Dynamic Pricing with Gain-Seeking Reference Price Effects

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We study a dynamic pricing problem of a firm facing consumers with reference price effects at an aggregate demand level, where demand is more sensitive to gains than losses. We analyze the dynamics of the myopic pricing strategy (i.e. the firm only maximizes the current period profit) and illustrate its connection with discontinuous maps, which suggests that under general parameter specifications, even the myopic pricing strategy leads to complicated reference price/price dynamics. We then show for a special case, validated by our empirical studies, that a cyclic skimming pricing strategy is optimal, and provide conditions to guarantee the optimality of high-low pricing strategies. Finally, we numerically demonstrate the performance of commonly used simple pricing strategies and the robustness of our findings.

Key words: dynamic pricing, gain-seeking, reference price effects, discontinuous maps

1. Introduction

Over the last two decades, dynamic pricing has attracted considerable attention from industry as well as academia. On the one hand, the scope of industries that adopt dynamic pricing strategies has widened remarkably, with examples ranging from airline, hotel industries whose use of dynamic pricing has long been a well established practice to many other industries such as retailing, manufacturing, cloud computing and energy, etc. On the other hand, the on-going research in academia has led the practice of dynamic pricing to grow more sophisticated over the years (see, for example, Chen and Simchi-Levi 2012, for a review). Yet, a major focus in the literature is still based on static demand models which assume that demand at a period depends only on the current price, despite of the fact that in a market with repeated purchases such as supermarkets, intertemporal changes in prices would have significant impacts on consumers' perception of the price and in turn influence consumers' purchasing decisions.

There are some attempts in modeling such impacts. Specifically, the concept of *reference price*, developed in the economics and marketing literature, argues that consumers form price expectations and use them to judge the current selling price. That is, reference price is used as an internal anchor formed in consumers' minds as a result of experience based on information such as prices in observed periods (Kalyanaram and Little 1994). Though in most cases, reference price cannot be physically observed, a large amount of literature is devoted to modeling the formation of reference prices and investigating the impact of reference prices on consumers' purchasing behavior. Among these models, a memory-based reference price model is validated by various empirical studies (Briesch et al. 1997, Hardie et al. 1993, Greenleaf 1995, and etc.). Reference price models are now accepted as an empirical generalization in the marketing literature (see the review paper Mazumdar et al. 2005).

Following a conceptual support from prospect theory (Tversky and Kahneman 1991), a widely beheld belief in the reference price literature postulates that consumers are loss-averse, i.e., they are more sensitive to losses than gains. Here, a purchasing instance is perceived by consumers as gains or losses based on whether the shelf price is considered as discounts or surcharges relative to reference prices. With such a belief, an aggregate level loss-averse demand model is studied in several papers in the literature (Kopalle et al. 1996, Fibich et al. 2003, Popescu and Wu 2007, Nasiry and Popescu 2011). Loss-aversion at a market level is interpreted as a more sensitive response to the negative part of the difference between reference price and price compared to the positive part in the aggregate demand function.

In this paper, we focus on the dynamic pricing problem of a firm facing gain-seeking demand at the aggregate level. Here, gain-seeking at a market level should be interpreted as a more sensitive response to the positive part of the difference between reference price and price compared to the negative part in the aggregate demand function. Although the term "loss-seeking" is used in the previous literature (see, for instance, Popescu and Wu 2007), we feel the term "gain-seeking" is more accurate in capturing the fact that consumers/aggregate demands are more sensitive to gains rather than losses and note that it has also been adopted in several recent papers (Aflaki and Popescu 2013, Kallio and Halme 2009, Kopalle et al. 2012) as well.

Unlike the common belief that loss-averse behavior is prevalent, it is admitted in the survey paper by Mazumdar et al. (2005) that "evidence of loss-aversion is mixed". In fact, there are empirical evidences that support gain-seeking behavior at both individual and aggregate level. At an individual level, two out of five papers reviewed by Mazumdar et al. (2005) that study the effects of loss-aversion show little or no evidence of loss-aversion. Specifically, Krishnamurthi et al. (1992) observe that consumers not loyal to any brand respond more strongly to gains than to losses in all six but one brand, and Bell and Lattin (2000) suggest through extensive empirical studies across different product categories that loss-aversion may not be a universal phenomenon in grocery products due to price response heterogeneity. More recently, Kopalle et al. (2012) find that for a large number of households the impact of a gain is greater than that of a loss and on average the effect of losses is almost negligible compared to that of gains. At an aggregate level, Greenleaf (1995) shows that the aggregate demand can be 10 times more responsive to gains than losses. Deploying the methodology of switching regression on the aggregate level data, Raman and Bass (2002) also encounter gain-seeking behavior in one of their brands.

One also needs to be cautious that loss-aversion at an individual level does not necessarily imply loss-aversion at an aggregate level. Kallio and Halme (2009) explicitly define the loss-aversion at an individual level as *loss averse in value* and loss-aversion at an aggregate level as *loss averse in demand* and give possible conditions under which loss averse in value does not imply loss averse in demand. One important implication of their conditions in a single product setting is that when market conditions are harsh (consumers have overall small probability of purchasing the product, i.e., mainly promotion-driven consumers), then the market should be more sensitive to gains which boost the market rather than losses which further make the market more miserable. Greenleaf

(1995) also points out that the presence of gain-seeking at an aggregate demand level does not necessarily contradict with prospect theory. Specifically, he argues that the market usually consists of "light" households who are price-sensitive and have a low probability of purchasing at the regular price, and "heavy" households who have a high purchase probability at the regular price even when there is a loss. As a result, during a promotion, when there is a gain, many more light households than heavy ones are attracted. Consequently, market demand is more sensitive to gains than losses even though each household may be more sensitive to losses than gains.

Therefore, one should realize the existence of gain-seeking at an aggregate level and it can be caused either by gain-seeking behavior at an individual level (with aggregation not changing the gain-loss asymmetry) or by aggregation under harsh market conditions. Even if, under certain market conditions, the gain-seeking model analyzed in this paper is not fully consistent with the model aggregating individual level behaviors, it may still be used as a prescriptive model to provide a plausible tractable approximation.

