$ and $i' > i$, there exists class $r \leq i'$ such that $\sum_{s=r}^{i'}(n_s - d_s) > 0$. Given $\hat{n}_{i'+1} > d_{i'+1} \geq 0$ and Lemma EC.6, there is $\sum_{s=r}^{i'}\hat{n}_s \geq \sum_{s=r}^{i'} n_s$ and $\sum_{s=r}^{i'}(\hat{n}_s - d_s) \geq \sum_{s=r}^{i'}(n_s - d_s) > 0$, which is a contradiction since $\sum_{s=r}^{i'}(\hat{n}_s - d_s) \leq 0$. Therefore, class i is the lowest separable class in $\hat{\mathbf{N}}-\mathbf{D}$.

The necessary condition can be similarly proved. This completes the proof. □

For any demand realization $\mathbf{D}$ in period t $(1 \leq t \leq T)$, let $\hat{\mathbf{N}}$ be the effective state of $\mathbf{N}$, Proposition EC.2 gives the relation between the outcomes of applying the PSR policy to initial states $(\mathbf{N}, \mathbf{D})$ and $(\hat{\mathbf{N}}, \mathbf{D})$.

Proposition EC.2. *Suppose* $\hat{\mathbf{N}} = (\hat{n}_1, \cdots, \hat{n}_N)$ *is the effective state of* $\mathbf{N} = (n_1, \cdots, n_N)$*, and the PSR policy solves the general upgrading problem in period* t $(1 \leq t \leq T)$*. For any demand realization* $\mathbf{D}$ *in period* t*, let* $\mathbf{N}' = (n'_1, \cdots, n'_N)$ *and* $\hat{\mathbf{N}}' = (\hat{n}'_1, \cdots, \hat{n}'_N)$ *be the effective states of the outcomes of applying the PSR policy to* $(\mathbf{N}, \mathbf{D})$ *and* $(\hat{\mathbf{N}}, \mathbf{D})$*, respectively. Then,* $\mathbf{N}' = \hat{\mathbf{N}}'$ *if classes* $(1, \cdots, N)$ *are not separable under* $\mathbf{N}-\mathbf{D}$*. Especially,* $\mathbf{N}'$ *and* $\hat{\mathbf{N}}'$ *are the outcomes of applying the PSR policy to* $(\mathbf{N}, \mathbf{D})$ *and* $(\hat{\mathbf{N}}, \mathbf{D})$ *in period* T*.*

Proof. From Proposition EC.1, classes $(1, \cdots, N)$ are also not separable under $\hat{\mathbf{N}}-\mathbf{D}$.

First, we must have $\mathbf{N}' \geq 0$. Suppose to the contrary that class k is the highest class with $n'_k < 0$. Since $\mathbf{N}'$ is the effective state of the outcome of applying the PSR to $(\mathbf{N}, \mathbf{D})$, there is $n'_1 = \cdots = n'_{k-1} = 0$. Note that any allocation decision is a transfer between two classes, which is true in both the PSR and the greedy upgrading. Thus, we have $0 > \sum_{s=r}^{k} n'_s \geq \sum_{s=r}^{k}(n_s - d_s)$ for any class $r < k$, where the second inequality is strict if there is any upgrade between classes $(1, \cdots, r-1)$ and $(r, \cdots, k)$ when applying the PSR or generating the effective state. Hence, class k is separable, which contradicts the assumption. Similarly, we know $\hat{\mathbf{N}}' \geq 0$.

Next, we show $\mathbf{N}' = \hat{\mathbf{N}}'$. Let class k be the lowest class such that $\hat{n}'_k \neq n'_k$. From the above argument, we have $\sum_{s=k}^{N} n'_s = \sum_{s=k}^{N}(n_s - d_s)$ if there is no upgrade between classes $(1, \cdots, k-1)$ and $(k, \cdots, N)$ in either solving $(\mathbf{N}, \mathbf{D})$ by the PSR or generating the effective state $\mathbf{N}'$. Furthermore, such an upgrade exists only if $n'_k = 0$ by the optimality of the PSR and the definition of the greedy upgrading, in which case $\sum_{s=k}^{N} n'_s > \sum_{s=k}^{N}(n_s - d_s)$. The same argument can be applied to $(\hat{\mathbf{N}}, \mathbf{D})$. With these observations, we derive contradictions for all possible cases.

1. $\sum_{s=k}^{N} \hat{n}_s = \sum_{s=k}^{N} n_s$.

(a) $\hat{n}'_k > 0, n'_k > 0$: For both initial states $(\mathbf{N}, \mathbf{D})$ and $(\hat{\mathbf{N}}, \mathbf{D})$, there is no upgrade between classes $(1, \cdots, k-1)$ and $(k, \cdots, N)$ in either applying the PSR or generating the effective state. Moreover, $\hat{n}_k \geq \hat{n}'_k > 0$ implies that $\sum_{s=k}^N \hat{n}_s = \sum_{s=k}^N n_s$ by Lemma EC.6. Thus,

$$\sum_{s=k}^{N}(\hat{n}_s - d_s) = \sum_{s=k}^{N} \hat{n}'_s \neq \sum_{s=k}^{N} n'_s = \sum_{s=k}^{N}(n_s - d_s),$$

which is a contradiction;

(b) $\hat{n}'_k > n'_k = 0$: Similar to the previous case, since $\hat{n}_k \geq \hat{n}'_k > 0$, then

$$\sum_{s=k}^{N} \hat{n}'_s = \sum_{s=k}^{N}(\hat{n}_s - d_s) = \sum_{s=k}^{N}(n_s - d_s) \leq \sum_{s=k}^{N} n'_s,$$

which violates the assumption of class k;

(c) $n'_k > \hat{n}'_k = 0$: From Lemma EC.6, we similarly have

$$\sum_{s=k}^{N} \hat{n}'_s \geq \sum_{s=k}^{N}(\hat{n}_s - d_s) \geq \sum_{s=k}^{N}(n_s - d_s) = \sum_{s=k}^{N} n'_s,$$

which is also a contradiction;

2. $\sum_{s=k}^N \hat{n}_s > \sum_{s=k}^N n_s$. In this case, $\hat{n}_k = 0$ by the definition of the effective state $\hat{\mathbf{N}}$. Since $\hat{n}'_k \geq 0$ from the previous discussion, there is $\hat{n}_k = \hat{n}'_k = 0$. Meanwhile, $n'_k \neq \hat{n}'_k$ by the assumption of k. From $n'_k \geq 0$, we must have $n'_k > \hat{n}'_k = 0$, and

$$\sum_{s=k}^{N} \hat{n}'_s \geq \sum_{s=k}^{N}(\hat{n}_s - d_s) > \sum_{s=k}^{N}(n_s - d_s) = \sum_{s=k}^{N} n'_s,$$

where the first inequality is from the fact that there might be upgrade between classes $(1, \cdots, k-1)$ and $(k, \cdots, N)$ while solving $(\hat{\mathbf{N}}, \mathbf{D})$ by the PSR and generating the effective state $\hat{\mathbf{N}}'$. This is a contradiction since $n'_k > \hat{n}'_k$ and $\sum_{s=k+1}^N n'_s = \sum_{s=k+1}^N \hat{n}'_s$ by assumption of k.

Therefore, $\mathbf{N} = \hat{\mathbf{N}}'$. Note that the PSR optimally solves the general upgrading problem with protection levels being 0 in period T by Lemma EC.3. Since the greedy upgrading is equivalent to the PSR with protection levels being 0, we know $\mathbf{N}'$ and $\hat{\mathbf{N}}'$ are the outcomes of applying the PSR policy to $(\mathbf{N}, \mathbf{D})$ and $(\hat{\mathbf{N}}, \mathbf{D})$ in period T, which completes the proof. □

Lemma EC.7 considers a general upgrading problem with special states in period T, which can be used to simplify the proof of Lemma EC.3.

LEMMA EC.7. *Consider an N-class general upgrading problem with states $\mathbf{N} = (n_1, \cdots, n_N)$ and demand realization $\mathbf{D}$ in period T. Suppose classes $(1, \cdots, N)$ are not separable based on $\mathbf{N} - \mathbf{D}$, $(n_{i+1}, \cdots, n_{j-1}) \leq 0$ and $n_j < 0$. Then, $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})$ and $\Delta_{ij}^{-+}\Theta^T(\mathbf{N}|\mathbf{D})$ are independent of the values of $(n_j, \cdots, n_N)$.*

Proof. Since $\Theta^T(\mathbf{N}|\mathbf{D})$ is piecewise linear and concave (see Murty 1983), both $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})$ and $\Delta_{ij}^{-+}\Theta^T(\mathbf{N}|\mathbf{D})$ exist.

