Operations Research Letters 43 (2015) 65–68

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

A new approach to two-location joint inventory and transshipment control via $L^\natural$-convexity

Xin Chen, Xiangyu Gao, Zhenyu Hu *

Department of Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

ARTICLE INFO

Article history:
Received 17 September 2014
Accepted 19 November 2014
Available online 3 December 2014

Keywords:
$L^\natural$-convexity
Dynamic programming
Transshipment
Inventory control
Uncertain capacity

ABSTRACT

We provide a new approach to the analysis of the optimal joint inventory and transshipment control with uncertain capacities by employing the concept of $L^\natural$-convexity. In this approach, we use variable transformation techniques and apply two recent results to establish the $L^\natural$-concavity of the profit-to-go functions, which significantly simplifies the analysis in the existing literature. Some variations of the basic model can also be analyzed using our approach with minor modifications.

1. Introduction

The concept of $L^\natural$-convexity was first introduced by Murota [7] to extend convex analysis from the real space to the space with integers. Later, $L^\natural$-convexity found its applications in many complicated inventory management problems. Lu and Song [6] develop a greedy algorithm based on the property of $L^\natural$-convexity for a cost optimization problem in assemble-to-order production systems. Zipkin [10] uses $L^\natural$-convexity to provide a new approach to the structural analysis of the standard, single-item lost-sales inventory system with a positive lead time. Huh and Janakiraman [5] apply $L^\natural$-convexity to establish properties of the optimal policy in serial systems with lost sales. By showing that the profit function is $L^\natural$-concave in transformed state variables, Pang et al. [8] partially characterize the structure of the optimal joint ordering and pricing policies in inventory-pricing models with positive lead time. Employing the concept of $L^\natural$-convexity, Chen et al. [3] significantly simplify the proof of a classical result and provide new insights to the structural analysis in perishable inventory models.

The purpose of this paper is to apply some of the state-of-the-art techniques in $L^\natural$-convexity to provide a new approach to the analysis of optimal joint inventory and transshipment control under uncertain capacity. Specifically, we consider the model studied in Hu et al. [4]. In this model, a firm operates two facilities in separate markets, where the firm produces the same product and sells at constant prices. Both facilities face uncertain demands and uncertain production capacities. The firm needs to determine the production quantities at the beginning of each period. The demand and production uncertainties are then revealed and the firm further decides how much inventory to be transshipped from one facility to another. Demands are satisfied after the transshipment and unfilled demands are lost.

Hu et al. [4] provide a characterization of the structure of the optimal transshipment and production policy. For this purpose, they identify several important properties of the profit-to-go functions, which play a pivotal role in the derivation of the structure of the optimal policy. They spent several pages through a very detailed and complicated analysis of derivatives to prove these properties. By employing the concept of $L^\natural$-convexity, or equivalently $L^\natural$-concavity, in this paper, we present a simple yet non-trivial proof of those properties. In particular, we realize that these properties of the profit-to-go functions are nothing but natural consequences of $L^\natural$-concave functions after a proper transformation of the original variables. However, to prove that the profit-to-go functions are $L^\natural$-concave (after variable transformation) is not straightforward. In fact, there are two bottlenecks in showing the $L^\natural$-concavity. First, in the transshipment stage, the equality constraint that guarantees the sum of inventory positions at two facilities must remain unchanged prohibits the feasible set to be sublattice. To tackle this difficulty, we apply a recent result by Chen et al. [1] that deals with parametric optimizations with nonlattice structures. Second, in the production stage, the realized production quantity is the minimum of the production quantity decision and the realized production capacity. As a result, the objective function is not concave in the decision variables. Interestingly, a transformation technique developed by Chen and Pang [2] provides us a tool to resolve this issue.

* Corresponding author.
E-mail addresses: xinchen@illinois.edu (X. Chen), xgao12@illinois.edu (X. Gao), hu48@illinois.edu (Z. Hu).

http://dx.doi.org/10.1016/j.orl.2014.11.007

Section 2 presents the basic notations as well as the concept and some key properties of $L^\natural$-convexity and introduces two recent results that are critical to the development of our new approach. In Section 3, the problem of joint inventory and transshipment control is described and the $L^\natural$-concavity of the profit-to-go functions is established. Section 4 concludes the paper with some extensions.