Despite the necessity of analyzing how the firm should respond if gain-seeking is present in the aggregate level demand, analytical results on the dynamic pricing problem of a firm in the gain-seeking case are very limited. To the best of our knowledge, the only result in this case is the non-optimality of the constant pricing strategy as observed by Kopalle et al. (1996) and Popescu and Wu (2007). Moreover, Popescu and Wu (2007) postulate that "High-low pricing, is provably optimal if consumers are focused on gains." However, we cannot find any rigorous proof in the literature. Indeed, there are many practical and interesting open questions left for the dynamic pricing problem. For example, is high-low pricing, in which only a regular price and a discount price are employed periodically, indeed optimal in general? If not, what are the conditions that guarantee its optimality? Furthermore, when high-low pricing is not optimal, what can we say about the optimal pricing strategy and the performance of the high-low pricing strategy?

This paper strives to answer the above questions. Specifically, we find that even the *myopic pricing strategy*, where the firm ignores the effect of current prices on future revenues and focuses on maximizing short-term revenue, does not always admit a cyclic high-low price pattern and its long-run behavior can be very complicated. We provide necessary and sufficient conditions for the existence of a cyclic high-low price pattern in the myopic pricing strategy. In addition, conditions are derived such that the myopic pricing strategy leads to either a cyclic penetrating pricing strategy in which the resulting reference prices increase within a cycle, or a cyclic skimming pricing strategy in which the resulting reference prices decrease within a cycle. Our numerical studies show that high-low pricing is generally not optimal and the dynamics of the optimal pricing strategy is likely to be significantly more complex than that resulted from the myopic pricing strategy. Interestingly, under the assumptions that consumers only remember the most recent price and the aggregate demand is insensitive to the negative part of the difference between reference price and price, we prove that the optimal pricing strategy is a cyclic skimming pricing strategy. In other words, when consumers have a low reference price, the firm should charge a high price and then gradually offer deeper and deeper discounts until consumers' reference price drops low again and repeat the cycle. The assumptions we made are also found to be very plausible in our empirical studies. We further provide sufficient conditions for the high-low pricing strategy to be optimal. Our numerical studies suggest that the high-low pricing strategy, when fails to be optimal, can still achieve over 90% of the optimal profit.

Our work is in sharp contrast to the stream of works in the dynamic pricing problem when loss-aversion is present at a market level. All these works arrive at the conclusion that a constant pricing strategy is optimal in the long run. Specifically, Kopalle et al. (1996) observe through numerical studies that optimal prices converge monotonically and conjecture that a constant price is optimal in the long run. Fibich et al. (2003) explicitly solve the optimal pricing strategy in a continuous time optimal control framework, and confirm the observation by Kopalle et al. (1996) when demand is piece-wise linear in price and reference price. Popescu and Wu (2007) extend the result of Fibich et al. (2003) to general demand functions in a discrete time infinite horizon setting. Nasiry and Popescu (2011) consider the dynamic pricing problem with a peak-end based reference price model; they also conclude the observation by Kopalle et al. (1996). Vast empirical literature, however, suggests that it is important for practitioners not to take loss-averse assumption as granted. When facing gain-seeking demands, our findings show that constant pricing strategy can result in as much as 50% loss in profit while simple cyclic pricing strategies (cyclic skimming or high-low pricing strategies) are optimal or close to optimal in many scenarios. This discrepancy in results between the loss-averse case and the gain-seeking case is due to the differences in the underlying structures of the optimization problems. In essence, under loss-aversion, the single period profit function, though non-smooth due to the asymmetric responses to losses and gains, is a concave function of the current price. The gain-seeking demand, on the other hand, changes the structure of the problem completely. The resulting optimization problem is neither smooth nor concave.

A few papers have considered dynamic pricing problems in various settings that also lead to cyclic pricing strategies. Conlisk et al. (1984) assume that consumers are strategic with two possible valuations and will remain in the market (possibly forever) until making a purchase. They establish that a cyclic skimming pricing strategy is optimal. Besbes and Lobel (2012) consider strategic consumers that are heterogeneous both in valuations and the time they may spend in the market. They prove that a cyclic pricing strategy is optimal but a cyclic penetrating or a cyclic skimming pricing strategy may yield arbitrary poor performance. Ahn et al. (2007) study both production and pricing decisions when consumers have uniformly distributed valuations and tend to buy the product as soon as the price drops below their valuations. They show, in a special case where consumers wait at most one period and the firm has no capacity constraint, that high-low pricing strategy is optimal. Liu and Cooper (2014) consider a pricing setting similar to Ahn et al. (2007) and demonstrate that a cyclic skimming pricing strategy is optimal even when consumers have general valuation distributions and can wait for multiple periods. We emphasize here some key distinctions of our work from these papers. In terms of modeling, the argument behind reference price models is that consumers' purchase decisions are affected by the prices in the past rather than anticipated prices in the future. In terms of proofs, we form a dynamic programming problem and identify the cyclic pricing strategy by analyzing properties of the value function whereas the above papers all tackle their problems directly by utilizing the notion of a regeneration point. Essentially, the driving force that leads to a cyclic pricing strategy in our model is the gain-seeking behavior of consumers while in these papers is the waiting behavior of consumers. Another related paper is Geng et al. (2010) who restrict to high-low pricing strategies and use a different approach by taking weighted average between the regular price and the discount price to model reference price. They show that when demand is gain-seeking, high-low pricing strategy outperforms constant pricing strategy. Our work following Fibich et al. (2003), Kopalle et al. (1996) and Popescu and Wu (2007), on the other hand, models the dynamics and intertemporal effects of reference prices explicitly and settles the conjecture in the existing literature by showing that high-low pricing strategy may not be optimal.