We focus on the proof of Δ_{ij}^{+-}, and the same argument applies to Δ_{ij}^{-+}. We consider the dual form of the general upgrading problem with initial state $(\mathbf{N},\mathbf{D})$, and let the dual variables be $(\lambda_1,\cdots,\lambda_N)$, where λ_i corresponds to the constraint of class i. The dual problem is

$$\begin{aligned}\min_{(\lambda_1,\cdots,\lambda_N)\geq 0} \quad & \sum_{s=1}^{N}|n_s-d_s|\lambda_s \\ \text{s.t.} \quad & \lambda_s+\lambda_r\geq\alpha_{sr}, \qquad s,r\in\{s,r|(\mathbf{N}-\mathbf{D})_s\geq 0,(\mathbf{N}-\mathbf{D})_r<0,\ 1\leq s<r\leq N\}.\end{aligned} \tag{EC.23}$$

1. $n_i\geq 0$: By Linear Programming theory, there is
$$\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=\begin{cases}\lambda_i^*+\lambda_j^*-g_j, & \text{if } n_i\geq d_i\\ -\lambda_i^*+\lambda_j^*+\alpha_{ii}-g_j, & \text{if } n_i<d_i,\end{cases}$$
where $\lambda^*=(\lambda_1^*,\cdots,\lambda_N^*)$ is optimal in the dual problem (EC.23).
   (a) $n_i>d_i$: Given classes $(1,\cdots,N)$ are not separable, we have $y_{kj}^*>0$ for some class k $(1\leq k\leq i)$. By the complementary slackness in the linear program, $\lambda_k^*+\lambda_j^*=\alpha_{kj}$. Assume without loss of generality that $i+1$ is the highest class s $(i+1\leq s\leq j-1)$ with $n_s-d_s<0$. Since it is optimal to first use class i's remaining capacity n_i-d_i to satisfy demands from $(i+1,\cdots,j)$, there is $y_{i,i+1}^*>0$, which implies $\lambda_i^*+\lambda_{i+1}^*=\alpha_{i,i+1}$. By examining constraints $\lambda_i^*+\lambda_j^*\geq\alpha_{ij}$, $\lambda_k^*+\lambda_{i+1}^*\geq\alpha_{k,i+1}$ in the dual problem (EC.23), as well as the assumption $\alpha_{kj}+\alpha_{i,i+1}=\alpha_{ij}+\alpha_{k,i+1}$, we have $\lambda_i^*+\lambda_j^*=\alpha_{ij}$ and $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=\lambda_i^*+\lambda_j^*-g_j=\alpha_{ij}-g_j$. Note that $y_{ij}>0$ if $n_s-d_s=0$ for all classes s $(i+1\leq s\leq j-1)$, which implies $\lambda_i^*+\lambda_j^*-g_j=\alpha_{ij}-g_j$.
   (b) $n_i<d_i$: The non-separable assumption implies that there exist classes r $(r<i)$ and k $(k<i)$ such that $y_{ri}^*>0$ and $y_{kj}^*>0$. Thus, $\lambda_k^*+\lambda_j^*=\alpha_{kj}$ and $\lambda_r^*+\lambda_i^*=\alpha_{ri}$. We similarly have $\lambda_r^*+\lambda_j^*=\alpha_{rj}$ and $\lambda_k^*+\lambda_i^*=\alpha_{ki}$ by using the constraints in (EC.23) and the assumption $\alpha_{kj}+\alpha_{ri}=\alpha_{rj}+\alpha_{ki}$. Thus, $-\lambda_i^*+\lambda_j^*=\alpha_{rj}-\alpha_{ri}$ and $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=-\lambda_i^*+\lambda_j^*+\alpha_{ii}-g_j=\alpha_{ij}-g_j$.

   Since $\Theta^T(\mathbf{N}|\mathbf{D})$ is piecewise linear in n_i and n_j, then $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=\alpha_{ij}-g_j$ when $n_i\geq 0$.
2. $n_i<0$: In this case,
$$\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=-\lambda_i^*+\lambda_j^*+g_i-g_j.$$
Note that this is similar to the case when $0\leq n_i<d_i$. Hence, $-\lambda_i^*+\lambda_j^*=\alpha_{rj}-\alpha_{ri}$ and $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=r_j-r_i$.

Hence, $\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})$ is independent of the values of $(n_j,\cdots,n_N)$, which concludes the proof. □

Suppose the PSR is optimal in period t. Then similar to Proposition EC.2, the following proposition shows the relation between the outcomes of $\mathbf{N}$ and its effective state $\hat{\mathbf{N}}$ after applying the PSR given any demand realization $\mathbf{D}$.

Proposition EC.3. *Suppose $\hat{\mathbf{N}} = (\hat{n}_1, \cdots, \hat{n}_N)$ is the effective state of $\mathbf{N} = (n_1, \cdots, n_N)$. If the PSR policy solves the general upgrading problem in period t, and the protection levels in period t have the independence property. For any demand realization $\mathbf{D}$, let $\mathbf{N}' = (n'_1, \cdots, n'_N)$ and $\hat{\mathbf{N}}' = (\hat{n}'_1, \cdots, \hat{n}'_N)$ be the outcomes of applying the PSR policy to $(\mathbf{N}, \mathbf{D})$ and $(\hat{\mathbf{N}}, \mathbf{D})$, respectively.*

Let k be the highest class in $\mathbf{N}'$ such that $(n'_k, \cdots, n'_N) \geq 0$ and $n'_k > 0$, where $k = N+1$ if such a class does not exist in $\mathbf{N}'$. $\hat{k}$ is similarly defined in $\hat{\mathbf{N}}'$. Then, $k = \hat{k}$ and $(\mathbf{N}')_{k,\cdots,N} = (\hat{\mathbf{N}}')_{k,\cdots,N}$ if classes $(1, \cdots, N)$ are not separable under $\mathbf{N} - \mathbf{D}$.

Proof. We first show $(\hat{\mathbf{N}}')_{k,\cdots,N} = (\mathbf{N}')_{k,\cdots,N}$. Let i $(k \leq i \leq N)$ be the lowest class such that $\hat{n}'_i \neq n'_i \geq 0$. There are three cases.

1. $\hat{n}'_i > n'_i \geq 0$: Since capacity i may be used when applying the PSR, we have $\hat{n}_i \geq \hat{n}'_i > 0$, which implies $\sum_{s=i}^N \hat{n}_s = \sum_{s=i}^N n_s$ by Lemma EC.6. Because $\hat{n}'_i > 0$, there is no upgrade from classes $(1, \cdots, i-1)$ to $(i, \cdots, N)$ when applying the PSR to $(\hat{\mathbf{N}}, \mathbf{D})$, and $\sum_{s=i}^N (\hat{n}_s - d_s) = \sum_{s=i}^N \hat{n}'_s$. On the other hand, $n'_i \geq 0$ implies that there could be upgrade from classes $(1, \cdots, i-1)$ to $(i, \cdots, N)$ when solving $(\mathbf{N}, \mathbf{D})$, thus $\sum_{s=i}^N n'_s \geq \sum_{s=i}^N (n_s - d_s)$. From the above, we have
$$\sum_{s=i}^N n'_s \geq \sum_{s=i}^N (n_s - d_s) = \sum_{s=i}^N (\hat{n}_s - d_s) = \sum_{s=i}^N \hat{n}'_s.$$
This is a contradiction given the assumption of class i.
2. $n'_i > \hat{n}'_i$ and $n'_i > 0$: In this case, $n'_i > 0$ implies that there is no upgrade from classes $(1, \cdots, i-1)$ to $(i, \cdots, N)$ when applying the PSR to $(\mathbf{N}, \mathbf{D})$. Thus, $\sum_{s=i}^N n'_s = \sum_{s=i}^N (n_s - d_s)$. However, there could be upgrade between classes $(1, \cdots, i-1)$ and $(i, \cdots, N)$ when generating $\hat{\mathbf{N}}$ as well as applying the PSR to $(\hat{\mathbf{N}}, \mathbf{D})$, thus
$$\sum_{s=i}^N n'_i = \sum_{s=i}^N (n_s - d_s) \leq \sum_{s=i}^N (\hat{n}_s - d_s) \leq \sum_{s=i}^N \hat{n}'_s, \tag{EC.24}$$
which is a contradiction.
3. $n'_i = 0 > \hat{n}'_i$ and $i > k$: From (EC.24), we only need to consider the case when $\sum_{s=i}^N n'_i > \sum_{s=i}^N (n_s - d_s)$, i.e., there is upgrade from classes $(1, \cdots, i-1)$ to $(i, \cdots, N)$ when applying the PSR to $(\mathbf{N}, \mathbf{D})$. Without loss of generality, we assume that $i-1$ is the highest class that upgrades the demands in classes $(i, \cdots, N)$ under initial state $(\mathbf{N}, \mathbf{D})$, and l $(l \geq i)$ is the lowest class being upgraded by capacity $i-1$. Since there is no upgrade from classes $(1, \cdots, i-2)$ to $(i-1, \cdots, N)$ when solving $(\mathbf{N}, \mathbf{D})$, similar to (EC.24), there is
$$\sum_{s=i-1}^N n'_i = \sum_{s=i-1}^N (n_s - d_s) \leq \sum_{s=i-1}^N (\hat{n}_s - d_s) \leq \sum_{s=i-1}^N \hat{n}'_s. \tag{EC.25}$$

Since $n'_i = 0 > \hat{n}'_i$ and $(\hat{\mathbf{N}}')_{i+1,\cdots,N} = (\mathbf{N}')_{i+1,\cdots,N} \geq 0$ by assumption of class i, (EC.25) implies $\hat{n}'_{i-1} > n'_{i-1} \geq 0$. Moreover, $\hat{n}_{i-1} > 0$ if $\hat{n}'_{i-1} > 0$.