2. $L^\natural$-convexity and preliminary results

In this section, we introduce the concept of $L^\natural$-convexity as well as other related notations and basic concepts. We also give a brief review of some commonly used properties of $L^\natural$-convexity.

Throughout this paper, we denote $\mathfrak{R}$ and $\mathfrak{R}_+$ as the real space and the set of nonnegative real numbers, and $\mathbb{Z}$ and $\mathbb{Z}_+$ as the space of integers and the set of nonnegative integers. We also use $\mathcal{F}$ to denote either the real space $\mathfrak{R}$ or the space of integers $\mathbb{Z}$. In addition, we denote $\bar{\mathfrak{R}} = \mathfrak{R} \cup \{\infty\}$ and define $e \in \mathfrak{R}^n$ to be the vector with all components 1.

Given any two vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathfrak{R}^n$, we denote $x^+ = (\max\{x_1, 0\}, \ldots, \max\{x_n, 0\})$, $x^- = (\max\{-x_1, 0\}, \ldots, \max\{-x_n, 0\})$, $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$ and $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$. In addition, a set $\mathcal{S}$ of $\mathfrak{R}^n$ is said to be convex if $\lambda x + (1-\lambda)y \in \mathcal{S}$ for all $x, y \in \mathcal{S}$ and $0 \le \lambda \le 1$, and $\mathcal{S}$ is said to be a sublattice of $\mathfrak{R}^n$ if for all $x, y \in \mathcal{S}$ it holds that $x \wedge y, x \vee y \in \mathcal{S}$.

Given a function f defined on a set $\mathcal{S}$ of $\mathfrak{R}^n$, f is supermodular if $\mathcal{S}$ is a sublattice and $f(x) + f(y) \le f(x \wedge y) + f(x \vee y)$ for all $x, y \in \mathcal{S}$, and f is convex if $\mathcal{S}$ is convex and $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ for all $0 \le \lambda \le 1$ and $x, y \in \mathcal{S}$. A function f is submodular (or concave) if $-f$ is supermodular (or convex).

We now introduce the definition of $L^\natural$-convexity by following Simchi-Levi et al. [9].

Definition 1 ($L^\natural$*-convexity*). A function $f : \mathcal{F}^n \to \bar{\mathfrak{R}}$ is $L^\natural$-convex if and only if $g(x, \xi) := f(x - \xi e)$ is submodular on $(x, \xi) \in \mathcal{F}^n \times \mathcal{S}$, where $\mathcal{S}$ is the intersection of $\mathcal{F}$ and any unbounded interval in $\mathfrak{R}$. A function f is $L^\natural$-concave if $-f$ is $L^\natural$-convex. A set $\mathcal{V} \subseteq \mathcal{F}^n$ is called $L^\natural$-convex if its indicator function $\delta_{\mathcal{V}}(\cdot)$, defined as $\delta_{\mathcal{V}}(x) = 0$ for $x \in \mathcal{V}$ and $+\infty$ otherwise, is $L^\natural$-convex.

For an $L^\natural$-convex function f, its effective domain denoted by $dom(f) = \{x \in \mathcal{F}^n | f(x) < +\infty\}$ is an $L^\natural$-convex set. A function f is said to be $L^\natural$-convex on a set $\mathcal{V} \subseteq \mathcal{F}^n$ if $\mathcal{V}$ is an $L^\natural$-convex set and the extension of f to the whole space $\mathcal{F}^n$ by defining $f(x) = +\infty$ for $x \notin \mathcal{V}$ is $L^\natural$-convex. It is straightforward to show that an $L^\natural$-convex function restricted to an $L^\natural$-convex set is also $L^\natural$-convex.

Some of the commonly used properties of $L^\natural$-convexity are listed in the following proposition.