Our work is also closely related to the on-going research in the one-dimensional discontinuous map in the dynamic system and chaos community. Sharkovsky and Chua (1993) examine a certain type of discontinuous maps that arise in electric circuits. They find that their class of discontinuous maps has strong temporal chaos and the behavior of trajectories can only be characterized by using probability language. Jain and Banerjee (2003) present a classification of border-collision bifurcations in discontinuous maps. Depending on parameters, the resulting dynamics can have various periodic orbits or chaos. Rajpathak et al. (2012) analyze in detail the stable periodic orbits of one type of discontinuous maps and explore the possible patterns exhibited by these orbits. It turns out that the myopic pricing strategy in our work can be reduced to the type of discontinuous maps analyzed in Rajpathak et al. (2012). However, to the best of our knowledge, the class of discontinuous maps with multiple discontinuous points, into which our optimal pricing strategy typically falls, has not been considered in the previous literature.

The remainder of this paper is organized as follows. In Section 2 the mathematical formulation of our model is presented. In Section 3, we analyze the dynamics of the myopic pricing strategy and relate it with the work in discontinuous maps. The structural results for the optimal pricing strategy are presented in Section 4. Section 5 presents an empirical study and conducts numerical experiments to test the performance of simple pricing strategies. Finally, we conclude the paper in the last section with some suggestions for future research. The proofs are all relegated to Appendix A.

2. Model

This section presents the dynamic pricing model. First we describe how consumers form reference price and how a gain-seeking reference price effect affects the demand for a firm's product.

We adopt one of the well-studied consumer behavioral pricing models in the marketing literature, where the impact of past prices on the demand is captured by the *reference price effect*. This type of models argues that consumers develop price expectations, called the *reference prices*, based on past observed prices and use them to judge the purchase price of a product (see Mazumdar et al. 2005, for a review). Among many different reference price models, a memory-based model, is commonly used and empirically validated on scanner panel data for a variety of products (see, for example, Greenleaf 1995). In this model, reference price is generated by exponentially weighting past prices. Specifically, starting with a given initial reference price r_0, the reference price at period t, denoted by r_t, evolves as

$$r_{t+1} = \alpha r_t + (1-\alpha)p_t, \quad t \geq 0. \tag{1}$$

In the above evolution equation, $p_t \in [0, U]$ is the price charged by the firm at period t, where U is the upper bound on feasible prices. The parameter $\alpha \in [0, 1)$ is called the memory factor or carryover constant (Kalyanaram and Little 1994). Observe that as α increases, "consumers change their reference prices more slowly to incorporate new price information (Greenleaf 1995)." When $\alpha = 1$, reference prices remain a constant over the whole planning horizon and consequently a constant pricing strategy is optimal irrespective of gain-loss asymmetry. We restrict $\alpha < 1$ to avoid such a case that past prices have no impact on demand. As reference prices are generated from historical prices, it is also reasonable to assume that $r_0 \in [0, U]$.

Following Greenleaf (1995), Kopalle and Winer (1996), Fibich et al. (2003) and Nasiry and Popescu (2011), the demand depends on the price p and reference price r via the model

$$D(r,p) = \begin{cases} b - ap + \eta^+(r-p), & r > p, \\ b - ap, & r = p, \\ b - ap + \eta^-(r-p), & r < p, \end{cases} \tag{2}$$

where $b, a > 0$ and $\eta^+, \eta^- \geq 0$. More concisely, we can write

$$D(r,p) = b - ap + \eta^+ \max\{r-p, 0\} + \eta^- \min\{r-p, 0\}.$$

Here, $D(p,p) = b - ap$ is the base demand independent of reference prices, $\eta^+(r-p)$ or $\eta^-(r-p)$ is the additional demand or demand loss induced by the reference price effect. To avoid negative demand, we further assume that $D(0,U) \geq 0$, i.e., $U \leq \frac{b}{a+\eta^-}$. The difference between reference price and selling price, i.e., $r - p$, in the above demand model is usually referred to as a perceived surcharge/discount. If $r < p$, consumers perceive this as a loss, while if $r > p$, they perceive it as a gain.

Consumers or the aggregate level demands are classified as loss averse, loss neutral and gain-seeking depending on whether $\eta^+ < \eta^-$, $\eta^+ = \eta^-$ or $\eta^+ > \eta^-$. In this paper we focus on the case when gains have greater impact than losses, i.e., $\eta^+ > \eta^-$.

The firm's one-period profit is denoted as $\Pi(r,p) = pD(r,p)$. Here, the marginal cost is assumed to be 0 for simplicity. All our results can be extended to cases with a non-zero marginal cost. We assume $U \geq \frac{b}{2a}$ such that $\Pi(p,p)$, called the base profit, is not monotone in $p \in [0,U]$. This assumption allows us to "rule out pathological boundary steady states (Popescu and Wu 2007).", but our analysis can be carried over similarly by distinguishing those boundary steady states when this assumption fails. Note that the assumptions $\eta^+ > \eta^-$ and $p \geq 0$ allow us to rewrite the one-period profit as

$$\Pi(r,p) = \max\{\Pi^+(r,p), \Pi^-(r,p)\},$$

where $\Pi^+(r,p) = p[b - ap + \eta^+(r-p)]$ and $\Pi^-(r,p) = p[b - ap + \eta^-(r-p)]$. Contrary to the loss-averse case, the one-period profit function is no longer a concave function in p.

Given an initial reference price r_0, the firm's long-term profit maximization problem is then:

$$V(r_0) = \max_{p_t \in [0,U]} \sum_{t=0}^{\infty} \gamma^t \Pi(r_t, p_t), \tag{3}$$

where $\gamma \in [0,1)$ is a discount factor and we interpret $0^0 = 1$. The infinite horizon problem is of particular interest in the literature since it is often more tractable than the finite horizon counterpart and provides valuable insights into the long-run behavior of the optimal pricing strategy, which in turn may shed light on the development of efficient heuristics for finite horizon models. It is worth noting here that two assumptions commonly imposed on optimization problems: differentiability and concavity in the decision variables, are both absent in the one-period profit function $\Pi(r,p)$, which makes the analysis of problem (3) quite challenging.