Next, we show that the profit can be increased by upgrading demand i by capacity $i-1$ under $(\hat{\mathbf{N}}, \mathbf{D})$, which violates the optimality assumption of the PSR. Since $\hat{n}'_{i-1} > 0$ and the assumption of class $i-1$, there is no upgrade between classes $(1, \cdots, i-2)$ and $(i-1, \cdots, N)$ when generating the effective state $\hat{\mathbf{N}}$ as well as applying the PSR to both $(\hat{\mathbf{N}}, \mathbf{D})$ and $(\mathbf{N}, \mathbf{D})$. From Proposition EC.2, given classes $(1, \cdots, N)$ are not separable under $\mathbf{N} - \mathbf{D}$, the effective states of $(\hat{\mathbf{N}}')_{1,\cdots,i-2}$ are the same as those of $(\mathbf{N}')_{1,\cdots,i-2}$.

If $l = i$, from the independence property of the protection levels, $p_{i-1,i}$ is the same for both $(\hat{\mathbf{N}}, \mathbf{D})$ and $(\mathbf{N}, \mathbf{D})$. Because $\hat{n}'_{i-1} > n'_{i-1}$ and capacity $i-1$ upgrades demand i under $(\mathbf{N}, \mathbf{D})$, it is also optimal to upgrade demand i by capacity $i-1$ under $(\hat{\mathbf{N}}, \mathbf{D})$.

If $l > i$, we have $(\hat{\mathbf{N}}')_{i+1,\cdots,l} = (\mathbf{N}')_{i+1,\cdots,l}$ by the assumption of class i. Moreover, $(\mathbf{N}')_{i+1,\cdots,l} = 0$ since capacity $i-1$ upgrades demand l under initial state $(\mathbf{N}, \mathbf{D})$, and n'_{i-1} is the remaining capacity after such upgrading. From the PSR, there is

$$\begin{aligned}
\alpha_{i-1,l} \geq& \Delta^{+-}_{i-1,l}\Theta^{t+1}(\mathbf{N}') \\
=& \Delta^{+-}_{i-1,l}\Theta^{t+1}\left((\mathbf{N}')_{1,\cdots,i-2}, n'_{i-1}, 0, \cdots, 0, (\mathbf{N}')_{l+1,\cdots,N}\right) \\
=& \Delta^{+-}_{i-1,l}\Theta^{t+1}((\hat{\mathbf{N}}')_{1,\cdots,i-2}, n'_{i-1}, 0, \cdots, 0, (\hat{\mathbf{N}}')_{l+1,\cdots,N}) \\
\geq& r_l - r_i + \partial^+_{i-1}\Theta^{t+1}((\hat{\mathbf{N}}')_{1,\cdots,i-2}, n'_{i-1}, 0, \cdots, 0, (\hat{\mathbf{N}}')_{l+1,\cdots,N}) \\
&- \partial^-_i\Theta^{t+1}((\hat{\mathbf{N}}')_{1,\cdots,i-2}, n'_{i-1}, 0, \cdots, 0, (\hat{\mathbf{N}}')_{l+1,\cdots,N}) \\
=& r_l - r_i + \Delta^{+-}_{i-1,i}\Theta^{t+1}((\hat{\mathbf{N}}')_{1,\cdots,i-2}, n'_{i-1}, 0, \cdots, 0, (\hat{\mathbf{N}}')_{l+1,\cdots,N}),
\end{aligned} \tag{EC.26}$$

where the second equality follows from the independence property of Θ^{t+1} and the fact that the effective states of $(\mathbf{N}')_{1,\cdots,i-2}$ and $(\hat{\mathbf{N}}')_{1,\cdots,i-2}$ are the same, and the last inequality is because of Lemma EC.2.

Since $\alpha_{i-1,i} - \alpha_{i-1,l} = r_i + g_i - r_l - g_l$, where $g_i > g_l$, $\hat{n}'_{i-1} > n'_{i-1}$, and Θ^{t+1} is concave, we have

$$\begin{aligned}
\alpha_{i-1,i} >& \Delta^{+-}_{i-1,i}\Theta^{t+1}((\hat{\mathbf{N}}')_{1,\cdots,i-2}, n'_{i-1}, 0, \cdots, 0, (\hat{\mathbf{N}}')_{l+1,\cdots,N}) \\
\geq& \Delta^{+-}_{i-1,i}\Theta^{t+1}((\hat{\mathbf{N}}')_{1,\cdots,i-2}, \hat{n}'_{i-1}, n'_{i-1} - \hat{n}'_{i-1}, 0, \cdots, 0, (\hat{\mathbf{N}}')_{l+1,\cdots,N}) \\
=& \Delta^{+-}_{i-1,i}\Theta^{t+1}((\hat{\mathbf{N}}')_{1,\cdots,i-2}, \hat{n}'_{i-1}, \hat{n}'_i, 0, \cdots, 0, (\hat{\mathbf{N}}')_{l+1,\cdots,N}).
\end{aligned} \tag{EC.27}$$

Thus, the profit can be increased by upgrading demand i with capacity $i-1$ under $(\hat{\mathbf{N}}, \mathbf{D})$, which contradicts the optimality assumption of the PSR policy.

Hence, $(\hat{\mathbf{N}}')_{k,\cdots,N} = (\mathbf{N}')_{k,\cdots,N}$. Similarly, we know $(\mathbf{N}')_{\hat{k},\cdots,N} = (\hat{\mathbf{N}}')_{\hat{k},\cdots,N}$, which concludes the proof. □

E.1. Monotonicity

To prove the monotonicity result in Proposition 5, we start with a basic property.

Under certain conditions, the following lemma states that the marginal values, $\Delta_{ij}^{+-}\Theta^t$ $(i<j)$ and $\Delta_{ij}^{-+}\Theta^t$, remain the same if capacity k $(k<i)$ is used to "optimally" upgrade the backlogged demand i. Note that such an upgrade can go beyond class k as long as there is unmet demand i.

Lemma EC.8. *Suppose $(\hat{n}_1,\cdots,\hat{n}_{i-1})$ is the effective state of $(n_1,\cdots,n_{i-1})$, and there exists class k $(1\le k<i)$ such that $\hat{n}_k>0$ and $\hat{n}_{k+1}=\cdots=\hat{n}_{i-1}=0$. If $(n_i,\cdots,n_j)\le 0$, $\delta>0$, and $n_i+\delta\le 0\le \hat{n}_k-\delta$, then*

$$\Delta_{ij}^{+-}\Theta^t(n_1,\cdots,n_N)=\Delta_{ij}^{+-}\Theta^t(\hat{n}_1,\cdots,\hat{n}_{k-1},\hat{n}_k-\delta,0,\cdots,0,n_i+\delta,n_{i+1},\cdots,n_N) \tag{EC.28}$$

and

$$\Delta_{ij}^{-+}\Theta^t(n_1,\cdots,n_N)=\Delta_{ij}^{-+}\Theta^t(\hat{n}_1,\cdots,\hat{n}_{k-1},\hat{n}_k-\delta,0,\cdots,0,n_i+\delta,n_{i+1},\cdots,n_N). \tag{EC.29}$$

Proof. It is sufficient to prove the equality in (EC.28). From Proposition 2, there is

$$\Delta_{ij}^{+-}\Theta^t(n_1,\cdots,n_N)=\Delta_{ij}^{+-}\Theta^t(\hat{n}_1,\cdots,\hat{n}_k,0,\cdots,0,n_i,\cdots,n_N).$$

Thus, for any demand realization $\mathbf{D}$ in period t, we use induction to show

$$\begin{aligned}&\Delta_{ij}^{+-}\Theta^t(\hat{n}_1,\cdots,\hat{n}_k,0,\cdots,0,n_i,\cdots,n_N|\mathbf{D})\\ =&\Delta_{ij}^{+-}\Theta^t(\hat{n}_1,\cdots,\hat{n}_{k-1},\hat{n}_k-\delta,0,\cdots,0,n_i+\delta,n_{i+1},\cdots,n_N|\mathbf{D})\end{aligned} \tag{EC.30}$$

under the conditions given in Lemma EC.8. To simplify our notations, let

$$\begin{aligned}\mathbf{N}&=(\hat{n}_1,\cdots,\hat{n}_k,0,\cdots,0,n_i,\cdots,n_N)\\ \bar{\mathbf{N}}&=(\hat{n}_1,\cdots,\hat{n}_{k-1},\hat{n}_k-\delta,0,\cdots,0,n_i+\delta,n_{i+1},\cdots,n_N).\end{aligned}$$

In period T, let r^* $(1\le r^*\le k)$ be the lowest class such that $n_i+\sum_{s=r^*}^k\hat{n}_s\ge\sum_{s=r^*}^i d_s$, i.e., r^* is the lowest class that satisfies the last unit of demand i. We analyze (EC.30) based on following cases.