Proposition 1. (a) *If $f_i : \mathcal{F}^n \to \bar{\mathfrak{R}}$, $i = 1, \ldots, n$ are all $L^\natural$-convex, then for any scalar $\alpha_i \ge 0$, $\sum_{i=1}^n \alpha_i f_i$ is also $L^\natural$-convex.*

(b) *If f_k is $L^\natural$-convex for $k = 1, 2, \ldots$ and $\lim_{k\to\infty} f_k(x) = f(x)$ for any $x \in \mathcal{F}^n$, then $f(x)$ is also $L^\natural$-convex.*

(c) *Let $f(\cdot,\cdot)$ be a function defined on the product space $\mathcal{F}^n \times \mathcal{F}^m$. If for any given $y \in \mathcal{F}^m$, $f(\cdot, y)$ is $L^\natural$-convex, then for a random vector ξ in $\mathcal{F}^m$, $E_\xi[f(x, \xi)]$ is $L^\natural$-convex, provided it is well defined.*

(d) *A smooth function $f : \mathfrak{R}^n \to \mathfrak{R}$ is $L^\natural$-convex if and only if its Hessian is a diagonally dominated M-matrix, where a matrix A with its ij-th component being a_{ij} is called a diagonally dominated M-matrix, if*

$$a_{ij} \le 0, \quad \forall i \ne j, \qquad a_{ii} \ge 0, \quad \text{and} \quad \sum_{j=1}^n a_{ij} \ge 0, \quad \forall i.$$

(e) *Assume that $\mathcal{A}$ is an $L^\natural$-convex set of $\mathcal{F}^n \times \mathcal{F}^m$ and $f(\cdot,\cdot) : \mathcal{F}^n \times \mathcal{F}^m \to \bar{\mathfrak{R}}$ is an $L^\natural$-convex function. Then the function*

$$g(x) = \inf_{y:(x,y)\in\mathcal{A}} f(x, y)$$

is $L^\natural$-convex over $\mathcal{F}^n$ if $g(x) \ne -\infty$ for any $x \in \mathcal{F}^n$.

(f) *A set with a representation $\{x \in \mathcal{F}^n : l \le x \le u, x_i - x_j \le v_{ij}, \forall i \ne j\}$, is $L^\natural$-convex in the space $\mathcal{F}^n$, where $l, u \in \mathcal{F}^n$ and $v_{ij} \in \mathcal{F}$ ($i \ne j$).*

In the following, we summarize two results developed in Chen et al. [1] and in Chen and Pang [2], respectively.

The first result comes from Corollary 4 part (c) of Chen et al. [1]. It establishes a preservation property of $L^\natural$-concavity under optimization operations when the constraint set may not be a sublattice.

Proposition 2. *Consider the following optimization problem parameterized by a two-dimensional vector x:*

$$f(x) = \max_{y_1,\ldots,y_N} \left\{ \sum_{n=1}^N f_n(y_n) : \sum_{n=1}^N y_n = x, \; y_n \in S_n, \forall n \right\},$$

where S_n are subsets of $\mathfrak{R}^2$, and f is defined on $S = \{\sum_{n=1}^N y_n : y_n \in S_n, \forall n\}$. If S_n is of the following form:

$$\{(x_1, x_2) \in \mathfrak{R}^2 : l_1 \le x_1 \le u_1, l_2 \le x_2 \le u_2, l_0 \le x_1 - x_2 \le u_0\},$$

and all f_n are $L^\natural$-concave on S_n, then f is $L^\natural$-concave on S.

The next proposition is from Proposition 3 of Chen and Pan [2]. It establishes the preservation of $L^\natural$-convexity when the objective function is not $L^\natural$-convex in the original decision variables.

Proposition 3. *Given $f(\cdot,\cdot) : \mathcal{F}^n \times \mathcal{F}^m \to \bar{\mathfrak{R}}$ and set $\mathcal{A} \subseteq \mathcal{F}^n \times \mathcal{F}^m$, define for any $x \in \mathcal{F}^n$*

$$g(x) = \inf_{(x,y)\in\mathcal{A}} E[f(x, y_1 \wedge \xi_1, \ldots, y_k \wedge \xi_k, y_{k+1} \vee \xi_{k+1}, \ldots, y_m \vee \xi_m)]. \tag{1}$$