The Bellman equation for problem (3) is

$$V(r) = \max_{p\in[0,U]} \Pi(r,p) + \gamma V(\alpha r + (1-\alpha)p). \tag{4}$$

A *pricing strategy* $p(r)$ is a function from $[0,U]$ to $[0,U]$ that specifies a feasible solution to (4) for a given reference price r. Given any pricing strategy $p(r)$, the sequence $\{r_t\}$ of reference prices which evolve according to $r_{t+1} = \alpha r_t + (1-\alpha)p(r_t)$, is referred to as the *reference price path* of the pricing strategy $p(r)$. We say $p(r)$ has a *periodic orbit* of period n or is a *cyclic pricing strategy* with cycle length n if and only if there exists $r_0 \in [0,U]$ such that the reference price path of $p(r)$ satisfies $r_n = r_0$ and $r_t \neq r_0$ for all $0 < t < n$. Clearly, if $r_n = r_0$, then by $r_{t+1} = \alpha r_t + (1-\alpha)p(r_t)$, the sequence $\{r_0, ..., r_{n-1}\}$ is repeated infinitely over time and this sequence is referred to as the *periodic orbit* of $p(r)$. In particular, when $n = 1$, we say $p(r)$ admits a *steady state* r_0 and when $n = 2$, $p(r)$ is a *high-low pricing strategy.* If there exists a periodic orbit that has the additional property that $r_0 < r_1 < ... < r_{n-1}$, then we refer $p(r)$ to as a *cyclic penetrating pricing strategy.* If, on the other hand, $r_0 > r_1 > ... > r_{n-1}$, then we refer $p(r)$ to as a *cyclic skimming pricing strategy.* Note that in the special case when $\alpha = 0$, $r_0 < r_1 < ... < r_{n-1}$ $(r_0 > r_1 > ... > r_{n-1})$ if and only if $p(r_{n-1}) < p(r_0) < ... < p(r_{n-2})$ $(p(r_{n-1}) > p(r_0) > ... > p(r_{n-2}))$, i.e., the monotonicity of reference prices is equivalent to the monotonicity of charged prices. However, for $\alpha > 0$, it is possible to have monotone reference prices with non-monotone charged prices. Recall that in practice, a skimming (penetrating) pricing strategy is used to describe pricing strategy with decreasing (increasing) price path overtime. Here, we use the term "skimming" ("penetrating") to reflect the fact that a skimming (penetrating) pricing strategy is usually designed to capture consumers with decreasing (increasing) valuations (an analogy of the notion "reference price" in our model) overtime. Since $[0,U]$ is compact and the objective function can be easily shown to be continuous, the *optimal pricing strategy* that solves (4) exists and is denoted by $p^*(r)$. As mentioned in Section 1, Kopalle et al. (1996) and Popescu and Wu (2007) prove that $p^*(r)$ does not admit a steady state. That is, for any $r \in [0,U]$, $p^*(r) \neq r$.

In the following, we will also use the term *pattern* to describe the existence of various monotonic structures within a periodic orbit. One is referred to Rajpathak et al. (2012) for a rigorous definition in the context of discontinuous maps. Here, we illustrate the term through a simple example. Consider a periodic orbit with period 4 that consists of four different reference prices $1, 2, 3, 4$. Then, depending on different orderings, the periodic orbit can demonstrate different patterns or monotonic structures as illustrated in Figure 1. The upper left and lower right panels show the reference price

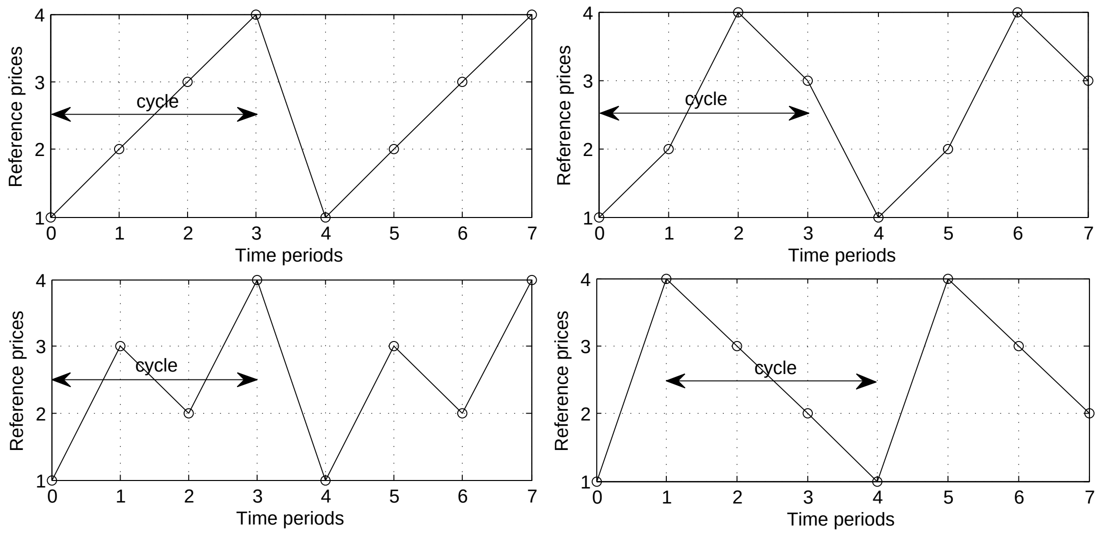

Figure 1 Possible patterns of a periodic orbit

patterns that are of penetrating and skimming pricing strategy respectively. However, the upper right panel shows a pattern that has reference prices increase in the first two periods accompanied by a decrease in the third period while the lower left panel shows a pattern that the reference prices alternate between increasing and decreasing in each period.

3. Dynamics of the Myopic Pricing Strategy

In this section we demonstrate the complicated nature of problem (3) by analyzing the dynamics of the myopic pricing strategy. We present conditions to guarantee that the myopic pricing strategy admits a high-low price pattern. We then reveal the complexity of the underlying dynamics by showing that even the myopic pricing strategy can result in a cyclic pricing strategy with a cycle length arbitrary long.