1. r^* does not exist: Then $n_i+\sum_{s=r}^k\hat{n}_s<\sum_{s=r}^i d_s$ and $n_i-\delta+\sum_{s=r}^k\hat{n}_s<-\delta+\sum_{s=r}^i d_s$ for all class r $(1\le r\le k)$. After applying the PSR, there is unmet demand i in both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$. Thus, given $(n_{i+1},\cdots,n_j)\le 0$, we have

$$\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=\Delta_{ij}^{+-}\Theta^T(\bar{\mathbf{N}}|\mathbf{D})=g_i-g_j.$$

2. r^* does exist: For both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$, since $n_i-\delta+\sum_{s=r^*}^{k}\hat{n}_s\ge -\delta+\sum_{s=r^*}^{i}d_s$, the last unit of demand i is fulfilled by capacity r^*. And the states of classes $(r^*,\cdots,i)$ after the last unit of demand i being satisfied are $(n_i+\sum_{s=r^*}^{k}\hat{n}_s-\sum_{s=r^*}^{i}d_s,0,\cdots,0)$. Hence,

$$\Delta_{ij}^{+-}\Theta^T(\mathbf{N}|\mathbf{D})=\Delta_{ij}^{+-}\Theta^T(\bar{\mathbf{N}}|\mathbf{D})=g_i-\alpha_{r^*i}+\frac{\partial}{\partial n_i^+}\Theta^T(\tilde{\mathbf{N}}|\tilde{\mathbf{D}})-\frac{\partial}{\partial n_j^-}\Theta^T(\tilde{\mathbf{N}}|\tilde{\mathbf{D}}),$$

where $\tilde{\mathbf{N}}=(\hat{n}_1,\cdots,\hat{n}_{r^*-1},n_i+\sum_{s=r^*}^{k}\hat{n}_s-\sum_{s=r^*}^{i}d_s,0,\cdots,0,n_{i+1},\cdots,n_N)$ and $\tilde{\mathbf{D}}=((\mathbf{D})_{1,\cdots,r^*-1},0,\cdots,0,(\mathbf{D})_{i+1,\cdots,N})$.

Hence, (EC.30) holds in period T.

In period $t<T$, we apply the PSR policy to the general upgrading problem with initial states $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$, and denote $\mathbf{N}'$ and $\bar{\mathbf{N}}'$ as the corresponding outcomes. We examine (EC.30) based on the states of class i in $\mathbf{N}'$ and $\bar{\mathbf{N}}'$.

1. $(\mathbf{N}')_i=(\bar{\mathbf{N}}')_i=0$: From the above analysis, the last unit of demand i is satisfied by class r^* in both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$, and we assume $r^*=k$ without loss of generality. Hence,

$$\Delta_{ij}^{+-}\Theta^t(\mathbf{N}|\mathbf{D})=\Delta_{ij}^{+-}\Theta^t(\bar{\mathbf{N}}|\mathbf{D})=g_i-\alpha_{ki}+\frac{\partial}{\partial n_i^+}\Theta^t(\tilde{\mathbf{N}}|\tilde{\mathbf{D}})-\frac{\partial}{\partial n_j^-}\Theta^t(\tilde{\mathbf{N}}|\tilde{\mathbf{D}}),$$

where $\tilde{\mathbf{N}}=(\hat{n}_1,\cdots,\hat{n}_{k-1},n_i+\hat{n}_k-\sum_{s=k}^{i}d_s,0,\cdots,0,n_{i+1},\cdots,n_N)$ and $\tilde{\mathbf{D}}=((\mathbf{D})_{1,\cdots,k-1},0,\cdots,0,(\mathbf{D})_{i+1,\cdots,N})$.

2. $(\mathbf{N}')_i<0$ and $(\bar{\mathbf{N}}')_i<0$: If there is no class r^* $(1\le r^*\le k)$ such that $n_i+\sum_{s=r^*}^{k}\hat{n}_s\ge\sum_{s=r^*}^{i}d_s$, demand i and j will never be satisfied in the remaining periods for both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$, which means

$$\Delta_{ij}^{+-}\Theta^t(\mathbf{N}|\mathbf{D})=(T-t+1)(g_i-g_j)=\Delta_{ij}^{+-}\Theta^t(\bar{\mathbf{N}}|\mathbf{D}).$$

Hence, we only need to consider the case when class r^* does exist. In this case, we assume $r^*=k$ without loss of generality. From Proposition 2, since $(\mathbf{N}')_i<0$ and $(\bar{\mathbf{N}}')_i<0$, MP_{ij} will not affect the optimal allocation decisions in period t under both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$. Thus,

$$\begin{aligned}\Delta_{ij}^{+-}\Theta^t(\mathbf{N}|\mathbf{D})&=g_i-g_j+\frac{\partial}{\partial n_i^+}\Theta^{t+1}(\mathbf{N}')-\frac{\partial}{\partial n_j^-}\Theta^{t+1}(\mathbf{N}'),\\ \Delta_{ij}^{+-}\Theta^t(\bar{\mathbf{N}}|\mathbf{D})&=g_i-g_j+\frac{\partial}{\partial n_i^+}\Theta^{t+1}(\bar{\mathbf{N}}')-\frac{\partial}{\partial n_j^-}\Theta^{t+1}(\bar{\mathbf{N}}').\end{aligned}\qquad\text{(EC.31)}$$

By the definition of class k, there is no upgrade between classes $(1,\cdots,k-1)$ and $(k,\cdots,N)$ when applying the PSR under both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$; otherwise, all capacity k should have been depleted before performing the aforementioned upgrade, which means there is no unmet demand i. Since the initial states of classes $(1,\cdots,k-1)$ are the same, the effective states of classes $(1,\cdots,k-1)$ in $\mathbf{N}'$ are the same as those in $\bar{\mathbf{N}}'$. Note that $(\mathbf{N}')_{i+1,\cdots,j}=(\bar{\mathbf{N}}')_{i+1,\cdots,j}=$

$(n_{i+1}-d_{i+1},\cdots,n_j-d_j)\leq 0$ and $n_i+\hat{n}_k-\sum_{s=k}^{i}d_s>0$ by assumption. Applying the induction assumption, we have

$$\begin{aligned}
&\frac{\partial}{\partial n_i^+}\Theta^{t+1}(\bar{\mathbf{N}}')-\frac{\partial}{\partial n_j^-}\Theta^{t+1}(\bar{\mathbf{N}}')\\
=&\Delta_{ij}^{+-}\Theta^{t+1}\Big((\mathbf{N}')_{1,\cdots,k-1},n_i+\hat{n}_k+\delta-\sum_{s=k}^{i}d_s,0,\cdots,0,-\delta,(\mathbf{N}')_{i+1,\cdots,N}\Big)\\
=&\Delta_{ij}^{+-}\Theta^{t+1}\Big((\bar{\mathbf{N}}')_{1,\cdots,k-1},n_i+\hat{n}_k+\delta-\sum_{s=k}^{i}d_s,0,\cdots,0,-\delta,(\bar{\mathbf{N}}')_{i+1,\cdots,N}\Big)\\
=&\frac{\partial}{\partial n_i^+}\Theta^{t+1}(\bar{\mathbf{N}}')-\frac{\partial}{\partial n_j^-}\Theta^{t+1}(\bar{\mathbf{N}}'),
\end{aligned}\tag{EC.32}$$

where $0<\delta<-\max\big((\mathbf{N}')_i,\ (\bar{\mathbf{N}}')_i\big)$ and the second equality follows from Proposition 2. This is a contradiction. Hence, $\Delta_{ij}^{+-}\Theta^t(\mathbf{N}|\mathbf{D})=\Delta_{ij}^{+-}\Theta^t(\bar{\mathbf{N}}|\mathbf{D})$ from (EC.31) and (EC.32).

3. $(\mathbf{N}')_i=0$ and $(\bar{\mathbf{N}}')_i<0$: In this case, there exists a class r^*, which can be assumed as $r^*=k$ without loss of generality. Moreover, the last unit of demand i is upgraded by capacity k when the PSR solves $(\mathbf{N},\mathbf{D})$.