Let $\mathcal{A}^\xi = \{(x, y_1 \wedge \xi_1, \ldots, y_k \wedge \xi_k, y_{k+1} \vee \xi_{k+1}, \ldots, y_m \vee \xi_m) : (x, y) \in \mathcal{A}, \xi \in Supp(\xi)\}$, where $Supp(\xi) \subseteq \mathcal{F}^m$ denotes support of the random vector ξ. Assume that $(x, y) \in \mathcal{A}$ if and only if $(x, y_1 \wedge \xi_1, \ldots, y_k \wedge \xi_k, y_{k+1} \vee \xi_{k+1}, \ldots, y_m \vee \xi_m) \in \mathcal{A}^\xi$ a.s. If f and $\mathcal{A}^\xi$ are $L^\natural$-convex, then g is also $L^\natural$-convex.

3. Main result

In this section, we describe the optimization problem of interest and present our new approach to prove the key properties of the profit-to-go functions in Hu et al. [4].

Consider a firm operating two manufacturing facilities in separate markets through multiple time periods. Each facility faces uncertain capacities that are independent in time and of each other. Facilities also face uncertain demands which are independent in time but can be correlated across the two facilities. In each period, the firm's decisions can be divided into two stages. The first stage is the production stage where the firm decides how much it will produce in each of the facilities. After the production stage, the capacities and demands are realized. The firm's actual production quantity, which is the minimum of the planned production quantity and the realized capacity, incurs a unit production cost. The firm then enters the transshipment stage where it decides how much inventory to be transshipped from one facility to another. Finally, the demands are met and unsatisfied demands are lost. The firm receives linear revenue on satisfied demands and pays linear holding and transshipment costs. The problem is then to find the optimal production and transshipment quantities in each period so that the firm maximizes the total discounted profit over the planning horizon.

We now introduce the dynamic programming formulation of the optimization problem in Hu et al. [4] as follows. Let $G_*^k(x_1^k, x_2^k)$ be the profit-to-go function when the current inventory levels at the two facilities are x_1^k and x_2^k respectively and there are k periods left in the planning horizon.

Production stage:

$$G_*^k(x_1^k, x_2^k) = \max_{y_1^k \ge x_1^k, y_2^k \ge x_2^k} E_{T_1^k, T_2^k, D_1^k, D_2^k}\{-c_1(y_1^k \wedge (x_1^k + T_1^k) - x_1^k) - c_2(y_2^k \wedge (x_2^k + T_2^k) - x_2^k) + r_1 D_1^k + r_2 D_2^k + J_*^k(y_1^k \wedge (x_1^k + T_1^k) - D_1^k, y_2^k \wedge (x_2^k + T_2^k) - D_2^k)\}. \quad (2)$$

Transshipment stage:

$$J_*^k(z_1^k, z_2^k) = \max_{\hat{z}_1^k + \hat{z}_2^k = z_1^k + z_2^k} J^k(z_1^k, z_2^k, \hat{z}_1^k, \hat{z}_2^k), \quad (3)$$

where

$$J^k(z_1^k, z_2^k, \hat{z}_1^k, \hat{z}_2^k) = -r_1(\hat{z}_1^k)^- - r_2(\hat{z}_2^k)^- - h_1(\hat{z}_1^k)^+ - h_2(\hat{z}_2^k)^+ - s_1(z_1^k - \hat{z}_1^k)^+ - s_2(z_2^k - \hat{z}_2^k)^+ + \alpha G_*^{k-1}((\hat{z}_1^k)^+, (\hat{z}_2^k)^+), \quad (4)$$

and $G_*^0(x_1^0, x_2^0) \equiv 0$.

In the production stage, in period k, the target inventory levels at the two facilities y_1^k and y_2^k are decided. They are constrained to be no smaller than the current inventory levels at the two facilities x_1^k and x_2^k. The first two terms on the right hand side of (2) are the production costs with c_1, c_2 and T_1^k, T_2^k representing the marginal production costs and random capacities at the two facilities respectively. The next two terms are the full revenue collected over the realized demands, where r_1, r_2 and D_1^k, D_2^k are marginal revenue and random demand respectively. The revenue for the lost sales is deducted in the transshipment stage.