By definition, the myopic pricing strategy $p^m(r)$ is given by solving the following problem:

$$p^m(r) = \arg\max_{p\in[0,U]} \Pi(r,p).$$

Define the constant

$$R = \frac{b}{a+\sqrt{(a+\eta^+)(a+\eta^-)}}. \tag{5}$$

LEMMA 1. *Let $R_U = \frac{2(a+\eta^-)U-b}{\eta^-}$. Then, if $R \le R_U$,*

$$p^m(r) = \begin{cases} \dfrac{\eta^- r + b}{2(a+\eta^-)}, & r \le R, \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & r > R. \end{cases} \tag{6}$$

If $R > R_U$, then

$$p^m(r) = \begin{cases} \dfrac{\eta^- r + b}{2(a+\eta^-)}, & r \le R_U, \\ U, & R_U < r \le R', \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & r > R', \end{cases}$$

where R' is the unique positive root for

$$\eta^+ r^2 + [2b\eta^+ - 4(a+\eta^+)\eta^- U]r + b^2 - 4(a+\eta^+)U[b-(a+\eta^-)U] = 0.$$

To keep the presentation clear and simple, we assume for the rest of this section that $R \le R_U$. That is, $p^m(r)$ is determined by (6). The analysis presented in this section can also be extended to the other case with additional discussions on whether U will appear on the periodic orbit or not.

Note that $p^m(r)$ is not a continuous function. As a result the dynamics of reference prices under the myopic pricing strategy

$$r_{t+1}(r_t) = \alpha r_t + (1-\alpha)p^m(r_t), \quad t = 0,1,..., \tag{7}$$

ends up with a *discontinuous map* from $[0,U]$ to $[0,U]$. The analysis of the dynamics (7) is not trivial at all. In fact, the study of dynamic systems with discontinuous maps is originated in the analysis of electrical circuits and is considered as "... a very complicated research subject and we can obtain useful and interesting results only for various special classes of maps (Sharkovsky and Chua 1993)." For example, let mod denote the modulo operation, then the dynamics of the famous doubling map $D:[0,1]\rightarrow[0,1]$ defined by $D(x)=2x \bmod 1$ is *chaotic*, which means such a dynamical system is highly sensitive to initial conditions (Hirsch et al. 2004).

Interestingly, we show that under some conditions, there exists a high-low price pattern under the myopic pricing strategy.

PROPOSITION 1. *$p^m(r)$ is a high-low pricing strategy with a periodic orbit $\{r_0, r_1\}$ for some $r_0, r_1 \in [0,U]$ and $r_0 \neq r_1$ if and only if the following inequality holds*

$$4(1-\alpha^2)a^2 + 4(1-\alpha-\alpha^2)a\eta^+ + 4a\eta^- - (1+\alpha)^2(\eta^+)^2 + 4\eta^+\eta^- \geq 0. \tag{8}$$

Notice that (8) holds when $\alpha=0$ (consumers only remember the price of the previous period) and when the direct price effect weakly dominates the reference price effect ($4a \geq \eta^+$).

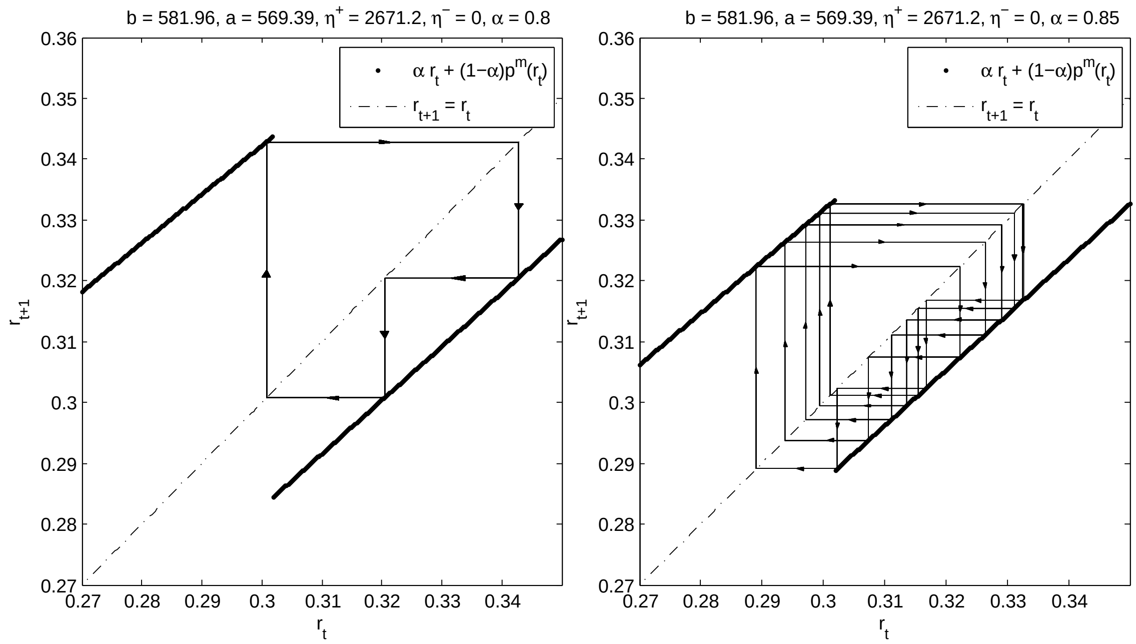

Figure 2 Discontinuous map (7) and periodic orbits for the myopic pricing strategies when $\alpha=0.8$ and $\alpha=0.85$ respectively

Unfortunately, condition (8) will be violated when α is close to 1. Figure 2 gives examples to illustrate the possible dynamics when condition (8) fails. The demand parameters used in Figure

2 come from our empirical examples in Section 5. In Figure 2, the bold lines represent the discontinuous map (7), which maps r_t to r_{t+1}. The arrowed lines illustrate a periodic orbit. Specifically, the vertical arrowed lines indicate that the trajectory evolves from r_t to r_{t+1} while the horizontal arrowed lines visually aid us in thinking the function value r_{t+1} as an argument of the next map. In each panel, the arrowed lines form a closed loop so it is indeed a periodic orbit. In fact, the periodic orbit in the left panel has period 3 and is of skimming pattern. The periodic orbit in the right panel, on the other hand, has period 16 and the myopic pricing strategy in this case is neither a cyclic penetrating nor a cyclic skimming pricing strategy. By comparing the two panels, one can see that a mere increment of 0.05 in α can result in dramatic changes in the dynamics.