   Given $(\bar{\mathbf{N}}')_i<0$, we must have $(\bar{\mathbf{N}}')_k>(\mathbf{N}')_k\geq 0$ since the total unmet demand after parallel allocation in classes $(k,\cdots,i)$ is the same for both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$. When the last unit of demand i is upgraded by capacity k in $(\mathbf{N},\mathbf{D})$, from the PSR, the upgrading decisions between classes $(1,\cdots,k-1)$ and $(i+1,\cdots,N)$ have not been considered yet. At that moment, the effective state of classes $(1,\cdots,k-1)$ in $(\mathbf{N},\mathbf{D})$ is the same as that in $\bar{\mathbf{N}}'$ because there is also no upgrade between classes $(1,\cdots,k-1)$ and $(k,\cdots,N)$ in $(\bar{\mathbf{N}},\mathbf{D})$ when applying the PSR. Hence, the protection levels between class k and the lower classes are the same for both $(\mathbf{N},\mathbf{D})$ and $(\bar{\mathbf{N}},\mathbf{D})$ from Proposition 2. Let h $(k<h\leq i)$ be the highest class with $(\bar{\mathbf{N}}')_h<0$. Similar to (EC.26) and (EC.27) in the proof of Proposition EC.3, we can show that the profit from solving $(\bar{\mathbf{N}},\mathbf{D})$ can be increased by upgrading demand h with capacity k. This contradicts the optimality of the PSR. Hence, this case cannot exist.

4. $(\mathbf{N}')_i<0$ and $(\bar{\mathbf{N}}')_i=0$: Similar to the previous case, this would lead to a contradiction.

Therefore, (EC.30) holds for any demand realization $\mathbf{D}$ in period t, and this completes the induction proof. □

The following lemma shows that the protection level p_{ij} $(1\leq i<j\leq N)$ in period $t-1$ is decreasing in the states of classes $(1,\cdots,,i-1)$ if the same monotonicity holds in period t.

LEMMA EC.9. *Consider an N-class upgrading problem in period t $(1\leq t<T)$ with $(n_{i+1},\cdots,n_j)\leq 0$. Let $\bar{\mathbf{N}}=\mathbf{N}+\epsilon\mathbf{e}_r$, where $1\leq r<i$ and $\epsilon>0$. Then,*

$$\Delta_{ij}^{+-}\Theta^t(\mathbf{N})\geq\Delta_{ij}^{+-}\Theta^t(\bar{\mathbf{N}}),\qquad \Delta_{ij}^{-+}\Theta^t(\mathbf{N})\geq\Delta_{ij}^{-+}\Theta^t(\bar{\mathbf{N}})\tag{EC.33}$$

if the same inequality holds for Θ^{t+1}.

Proof. To prove (EC.33), it is sufficient to show

$$\Theta^t(\mathbf{N}_{ij}) - \Theta^t(\mathbf{N}) \geq \Theta^t(\bar{\mathbf{N}}_{ij}) - \Theta^t(\bar{\mathbf{N}}), \tag{EC.34}$$

where

$$\begin{aligned}
\mathbf{N}_{ij} &= (n_1, \cdots, n_{i-1}, n_i + 1, n_{i+1}, \cdots, n_{j-1}, n_j - 1, n_{j+1}, \cdots, n_N), \\
\bar{\mathbf{N}} &= (n_1, \cdots, n_{r-1}, n_r + 1, n_{r+1}, \cdots, n_N), \\
\bar{\mathbf{N}}_{ij} &= (n_1, \cdots, n_{r-1}, n_r + 1, n_{r+1}, \cdots, n_{i-1}, n_i + 1, n_{i+1}, \cdots, n_{j-1}, n_j - 1, n_{j+1}, \cdots, n_N).
\end{aligned}$$

In each period t, given any demand realization $\mathbf{D} = (d_1, \cdots, d_N)$, we next show

$$\Delta = \Theta^t(\mathbf{N}_{ij}|\mathbf{D}) - \Theta^t(\mathbf{N}|\mathbf{D}) \geq \Theta^t(\bar{\mathbf{N}}_{ij}|\mathbf{D}) - \Theta^t(\bar{\mathbf{N}}|\mathbf{D}) = \bar{\Delta}, \tag{EC.35}$$

which proves (EC.34).

To compare Δ and $\bar{\Delta}$, we consider upgrading decisions in period t. Denote $\mathbf{R}$ as the resulting states of classes $(1, \cdots, N)$ after applying the PSR under initial state $(\mathbf{N}, \mathbf{D})$, let h be the highest capacity class which upgrades the demand in classes $(i, \cdots, N)$, and l is the lowest demand class that is upgraded by classes $(1, \cdots, i)$. By the definition of classes h and l, we have $h \leq l$, and $h = l$ only if $h = i$ and $l = i$. From the PSR policy, there is neither unmet demand nor remaining capacity between classes h and l in $\mathbf{R}$, i.e. $(\mathbf{R})_{h+1,\cdots,l-1} = \mathbf{0}$. $\bar{\mathbf{R}}$, $\bar{h}$ and $\bar{l}$ are similarly defined under initial state $(\bar{\mathbf{N}}, \mathbf{D})$.

For any classes $1 \leq k < s \leq N$ in period t, the protection level p_{ks} defined in (8) are decreasing in $(n_1, \cdots, n_{k-1})$ since (EC.33) is true for Θ^{t+1}, thus upgrade is more likely to happen under initial state $(\bar{\mathbf{N}}, \mathbf{D})$ rather than $(\mathbf{N}, \mathbf{D})$, i.e., $l \leq \bar{l}$.

Switching from $\mathbf{N}$ ($\bar{\mathbf{N}}$) to $\mathbf{N}_{ij}$ ($\bar{\mathbf{N}}_{ij}$), we not only change the current revenues in period t, but also the result $\mathbf{R}$ ($\bar{\mathbf{R}}$), which is the initial states in period $t+1$. Denote $\mathbf{R}'$ and $\bar{\mathbf{R}}'$ as the outcomes after applying the PSR under $(\mathbf{N}_{ij}, \mathbf{D})$ and $(\bar{\mathbf{N}}_{ij}, \mathbf{D})$, respectively. Then,

$$\Delta = \delta + \Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R}), \qquad \bar{\Delta} = \bar{\delta} + \Theta^{t+1}(\bar{\mathbf{R}}') - \Theta^{t+1}(\bar{\mathbf{R}}),$$

where δ and $\bar{\delta}$ are the corresponding differences of the current period revenues in period t under $(\mathbf{N}, \mathbf{D})$ and $(\bar{\mathbf{N}}, \mathbf{D})$, respectively.

When the initial states change from $\mathbf{N}$ to $\mathbf{N}_{ij}$, there are four cases which differ in the allocation decisions in period t. Note that the analogy applies when initial states change form $\bar{\mathbf{N}}$ to $\bar{\mathbf{N}}_{ij}$. For simplicity, we assume without loss of generality that $(\mathbf{R})_{l+1} < 0$.

Case 1: An extra unit of demand l is satisfied when $l < j$.

Case 2: An extra unit of capacity h is passed along to period $t+1$.

Case 3: An extra unit of demand $l+1$ is satisfied when $l+1 < j$.

Case 4: An extra unit of demand j is satisfied if $l \geq j$.

Here, we explain the above cases in detail by recalling the "chain reaction" described in the proof of Proposition 2.

Case 1: There is unmet demand l in $\mathbf{R}$ in this case. Note that capacity h is the highest class that upgrades demand l under $(\mathbf{N}, \mathbf{D})$. And the upgrade between classes h and l is bounded because either there is no capacity h remaining or the protection level p_{hl} is reached. When the initial state changes from $\mathbf{N}$ to $\mathbf{N}_{ij}$, from the chain reaction, there is an additional unit of capacity h which will upgrade the remaining demand l.

Case 2: In this case, class l demand has been fully satisfied in $\mathbf{R}$; otherwise, the analysis of Case 1 gives a contradiction.

Case 3: If $l+1<j$, similar to *Case 1*, it is possible that an additional unit of demand $l+1$ is upgraded by capacity h under $(\mathbf{N}_{ij}, \mathbf{D})$, in which case all demand l has been satisfied in $\mathbf{R}$.

Case 4: Suppose that k_j is the highest class that upgrades demand j under $(\mathbf{N}, \mathbf{D})$. Because increasing n_i simultaneously decreases n_j by the same amount, there will be an additional unit of both capacity k_j and unmet demand j from the chain reaction. From the PSR, it is optimal to upgrade such an additional demand j by capacity k_j, and the outcome $\mathbf{R}' = \mathbf{R}$ in this case.

To compare Δ and $\bar{\Delta}$, we start with Case 4, where $l \geq j$ and $\mathbf{R}' = \mathbf{R}$ from the above discussion.