In the transshipment stage, in period k, the transshipment quantities or equivalently, the inventory levels after transshipment $\hat{z}_1^k$ and $\hat{z}_2^k$ are decided, whose sum is constrained to be equal to the inventory levels before transshipment (but after demands realization) z_1^k and z_2^k. The first two terms on the right hand side of (4) are the deducted revenue for the lost sales. The next two terms are the holding costs, where h_1 and h_2 are unit holding costs at the two facilities respectively. The two terms following are transshipment costs with s_1 (s_2) being the unit transshipment cost from facility 1 (2) to 2 (1). Finally, α in (4) is the discount factor.

Hu et al. [4], under the assumption of continuous demands and capacities, prove the following properties on the profit-to-go function $G_*^k(x_1, x_2)$, which are essential for their derivation of the optimal transshipment and production policies.

$\mathbb{A}_1$: $G_*^{k-1}(x_1, x_2)$ is jointly concave in x_1 and x_2, and

$$\frac{\partial^2}{\partial x_1^2} G_*^{k-1}(x_1, x_2) \le \frac{\partial^2}{\partial x_1 \partial x_2} G_*^{k-1}(x_1, x_2),$$
$$\frac{\partial^2}{\partial x_2^2} G_*^{k-1}(x_1, x_2) \le \frac{\partial^2}{\partial x_2 \partial x_1} G_*^{k-1}(x_1, x_2);$$

$\mathbb{A}_2$: $G_*^{k-1}(x_1, x_2)$ is submodular and

$$\frac{\partial^2}{\partial x_1 \partial x_2} G_*^{k-1}(x_1, x_2) = \frac{\partial^2}{\partial x_2 \partial x_1} G_*^{k-1}(x_1, x_2).$$

Through an inductive argument, their proof relies on a full characterization of the optimal transshipment policy and a rather involved analysis of the derivatives which spans several pages. In the following, we present our new approach by using what we have introduced in Section 2. Interestingly, our approach does not rely on the characterization of the optimal policy and it applies to discrete demands as well as capacities without any further efforts.

Denote d_i^k as the realization of demand for facility i in period k, and define $q_i^k = z_i^k + d_i^k$, $w_i^k = \hat{z}_i^k + d_i^k$. Furthermore, we change variables by letting $\tilde{y}_2^k = -y_2^k$, $\tilde{x}_2^k = -x_2^k$, $\tilde{T}_2^k = -T_2^k$, $\tilde{q}_2^k = -q_2^k$, $\tilde{w}_2^k = -w_2^k$. Then the original problem can be equivalently reformulated as

$$\tilde{G}_*^k(x_1^k, \tilde{x}_2^k) = \max_{y_1^k \ge x_1^k, \tilde{y}_2^k \le \tilde{x}_2^k} E_{T_1^k, \tilde{T}_2^k, D_1^k, D_2^k}\{-c_1(y_1^k \wedge (x_1^k + T_1^k) - x_1^k) + c_2(\tilde{y}_2^k \vee (\tilde{x}_2^k + \tilde{T}_2^k) - \tilde{x}_2^k) + \tilde{J}_*^k(y_1^k \wedge (x_1^k + T_1^k), \tilde{y}_2^k \vee (\tilde{x}_2^k + \tilde{T}_2^k))\}, \quad (5)$$

where $\tilde{G}_*^k(x_1^k, \tilde{x}_2^k) = G_*^k(x_1^k, -\tilde{x}_2^k)$ and by introducing a new variable v

$$\tilde{J}_*^k(q_1^k, \tilde{q}_2^k) = \max_{w_1^k, \tilde{w}_2^k, v} \tilde{J}(w_1^k, \tilde{w}_2^k, v) \quad \text{s.t.} \quad w_1^k + v = q_1^k, \quad \tilde{w}_2^k + v = \tilde{q}_2^k, \quad (6)$$

where

$$\tilde{J}(w_1^k, \tilde{w}_2^k, v) = r_1(w_1^k \wedge d_1^k) + r_2((-\tilde{w}_2^k) \wedge d_2^k) - h_1(w_1^k - d_1^k)^+ - h_2(-\tilde{w}_2^k - d_2^k)^+ - s_1 v^+ - s_2(-v)^+ + \alpha \tilde{G}_*^{k-1}((w_1^k - d_1^k)^+, -(-\tilde{w}_2^k - d_2^k)^+). \quad (7)$$

Now we are ready to state and prove our main result, which offers a new approach that proves the key properties $\mathbb{A}_1$ and $\mathbb{A}_2$ when demands and capacities are continuous.