As illustrated in Figure 1, when the period is greater than or equal to 4, there could exist other patterns rather than penetrating or skimming pattern that have the same period. Our numerical experiments illustrate that both the period and pattern of a periodic orbit can be very sensitive to the changes in parameters, a phenomenon termed as *border collision bifurcation* in the dynamic system and chaos community (Jain and Banerjee 2003). In the following, we identify necessary and sufficient conditions for the myopic pricing strategy to admit a cyclic penetrating pattern and a cyclic skimming pattern with cycle length n respectively. To simplify the expressions of our conditions, define constants

$$\mu = \frac{a+\eta^+ - \sqrt{a^2 + a\eta^+ + a\eta^- + \eta^+\eta^-}}{\eta^+ - \eta^-}, \quad A = 1+\alpha-\alpha\frac{\eta^-}{2(a+\eta^-)}, \quad B = 1+\alpha-\alpha\frac{\eta^+}{2(a+\eta^+)},$$

and denote for $n \geq 0$ the sum of geometric series $\sum_{i=0}^{n} k^i$ by S_n^k (for $n<0$, let $S_n^k = 0$).

PROPOSITION 2. *For $n \geq 2$, $p^m(r)$ is a cyclic penetrating pricing strategy with cycle length n if and only if the following inequalities hold*

$$\frac{A^{n-1}}{S_{n-1}^A} < \mu \leq \frac{A^{n-2}}{A^{n-2}B + S_{n-2}^A}. \tag{9}$$

On the other hand, $p^m(r)$ is a cyclic skimming pricing strategy with cycle length n if and only if the following inequalities hold

$$\frac{AB^{n-2} + S_{n-3}^B}{AB^{n-2} + S_{n-2}^B} < \mu \leq \frac{S_{n-2}^B}{S_{n-1}^B}. \tag{10}$$

Note that when $n = 2$, both (9) and (10) reduce to $\frac{A}{1+A} < \mu \leq \frac{1}{1+B}$, which can be simplified to (8) by substituting the expressions for A, B and μ, and by definition, a high-low pricing strategy is both a cyclic penetrating and a cyclic skimming pricing strategy. For n large enough, there exists other range of parameters other than (9) and (10) resulting periodic orbits of period n that are neither penetrating nor skimming. In fact, corresponding to different parameter regions, there are exactly $\phi(n)$ different patterns of periodic orbits of period n, where $\phi(\cdot)$ is the so-called Euler totient function (Rajpathak et al. 2012).

4. Optimal Pricing Strategy

Unlike the myopic pricing strategy, we do not have an explicit solution for the optimal pricing strategy, which makes the analysis significantly more challenging. To illustrate the difficulty, we first present a few properties on the value function and optimal solution.

Since $\Pi(r,p) = \max\{\Pi^+(r,p), \Pi^-(r,p)\}$, the Bellman equation can be correspondingly rewritten as

$$V(r) = \max_{p \in [0,U]} \{\max\{\Pi^+(r,p), \Pi^-(r,p)\} + \gamma V(\alpha r + (1-\alpha)p)\}. \tag{11}$$

We assume, without loss of generality, that $p^*(r)$ and the optimal solutions for other optimization problems in this section always take the largest one among multiple solutions.

Consider the following two problems:

$$V^+(r) = \max_{p \in [0,U]} \Pi^+(r,p) + \gamma V(\alpha r + (1-\alpha)p), \tag{12a}$$

$$V^-(r) = \max_{p \in [0,U]} \Pi^-(r,p) + \gamma V(\alpha r + (1-\alpha)p). \tag{12b}$$

The solutions of (12a) and (12b) are denoted respectively as $p^+(r)$ and $p^-(r)$. An observation here is that $V(r) = \max\{V^+(r), V^-(r)\}$ and $p^*(r) \in \{p^+(r), p^-(r)\}$. We next characterize properties of $V^{\pm}(r)$ and $p^{\pm}(r)$.

LEMMA 2. *Both $V^+(r)$ and $V^-(r)$ are increasing and convex in r while $p^+(r)$ and $p^-(r)$ are increasing in r.*

Although problem (11) is difficult to analyze due to the term $\max\{\Pi^+(r,p),\Pi^-(r,p)\}$, Lemma 2 shows that the decomposed problems (12a) and (12b) have some desired properties, i.e., monotonic solutions and optimal objective values. We are interested in how $p^+(r)$ and $p^-(r)$ relate to the optimal pricing strategy $p^*(r)$, and whether it is possible to obtain simple characterizations for $p^+(r)$ and $p^-(r)$. For this purpose, we draw in Figure 3 the (numerically approximated) optimal pricing strategy for the example used in the left panel of Figure 2 with a discount factor $\gamma = 0.9$. In

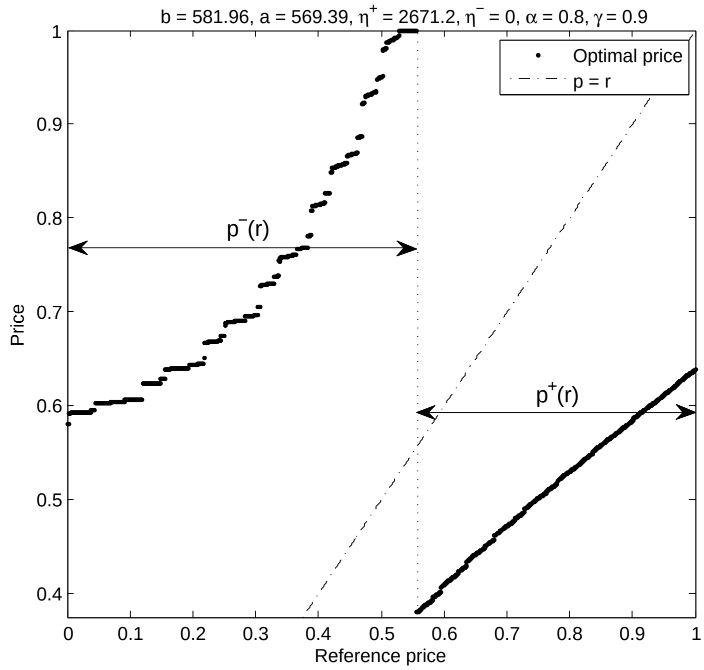

Figure 3 Optimal pricing strategy when $\alpha = 0.8, \eta^- = 0$ and $\gamma = 0.9$