1. $n_i < 0$: Suppose the last unit of demand i is upgraded by class k_i, then $\Delta = g_i - g_j - \alpha_{k_i i} + \alpha_{k_i j} = r_j - r_i$;
2. $0 \leq n_i < d_i$: Similar to the previous case, we have $\Delta = -g_j - \alpha_{k_i i} + \alpha_{k_i j} = r_j - r_i - g_i$;
3. $n_i \geq d_i$: Given the chain reaction, the overall effect is equivalent to upgrading demand j with capacity i. Then, $\Delta = -g_j + \alpha_{ij} = r_j - u_i$.

To summarize, if $l \geq j$, we have

$$\Delta = \begin{cases} r_j - r_i, & \text{if } n_i < 0 \\ r_j - r_i - g_i, & \text{if } 0 \leq n_i < d_i \\ r_j - u_i, & \text{if } n_i \geq d_i. \end{cases} \tag{EC.36}$$

Note that (EC.36) also holds for $\bar{\Delta}$ if $\bar{l} \geq j$. Therefore, $\Delta = \bar{\Delta}$ when $j \leq l \leq \bar{l}$.

Next, we compare Δ and $\bar{\Delta}$ when both $l < j$ and $\bar{l} < j$. We categorize different situations based on Case 1, Case 2 and Case 3 as follows:

1. Case 1 for both $\mathbf{N}$ and $\bar{\mathbf{N}}$: Notice that class l here is similar to class j in (EC.36) in Case 4. Then, we have

$$\delta = \begin{cases} g_i - g_j + \alpha_{k_i l} - \alpha_{k_i i} = g_l - g_j - r_i + r_l, & \text{if } n_i < 0 \\ -g_j + \alpha_{k_i l} - \alpha_{k_i i} = -g_j - g_i + g_l - r_i + r_l, & \text{if } 0 \leq n_i < d_i \\ -g_j + \alpha_{il}, & \text{if } n_i \geq d_i, \end{cases} \tag{EC.37}$$

and

$$\bar{\delta} = \begin{cases} g_i - g_j + \alpha_{\bar{k}_i\bar{l}} - \alpha_{\bar{k}_i i} = g_{\bar{l}} - g_j - r_i + r_{\bar{l}}, & \text{if } n_i < 0 \\ -g_j + \alpha_{\bar{k}_i\bar{l}} - \alpha_{\bar{k}_i i} = -g_j - g_i + g_{\bar{l}} - r_i + r_{\bar{l}}, & \text{if } 0 \le n_i < d_i \\ -g_j + \alpha_{i\bar{l}}, & \text{if } n_i \ge d_i. \end{cases} \quad \text{(EC.38)}$$

Thus, $\delta - \bar{\delta} = r_l + g_l - r_{\bar{l}} - g_{\bar{l}}$.

Furthermore, $\bar{\mathbf{R}}' = \bar{\mathbf{R}}_{\bar{l}j}$ by the assumption of this case. We next show

$$\Theta^{t+1}(\bar{\mathbf{R}}') - \Theta^{t+1}(\bar{\mathbf{R}}) = \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{l}j}) - \Theta^{t+1}(\bar{\mathbf{R}}) = \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) - \Theta^{t+1}(\bar{\mathbf{N}}). \quad \text{(EC.39)}$$

From the assumption, there is no upgrade between classes $(1, \cdots, \bar{h}-1)$ and $(\bar{h}, \cdots, N)$ when applying the PSR under $(\bar{\mathbf{N}}, \mathbf{D})$, whose result is $\bar{\mathbf{R}}$. Thus, the effective states of classes $(1, \cdots, \bar{h}-1)$ in $\bar{\mathbf{R}}$ are the same as those in $\bar{\mathbf{N}}$ by Proposition EC.2. Moreover, note that $\bar{h}$ is the highest class upgrading demand $\bar{l}$ by assumption. Without loss of generality, we assume $\bar{h}$ is also the lowest class upgrading demand $\bar{l}$, then the effective state of classes $(\bar{h}, \cdots, \bar{l}-1)$ in $\bar{\mathbf{N}}$ is $((\bar{\mathbf{R}})_{\bar{h}} + y_{\bar{h}\bar{l}}, 0, \cdots, 0)$, where $y_{\bar{h}\bar{l}}$ is the upgrade between classes $\bar{h}$ and $\bar{l}$ under initial state $(\bar{\mathbf{N}}, \mathbf{D})$. Thus, classes $\bar{h}$ and $\bar{l}$ correspond to classes k and i in Lemma EC.8, which proves (EC.39).

Similarly, since $\mathbf{R}' = \mathbf{R}_{lj}$ in this case, there is

$$\Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R}) = \Theta^{t+1}(\mathbf{R}_{lj}) - \Theta^{t+1}(\mathbf{R}) = \Theta^{t+1}(\mathbf{N}_{lj}) - \Theta^{t+1}(\mathbf{N}). \quad \text{(EC.40)}$$

Moreover,

$$\Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R}) = \Theta^{t+1}(\mathbf{N}_{lj}) - \Theta^{t+1}(\mathbf{N}) \ge \Theta^{t+1}(\bar{\mathbf{N}}_{lj}) - \Theta^{t+1}(\bar{\mathbf{N}}) \quad \text{(EC.41)}$$

from the induction assumption.

To complete the proof in this case, from Lemma EC.2 and the fact that $l \le \bar{l}$, there is

$$\Theta^{t+1}(\bar{\mathbf{N}}_{lj}) - \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) \ge r_{\bar{l}} - r_l,$$

which implies $\Delta - \bar{\Delta} = \delta - \bar{\delta} + r_{\bar{l}} - r_l$ by (EC.39) and (EC.41). Since $\delta - \bar{\delta} = r_l + g_l - r_{\bar{l}} - g_{\bar{l}}$ by (EC.37) and (EC.38), we have $\Delta - \bar{\Delta} = g_l - g_{\bar{l}} \ge 0$.

In the remaining cases, we apply similar arguments to prove (EC.35). For simplicity, we will omit some details and only present the primary results.

2. Case 2 for both $\mathbf{N}$ and $\bar{\mathbf{N}}$: We have

$$\delta = \begin{cases} g_i - g_j + (r_l + g_l - r_i - g_i) - \alpha_{hl} = -g_j - r_i + u_h, & \text{if } n_i < 0 \\ -g_j + (r_l + g_l - r_i - g_i) - \alpha_{hl} = -g_j - g_i - r_i + u_h, & \text{if } 0 \le n_i < d_i \\ -g_j + \alpha_{il} - \alpha_{hl} = -g_j - u_i + u_h, & \text{if } n_i \ge d_i \end{cases} \quad \text{(EC.42)}$$

and

$$\bar{\delta} = \begin{cases} g_i - g_j + (r_{\bar{l}} + g_{\bar{l}} - r_i - g_i) - \alpha_{\bar{h}\bar{l}} = -g_j - r_i + u_{\bar{h}}, & \text{if } n_i < 0 \\ -g_j + (r_{\bar{l}} + g_{\bar{l}} - r_i - g_i) - \alpha_{\bar{h}\bar{l}} = -g_j - g_i - r_i + u_{\bar{h}}, & \text{if } 0 \le n_i < d_i \\ -g_j + \alpha_{i\bar{l}} - \alpha_{\bar{h}\bar{l}} = -g_j - u_i + u_{\bar{h}}, & \text{if } n_i \ge d_i. \end{cases} \quad \text{(EC.43)}$$

Note that $\delta - \bar{\delta} = u_h - u_{\bar{h}}$ in all cases.

(a) $\bar{l} = l$: From the assumption, all backlogged demands in classes $(i, \cdots, l)$, which are the same for both initial states $(\mathbf{N}, \mathbf{D})$ and $(\bar{\mathbf{N}}, \mathbf{D})$, have been satisfied in period t. Meanwhile, $\mathbf{D}$ is the same for both initial states in period t. Thus, the total demands satisfied are the same for both $(\mathbf{N}, \mathbf{D})$ and $(\bar{\mathbf{N}}, \mathbf{D})$, and we have $\bar{h} = h \ge r$ or $h \le \bar{h} \le r$.

By assumption, $\mathbf{R}' = \mathbf{R}_{hj}$ and $\bar{\mathbf{R}}' = \bar{\mathbf{R}}_{\bar{h}j}$ in this case, then

$$\Theta^{t+1}(\bar{\mathbf{R}}') - \Theta^{t+1}(\bar{\mathbf{R}}) = \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) - \Theta^{t+1}(\bar{\mathbf{R}}), \quad \Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R}) = \Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R}). \quad \text{(EC.44)}$$

Moreover, we define $\tilde{\mathbf{R}}$ as follows:

$$\tilde{\mathbf{R}} = \begin{cases} \mathbf{R} + \mathbf{e}_r, & \text{if } r < h \\ \mathbf{R} + \mathbf{e}_h, & \text{if } r \ge h. \end{cases}$$

Note that $\tilde{\mathbf{R}} = \bar{\mathbf{R}}$ from the definition. If $r < h$, given $(\mathbf{R})_{h+1,\cdots,j-1} = (\mathbf{R}')_{h+1,\cdots,j-1} \le 0$, we have

$$\Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R}) \ge \Theta^{t+1}(\tilde{\mathbf{R}}_{hj}) - \Theta^{t+1}(\tilde{\mathbf{R}}) \quad \text{(EC.45)}$$

from the induction assumption. On the other hand, if $r \ge h$, (EC.45) still holds because of the concavity in Proposition 1.