Theorem 1. *Suppose that $\tilde{G}_*^{k-1}(\cdot,\cdot)$ is $L^\natural$-concave, then $\tilde{G}_*^k(\cdot,\cdot)$ is also $L^\natural$-concave.*

Proof. For notational brevity, we omit the superscript k in the proof when there is no ambiguity. Define u_1, u_2 as the inventory level after the sales assuming that *the firm can hold inventory with some demand unsatisfied*. Then the realized sales are given by $w_1 - u_1$ and $w_2 - u_2$ at the two facilities respectively. By letting $\tilde{u}_2 = -u_2$, we claim that $\tilde{J}(w_1, \tilde{w}_2, v)$ equals the optimal objective value of the following problem:

$$\max_{u_1, \tilde{u}_2} r_1(w_1 - u_1) - r_2(\tilde{w}_2 - \tilde{u}_2) - h_1 u_1 + h_2 \tilde{u}_2 - s_1 v^+ - s_2(-v)^+ + \alpha \tilde{G}_*^{k-1}(u_1, \tilde{u}_2)$$
$$\text{s.t.} \quad 0 \le u_1, \quad u_1 - w_1 \le 0, \quad \tilde{w}_2 - \tilde{u}_2 \le 0, \quad \tilde{u}_2 \le 0, \quad w_1 - u_1 \le d_1, \quad u_2 - \tilde{w}_2 \le d_2. \quad (8)$$

Note that facing a stationary system, the firm should never hold inventory and reject demand at the same time since it is always more profitable to satisfy the current demand than holding the inventory to fulfill future demands. Therefore, the optimal solution is $u_1 = (w_1 - d_1)^+$, $u_2 = -(-\tilde{w}_2 - d_2)^+$ and our claim is correct.

We further claim that the objective function of the problem (8) is $L^\natural$-concave in $(w_1, \tilde{w}_2, v, u_1, \tilde{u}_2)$. To see this, note that $\tilde{G}_*^{k-1}(u_1, \tilde{u}_2)$ is $L^\natural$-concave by our induction hypothesis. The $L^\natural$-concavity of the rest of terms in the objective function is straightforward to verify. The constraint set is $L^\natural$-convex according to Proposition 1 part (f). Then the $L^\natural$-concavity of $\tilde{J}(w_1, \tilde{w}_2, v)$ follows from Proposition 1 part (e).

Note that the objective function in (6) is separable in variables $(w_1, \tilde{w}_2)$ and (v, v). Thus, the $L^\natural$-concavity of $\tilde{J}_*^k(q_1, \tilde{q}_2)$ follows from Proposition 2.

By defining $\tilde{G}(y_1, y_2) = E_{D_1, D_2}\{-c_1 y_1 + c_2 y_2 + \tilde{J}_*^k(y_1, y_2)\}$, (5) can be expressed as

$$\tilde{G}_*(x_1, \tilde{x}_2) = \max_{y_1 \ge x_1, \tilde{y}_2 \le \tilde{x}_2} E_{T_1, \tilde{T}_2}\{\tilde{G}(y_1 \wedge (x_1 + T_1), \tilde{y}_2 \vee (\tilde{x}_2 + \tilde{T}_2))\} + c_1 x_1 - c_2 \tilde{x}_2.$$

Clearly $\tilde{G}(y_1, y_2)$ is $L^\natural$-concave in (y_1, y_2). Moreover, $y_1 \wedge (x_1 + T_1) = (y_1 - x_1) \wedge T_1 + x_1$ and $\tilde{y}_2 \vee (\tilde{x}_2 + \tilde{T}_2) = (\tilde{y}_2 - \tilde{x}_2) \vee \tilde{T}_2 + \tilde{x}_2$. It is easy to see that by transforming the variables $\hat{y}_1 = y_1 - x_1$ and $\hat{y}_2 = \tilde{y}_2 - \tilde{x}_2$, the above problem can be expressed in the form of (1). Then Proposition 3 implies that the profit-to-go function $\tilde{G}_*(x_1, \tilde{x}_2)$ is $L^\natural$-concave. □

Using Proposition 1 part (d), it is straightforward to check that Theorem 1 then implies the properties $\mathbb{A}_1$ and $\mathbb{A}_2$ of $G_*^k(\cdot, \cdot)$ when demands and capacities are continuous. We also point out that the structure of the optimal policies can be derived from Theorem 1 for both continuous and discrete demands as well as capacities with some minor modifications of the analysis in Hu et al. [4].