Figure 3, we observe that, similar to the myopic pricing strategy, there exists a point $\hat{r} \in [0, U]$ such that $p^*(r)$ is given by $p^-(r)$ for $r \le \hat{r}$ and $p^+(r)$ for $r > \hat{r}$. Unfortunately, though this observation seems to hold in all our numerical experiments, we do not have a proof for general parameter configurations. Of course, even if this observation is indeed true, Figure 3 suggests that both $p^+(r)$ and $p^-(r)$ can have numerous discontinuous points, in contrast to the two linear pieces in the myopic pricing strategy. This implies that they may not admit any simple characterizations.

In terms of dynamics, we have already demonstrated the dynamics of the myopic pricing strategy (which has only one discontinuous point and admits explicit expression) in Section 3. Here, we further give a side-by-side comparison of the dynamics of the optimal pricing strategies and their periodic orbits in Figure 4 for the examples used in Figure 2 with the discount factor $\gamma = 0.9$. The bold lines in Figure 4 represent the map $\alpha r_t + (1-\alpha)p^*(r_t)$ and as α is close to one in both instances, many discontinuous points observed in Figure 3 become difficult to be distinguished in Figure 4 through visual inspection. When comparing Figure 2 and Figure 4, it is clear that the period lengths under optimal pricing strategies can be much larger than these under myopic pricing strategies, which indicates that the dynamics of the optimal pricing strategy can be much more complicated and the analysis, if possible, is likely to be significantly more challenging than that for the myopic pricing strategy.

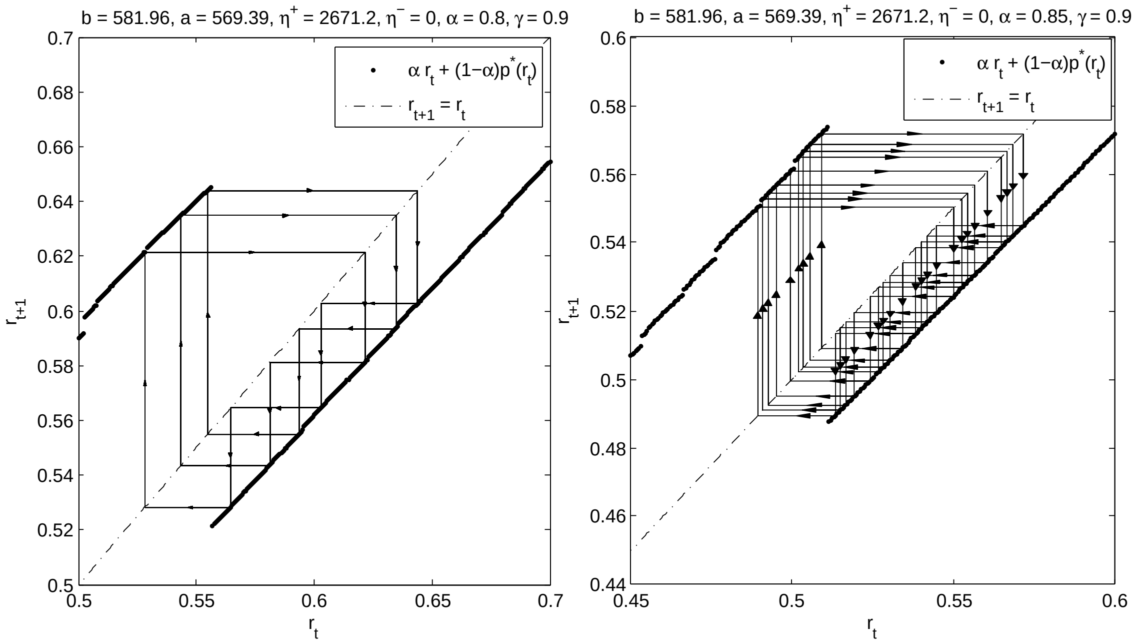

Figure 4 Periodic orbits for the optimal pricing strategies when $\alpha = 0.8$ and $\alpha = 0.85$ respectively

Thus, for the rest of this section, we focus on a special case satisfying the following assumption.

ASSUMPTION 1. *Consumers only remember the most recent price (i.e. $\alpha = 0$) and the demand is insensitive to the perceived surcharge (i.e. $\eta^- = 0$).*

Assumption 1 seems restrictive but has very plausible explanations. First of all, consumers are unlikely to remember many historical prices and form reference price by averaging them. Several papers, for example, Raman and Bass (2002), Krishnamurthi et al. (1992), Mayhew and Winer

(1992), also assume that $\alpha = 0$. It captures the fact that "consumers... experience considerable difficulty in recalling accurately even the most recently encountered prices ... Thus, it is unlikely that consumers would retrieve from memory and use prices encountered much beyond the immediate past purchase occasion (Krishnamurthi et al. 1992)." On the other hand, $\eta^- = 0$ models the market of promotion-driven products, where the demand of product consists of the base demand $b - ap$ and promotion stimulated demand $\eta^+ \max\{r - p, 0\}$. This also reflects a harsh market condition under which the firm is very likely to face a gain-seeking demand (Kallio and Halme 2009).

Assumption 1 also provides good approximation to some practical scenarios. Specifically, we provide empirical examples in Section 5, in which by imposing Assumption 1 does not result in much loss in the goodness of fit of the model.

An immediate consequence from Assumption 1 is that both $V^-(r)$ and $p^-(r)$ are now constant functions. In the sequel, we will use constants p^- and V^- to denote the function values of $p^-(r)$ and $V^-(r)$. $V^-(r)$ being a constant function is critical for the simplification of the problem, as it allows us to relate $p^+(r)$ and $p^-(r)$ with $p^*(r)$ in a simple way as demonstrated in the following lemma.