Since $\tilde{\mathbf{R}} = \bar{\mathbf{R}}$ and $h \le \bar{h}$, there is $\Theta^{t+1}(\tilde{\mathbf{R}}_{hj}) - \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) \ge u_{\bar{h}} - u_h$ by Lemma EC.1. Therefore, from (EC.44) and (EC.45), we have

$$\Delta - \bar{\Delta} \ge \delta - \bar{\delta} + \Theta^{t+1}(\tilde{\mathbf{R}}_{hj}) - \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) \ge 0,$$

where $\delta - \bar{\delta} = u_h - u_{\bar{h}}$ by (EC.42) and (EC.43).

(b) $l < \bar{l}$: From the above discussion of Case 2, $\bar{\mathbf{R}}'$ has one more unit of capacity $\bar{h}$ than $\bar{\mathbf{R}}$ after the chain reaction. Note that class $\bar{h}$ would have upgraded demand $\bar{l}$ if there exists unmet demand $\bar{l}$ under $\bar{\mathbf{R}}'$, which implies that the expected value of such a unit of capacity $\bar{h}$ is smaller than $\alpha_{\bar{h}\bar{l}}$. Thus,

$$\begin{aligned} &\Theta^{t+1}(\bar{\mathbf{R}}') - \Theta^{t+1}(\bar{\mathbf{R}}) = \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) - \Theta^{t+1}(\bar{\mathbf{R}}) \\ =&\Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) - \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}\bar{l}}) + \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}\bar{l}}) - \Theta^{t+1}(\bar{\mathbf{R}}) \\ \le&\alpha_{\bar{h}\bar{l}} + \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) - \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}\bar{l}}), \end{aligned} \quad \text{(EC.46)}$$

Moreover, similar to (EC.39), we can apply Lemma EC.8 to (EC.46) as $(\bar{\mathbf{R}})_{\bar{h}+1,\cdots,\bar{l}} = 0$, then

$$\Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) - \Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}\bar{l}}) = \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) - \Theta^{t+1}(\bar{\mathbf{N}})$$

and

$$\Theta^{t+1}(\bar{\mathbf{R}}') - \Theta^{t+1}(\bar{\mathbf{R}}) \le \alpha_{\bar{h}\bar{l}} + \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) - \Theta^{t+1}(\bar{\mathbf{N}}). \tag{EC.47}$$

For initial state $(\mathbf{N}, \mathbf{D})$, since $l < \bar{l}$ and $(\mathbf{R})_{l+1} < 0$, after the chain reaction, there is an additional unit of capacity h in $\mathbf{R}'$ which can be used to upgrade demand $l+1$. However, upgrading demand $l+1$ by capacity h is not optimal under $\mathbf{R}'$, i.e., the expected value of such a unit of capacity h is higher than $\alpha_{h,l+1}$. Then,

$$\begin{aligned}&\Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R}) = \Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R})\\ =&\Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R}_{h,l+1}) + \Theta^{t+1}(\mathbf{R}_{h,l+1}) - \Theta^{t+1}(\mathbf{R})\\ \ge&\alpha_{h,l+1} + \Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R}_{h,l+1}).\end{aligned} \tag{EC.48}$$

From the definition of $\tilde{\mathbf{R}}$ and the induction assumption, we have

$$\Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R}_{h,l+1}) \ge \Theta^{t+1}(\tilde{\mathbf{R}}_{hj}) - \Theta^{t+1}(\tilde{\mathbf{R}}_{h,l+1})$$

because $(\mathbf{R})_{l+2,\cdots,j-1} = (\tilde{\mathbf{R}})_{l+2,\cdots,j-1} \le 0$. Moreover, $(\tilde{\mathbf{R}})_{h+1,\cdots,l} = (\mathbf{R})_{h+1,\cdots,l} = 0$ by the assumption of this case, from Lemma EC.8, we similarly have

$$\Theta^{t+1}(\tilde{\mathbf{R}}_{hj}) - \Theta^{t+1}(\tilde{\mathbf{R}}_{h,l+1}) = \Theta^{t+1}(\bar{\mathbf{N}}_{l+1,j}) - \Theta^{t+1}(\bar{\mathbf{N}}).$$

Thus,

$$\Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R}) \ge \alpha_{h,l+1} + \Theta^{t+1}(\bar{\mathbf{N}}_{l+1,j}) - \Theta^{t+1}(\bar{\mathbf{N}}). \tag{EC.49}$$

Given $l < \bar{l}$, we have $\Theta^{t+1}(\bar{\mathbf{N}}_{l+1,j}) - \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) \ge r_{\bar{l}} - r_{l+1}$ from Lemma EC.2. Since $\delta - \bar{\delta} = u_h - u_{\bar{h}}$ by (EC.42) and (EC.43), (EC.47) and (EC.49) imply that $\Delta \ge \bar{\Delta}$ as $g_{l+1} \ge g_{\bar{l}}$.

3. Case 3 for both $\mathbf{N}$ and $\bar{\mathbf{N}}$: Since $l \le \bar{l}$, the same proof of "Case 1 for both $\mathbf{N}$ and $\bar{\mathbf{N}}$" can be applied.
4. Case 1 for $\mathbf{N}$ and Case 2 for $\bar{\mathbf{N}}$: Note that (EC.41) and (EC.47) still hold, meanwhile, δ and $\bar{\delta}$ are given in (EC.37) and (EC.43), respectively. We have

$$\delta - (\bar{\delta} + \alpha_{\bar{h}\bar{l}}) = r_l + g_l - r_{\bar{l}} - g_{\bar{l}}$$

and

$$\begin{aligned}&\left(\Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R})\right) - \left(\Theta^{t+1}(\bar{\mathbf{R}}') - \Theta^{t+1}(\bar{\mathbf{R}})\right)\\ =&\left(\Theta^{t+1}(\mathbf{R}_{lj}) - \Theta^{t+1}(\mathbf{R})\right) - \left(\Theta^{t+1}(\bar{\mathbf{R}}_{\bar{h}j}) - \Theta^{t+1}(\bar{\mathbf{R}})\right)\\ \ge&\left(\Theta^{t+1}(\bar{\mathbf{N}}_{lj}) - \Theta^{t+1}(\bar{\mathbf{N}})\right) - \left(\alpha_{\bar{h}\bar{l}} + \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) - \Theta^{t+1}(\bar{\mathbf{N}})\right)\\ =&\Theta^{t+1}(\bar{\mathbf{N}}_{lj}) - \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) - \alpha_{\bar{h}\bar{l}}.\end{aligned}$$

Given $l \le \bar{l}$, $\Theta^{t+1}(\bar{\mathbf{N}}_{lj}) - \Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) \ge r_{\bar{l}} - r_l$ by Lemma EC.2. Then, $\Delta \ge \bar{\Delta}$ since $g_l \ge g_{\bar{l}}$.

5. Case 1 for $\mathbf{N}$ and Case 3 for $\bar{\mathbf{N}}$: Note that $l \le \bar{l} < \bar{l}+1$, the same proof of "Case 1 for both $\mathbf{N}$ and $\bar{\mathbf{N}}$" can be applied.
6. Case 2 for $\mathbf{N}$ and Case 1 for $\bar{\mathbf{N}}$: In this case, class $\bar{l}$ in $(\bar{\mathbf{N}}, \mathbf{D})$ still has unmet demand while demand l is fully satisfied in $(\mathbf{N}, \mathbf{D})$ by assumption. From the induction assumption, upgrade is more likely to happen under initial state $(\bar{\mathbf{N}}, \mathbf{D})$, thus $l < \bar{l}$.