4. Conclusion and extensions

In this paper, we introduce the concept of $L^\natural$-convexity and combine two recent results in the literature in a novel way that allows us to provide a new approach to a joint inventory and transshipment control problem with uncertain capacity. Our approach not only significantly simplifies the structural analysis but also can be easily applied to some extensions as we discuss below, which otherwise may require considerable amount of additional effort.

(1) *Backorder case.* In the case of backorder instead of lost sales, we replace the term $\alpha\tilde{G}_*^{k-1}((w_1^k - d_1^k)^+, -(-\tilde{w}_2^k - d_2^k)^+)$ in (7) by $\alpha\tilde{G}_*^{k-1}((w_1^k - d_1^k), -(-\tilde{w}_2^k - d_2^k))$, which is still $L^\natural$-concave by induction hypothesis. Similarly, it is easy to show that adding shortage cost does not change the $L^\natural$-concavity in (7). Thus, Theorem 1 holds in this case.

(2) *Capacities on the transshipment quantities.* In many practical scenarios, a firm may not have the luxury to transship any arbitrary large amount of quantities from one facility to another because for instance, it has only a few fleet vehicles. In some settings, the transshipment can be restricted to a single direction, i.e., one of the transshipment capacity is zero. Let S_1 (S_2) be the capacity on the transshipment quantities from facility 1 (2) to facility 2 (1). Then problem (6) is now reformulated as

$$\tilde{J}_*^k(q_1, \tilde{q}_2) = \max_{w_1, \tilde{w}_2, v} \tilde{J}(w_1, \tilde{w}_2, v)$$

$$\text{s.t.} \quad w_1 + v = q_1, \quad \tilde{w}_2 + v = \tilde{q}_2, \quad -S_2 \le v \le S_1.$$

It is straightforward to check that Proposition 2 still applies and consequently our conclusion still holds.

Acknowledgments

This research is partly supported by National Science Foundation Grant CMMI-1363261 and National Natural Science Foundation of China Grant 71228203.

References

[1] X. Chen, P. Hu, S. He, Technical note-preservation of supermodularity in parametric optimization problems with nonlattice structures, Oper. Res. 61 (5) (2013) 1166–1173.
[2] X. Chen, Z. Pang, Dual sourcing under supply capacity uncertainty, Working Paper, 2014.
[3] X. Chen, Z. Pang, L. Pan, Coordinating inventory control and pricing strategies for perishable products, Oper. Res. 62 (2) (2014) 284–300.
[4] X. Hu, I. Duenyas, R. Kapuscinski, Optimal joint inventory and transshipment control under uncertainty capacity, Oper. Res. 56 (4) (2008) 881–897.
[5] W.T. Huh, G. Janakiraman, On the optimal policy structure in serial inventory systems with lost sales, Oper. Res. 58 (2) (2010) 486–491.
[6] Y. Lu, J.-S. Song, Order-based cost optimization in assemble-to-order systems, Oper. Res. 53 (1) (2005) 151–169.
[7] K. Murota, Discrete convex analysis, Math. Program. 83 (1998) 313–371.
[8] Z. Pang, F.Y. Chen, Y. Feng, Technical note-a note on the structure of joint inventory-pricing control with leadtimes, Oper. Res. 60 (3) (2012) 581–587.
[9] D. Simchi-Levi, X. Chen, J. Bramel, The Logic of Logistics: Theory, Algorithms, and Applications for Logistics Management, third ed., Springer, New York, 2014.
[10] P. Zipkin, On the structure of lost-sales inventory models, Oper. Res. 56 (4) (2008) 937–944.