   Given that (EC.38, EC.39, EC.42, EC.49) all hold, there is

$$(\delta + \alpha_{h,l+1}) - \bar{\delta} = r_{l+1} + g_{l+1} - r_{\bar{l}} - g_{\bar{l}}.$$

   Since $l+1 \le \bar{l}$,

$$\begin{aligned}
&\big(\Theta^{t+1}(\mathbf{R}') - \Theta^{t+1}(\mathbf{R})\big) - \big(\Theta^{t+1}(\bar{\mathbf{R}}') - \Theta^{t+1}(\bar{\mathbf{R}})\big) \\
=&\big(\Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R})\big) - \big(\Theta^{t+1}(\bar{\mathbf{R}}_{\bar{l}j}) - \Theta^{t+1}(\bar{\mathbf{R}})\big) \\
\ge&\big(\alpha_{h,l+1} + \Theta^{t+1}(\bar{\mathbf{N}}_{l+1,j}) - \Theta^{t+1}(\bar{\mathbf{N}})\big) - \big(\Theta^{t+1}(\bar{\mathbf{N}}_{\bar{l}j}) - \Theta^{t+1}(\bar{\mathbf{N}})\big) \\
\ge&\alpha_{h,l+1} + r_{\bar{l}} - r_{l+1}
\end{aligned}$$

   by Lemma EC.2. Then, $\Delta \ge \bar{\Delta}$ since $g_{l+1} \ge g_{\bar{l}}$.
7. Case 2 for $\mathbf{N}$ and Case 3 for $\bar{\mathbf{N}}$: Note that $l \le \bar{l} < \bar{l}+1$, the same proof of "Case 2 for $\mathbf{N}$ and Case 1 for $\bar{\mathbf{N}}$" can be applied.
8. Case 3 for $\mathbf{N}$ and Case 1 for $\bar{\mathbf{N}}$: To apply the same proof of "Case 1 for both $\mathbf{N}$ and $\bar{\mathbf{N}}$", we only need to show $l+1 \le \bar{l}$. Suppose to the contrary that $l+1 > \bar{l}$, then $l \ge \bar{l}$. Note that upgrade is more likely to happen under initial state $(\bar{\mathbf{N}}, \mathbf{D})$ by assumption. Recall the discussions about Case 1 and Case 3, there is unmet demand $\bar{l}$ remaining in $\bar{\mathbf{R}}'$, but all demand l has been satisfied in $\mathbf{R}$. This is a contradiction.
9. Case 3 for $\mathbf{N}$ and Case 2 for $\bar{\mathbf{N}}$: To apply the same proof of "Case 1 for $\mathbf{N}$ and Case 2 for $\bar{\mathbf{N}}$", we need to show $l+1 \le \bar{l}$. Similar to the above discussion, we suppose $l+1 > \bar{l}$. Note that all demand l has been satisfied in $\mathbf{R}$ and some of the lower class demand $l+1$ is also satisfied in $\mathbf{R}'$. Meanwhile, the demand lower than class $\bar{l}$ is not upgraded under both $\bar{\mathbf{R}}$ and $\bar{\mathbf{R}}'$. This is contradiction.

To complete this proof, we need to consider the case when $l < j$ and $\bar{l} \ge j$, where $\bar{l} \ge j$ means $\mathbf{R}' = \mathbf{R}$ and (EC.36) is true.

1. Case 1 for $\mathbf{N}$: From (EC.40),

$$\Theta^{t+1}(\mathbf{R}_{lj}) - \Theta^{t+1}(\mathbf{R}) \ge r_j - r_l$$

   by Lemma EC.2. Since $\bar{\Delta}$ is given in (EC.36), we have $\Delta \ge \bar{\Delta}$ from δ in (EC.37).

2. Case 2 for $\mathbf{N}$: From (EC.48) and the fact $l+1 \le j$, there is

$$\Theta^{t+1}(\mathbf{R}_{hj}) - \Theta^{t+1}(\mathbf{R}_{h,l+1}) \ge r_j - r_{l+1}$$

   by Lemma EC.2. With $\bar{\Delta}$ in (EC.36) and δ in (EC.42), we have $\delta + \alpha_{h,l+1} + r_j + g_j - r_{l+1} - g_{l+1} = \bar{\Delta}$. Hence, $\Delta \ge \bar{\Delta}$.
3. Case 3 for $\mathbf{N}$: Note that $l+1 < j$ in this case. Then, the same proof of "Case 1 for $\mathbf{N}$" can be applied.

This completes the proof. □

The next lemma states that the protection level p_{ij} $(1 \le i < j \le N)$ in period $T-1$ decease in the states of classes $(1, \cdots, i-1)$.

Lemma EC.10. *Consider an N-class upgrading problem in period T with $(n_{i+1}, \cdots, n_j) \le 0$. Let $\bar{\mathbf{N}} = \mathbf{N} + \epsilon \mathbf{e}_r$, where $1 \le r < i$ and $\epsilon > 0$. Then,*

$$\Delta_{ij}^{+-}\Theta^T(\mathbf{N}) \ge \Delta_{ij}^{+-}\Theta^T(\bar{\mathbf{N}}), \qquad \Delta_{ij}^{-+}\Theta^T(\mathbf{N}) \ge \Delta_{ij}^{-+}\Theta^T(\bar{\mathbf{N}}).$$

Proof. Following the notations in the proof of Lemma EC.9, in this proof we only need to consider Case 1, Case 3 and Case 4 for Θ^T since the additional unit of capacity h $(\bar{h})$ will not be passed to the next period. Note that $\Delta = \delta$ and $\bar{\Delta} = \bar{\delta}$ since $\Theta^{T+1} \equiv 0$. Also, the protection levels are zero in period T.

Recall the similarity of Case 1 and Case 3. In the proof of Lemma EC.9, we have shown that $l+1 \le \bar{l}$ if "Case 3 for $\mathbf{N}$ and Case 1 for $\bar{\mathbf{N}}$". Therefore, we only have three different cases in period T.

1. $j \le l \le \bar{l}$: Since (EC.36) still holds, we have $\Delta = \bar{\Delta}$.
2. $l \le \bar{l} < j$: From (EC.37) and (EC.38), there is $\Delta - \bar{\Delta} = r_l + g_l - r_{\bar{l}} - g_{\bar{l}} \ge 0$ since $r_l \ge r_{\bar{l}}$ and $g_l \ge g_{\bar{l}}$.
3. $l < j \le \bar{l}$: From (EC.36) and (EC.37), we have $\Delta - \bar{\Delta} = r_l + g_l - r_j - g_j > 0$ since $r_l > r_j$ and $g_l > g_j$.

Hence, the desired result holds in period T for any demand realization, which completes the proof. □

With the previous two lemmas, we can prove the monotonicity result.

Proposition 5 *The optimal protection level p_{ij} $(1 \le i < j \le N)$ in period t $(1 \le t \le T)$ are decreasing in $(n_1^t, \cdots, n_{i-1}^t)$.*

Proof. Given the definition of the protection level in (8), this proposition can be inductively proved using Lemmas EC.9 and EC.10. □

E.2. Non-concave Example

For the RCEC heuristic, although the allocation decisions in each period can be solved by a concave function $\bar{\Theta}^{t+1}_{\mathrm{RCEC}}$, $\Pi_{\mathrm{RCEC}}(\mathbf{X})$, the firm's total revenue with initial capacity $\mathbf{X}$ under the RCEC heuristic, is in general not concave or even quasi-concave. To illustrate this, we provide the following 2-product and 3-period example. In this example, the revenue is $(r_1, r_2) = (5.5, 2.5)$, the goodwill cost is $(g_1, g_2) = (1.5, 0)$, the usage cost is $(u_1, u_2) = (1.5, 0)$, and the capacity cost is $(c_1, c_2) = (0.2, 0.1)$. Hence, the profit margins are $(\alpha_{11}, \alpha_{22}, \alpha_{12}) = (5.5, 2.5, 1)$. We assume that there are only two possible demand realizations with equal probabilities $\frac{1}{2}$ in each period, i.e.,

$$\mathbf{D}^1 = \begin{cases} (1,4)^\intercal & \text{with probability } 0.5; \\ (1,0)^\intercal & \text{with probability } 0.5. \end{cases}$$

$$\mathbf{D}^2 = \begin{cases} (1,0)^\intercal & \text{with probability } 0.5; \\ (0,4)^\intercal & \text{with probability } 0.5. \end{cases}$$

$$\mathbf{D}^3 = \begin{cases} (1,4)^\intercal & \text{with probability } 0.5; \\ (0,0)^\intercal & \text{with probability } 0.5. \end{cases}$$

Let $x_2 = 1$, then Figure EC.1 illustrates the non-concavity of $\Pi_{\mathrm{RCEC}}(\mathbf{X})$ with respect to the initial capacity x_1.

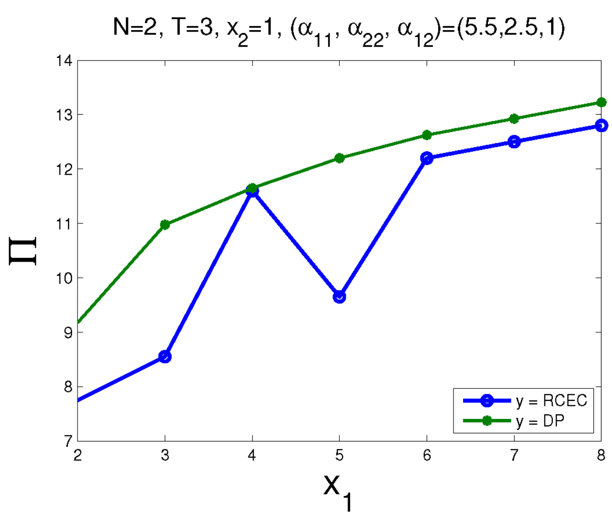

Figure EC.1 $\Pi_{\mathrm{RCEC}}(\mathbf{X})$