LEMMA 3. *Under Assumption 1, there exists* $R_0 \in (0, U)$ *such that if* $r \le R_0$*, then* $V(r) = V^-$ *and* $p^*(r) = p^- > r$*. If* $r > R_0$*, then* $V(r) = V^+(r)$ *and* $p^*(r) = p^+(r) < r$*.*

Lemma 3 gives us a broad picture of what $p^*(r)$ looks like. That is, when $r \le R_0$, $p^*(r)$ is a constant function and is always above r. At the point R_0, there is a "downward jump" from $p^- > R_0$ to $p^+(R_0) < R_0$. When $r > R_0$, $p^*(r)$ is then monotonically increasing in r. In the sequel, we will briefly sketch the idea of how to characterize $p^+(r)$, which leads to a complete characterization of the optimal pricing strategy $p^*(r)$.

Let us reconsider problem (12a) when $r \in [R_0, U]$ and keep in mind that $p^+(R_0) < R_0$ and $p^+(r)$ is increasing on $[R_0, U]$. We distinguish between two cases:

Case 1: For any $r \in [R_0, U]$, $p^+(r) \le R_0$. In this case, $V(p^+(r)) = V^-$ and there is no loss of optimality to write (12a) as

$$V^+(r) = \max_{p \in [0,U]} \Pi^+(r, p) + \gamma V^-,$$

from which we can explicitly solve

$$p^+(r) = \frac{\eta^+ r + b}{2(a+\eta^+)}.$$

Then, we have completely characterized the optimal solution as

$$p^*(r) = \begin{cases} p^-, & 0 \le r \le R_0, \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & r > R_0. \end{cases}$$

Case 2: There exists $R_1 \in (R_0, U)$ such that $p^+(r) \le R_0$ for any $r \in [R_0, R_1]$ and $p^+(r) \ge R_0$ for any $r \in [R_1, U]$. In this case, going through the same argument in Case 1, we arrive at

$$p^+(r) = \frac{\eta^+ r + b}{2(a+\eta^+)}, \quad r \in [R_0, R_1],$$

and

$$V^+(r) = \frac{(\eta^+ r + b)^2}{4(a+\eta^+)} + \gamma V^-, \quad r \in [R_0, R_1].$$

When $r \in [R_1, U]$, however, we need to again distinguish between two cases based on whether $p^+(r) < R_1$ for all $r \in [R_1, U]$ or not.

Essentially, repeating the analysis sketched above, we can arrive at the main result of this section. Let $m_1 = 0$ and for $k > 1$, $m_k = \frac{\gamma \eta^+}{2(a+\eta^+) - m_{k-1}\eta^+}$.

Proposition 3. *Under Assumption 1, there exists an integer $N \ge 0$ and $0 < R_0 < R_1 < ... < R_N < U = R_{N+1}$ such that*

$$p^*(r) = \begin{cases} p^-, & 0 \le r \le R_0, \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & R_0 < r < R_1, \\ \dfrac{\eta^+ r + b + \sum_{i=0}^{k} (\prod_{j=0}^{i} m_{k+1-j}) b}{2(a+\eta^+) - m_{k+1}\eta^+}, & R_k \le r < R_{k+1}, \quad k = 1, ..., N. \end{cases}$$

Now we have a complete picture of the optimal pricing strategy $p^*(r)$. After a "downward jump" at R_0, $p^*(r)$ follows a piece-wise linear function with finitely many "upward jumps" at R_k for $k = 1, ..., N$. Moreover, the slopes of these linear pieces increase after each of the "upward jump" by the expression of $p^*(r)$ given in Proposition 3.

Another important insight from our analysis is that for $k = 1, ..., N$, $p^*(r)$ maps $[R_k, R_{k+1}]$ to $[R_{k-1}, R_k]$ and finally $p^*(r)$ maps $[R_0, R_1]$ to $[0, R_0]$. This leads us to the study of the dynamics of $p^*(r)$, which is a discontinuous map with more than one discontinuous point. Even though the dynamics of a discontinuous map with only one discontinuous point is already complicated, the characterization presented in Proposition 3 allows us to find simple dynamics for $p^*(r)$. Denote $p_1^*(r) = p^*(r)$ and $p_i^*(r) = p^*(p_{i-1}^*(r))$ for $i > 1$. To make the notation clearer, in the following, we alternatively use r^* to denote the constant p^-.

Proposition 4. *Let N be the integer in Proposition 3. Under Assumption 1, there exists an integer n with $2 \le n \le N+2$, such that $p_n^*(r^*) = r^*$ and for all $r_0 \in [0, U]$, the optimal reference price path r_t^* converges in at most $N+2$ periods to the unique periodic orbit: $\{r^*, p_1^*(r^*), ..., p_{n-1}^*(r^*)\}$, i.e., there exists $0 \le \tau \le N+2$ such that $r_\tau^* = r^*$. Moreover, the periodic orbit has the property $r^* > p_1^*(r^*) > ... > p_{n-1}^*(r^*)$, i.e., $p^*(r)$ is a cyclic skimming pricing strategy.*

Proposition 4 suggests that the following pricing strategy for practitioners when the demand they face is gain-seeking and promotion-driven: when consumers' initial reference price is low (below R_0), the firm should use a regular price (r^*). Then the firm applies a skimming pricing strategy by gradually discounting the regular price over time until consumers' reference price falls below R_0, and repeats such pricing strategy. The intuition behind is easy to understand. When consumers have a low reference price, as they are gain-seeking, it will not hurt the firm too much by setting a high price in order to drag consumers' reference price to a higher level. After such manipulation, the firm will benefit greatly by offering discounts since consumers are sensitive to gains and there will be a boost in demand.

An illustration of the optimal pricing strategy and the periodic orbit is provided in Figure 5. The parameters used for Figure 5 are taken from one of the empirical examples provided in Section 5 (also one can refer to Appendix C), except $\gamma = 0.1$ here. One can see that, indeed, there are more than one discontinuous point and in this particular example the periodic orbit has period 3 ($n = 3$).

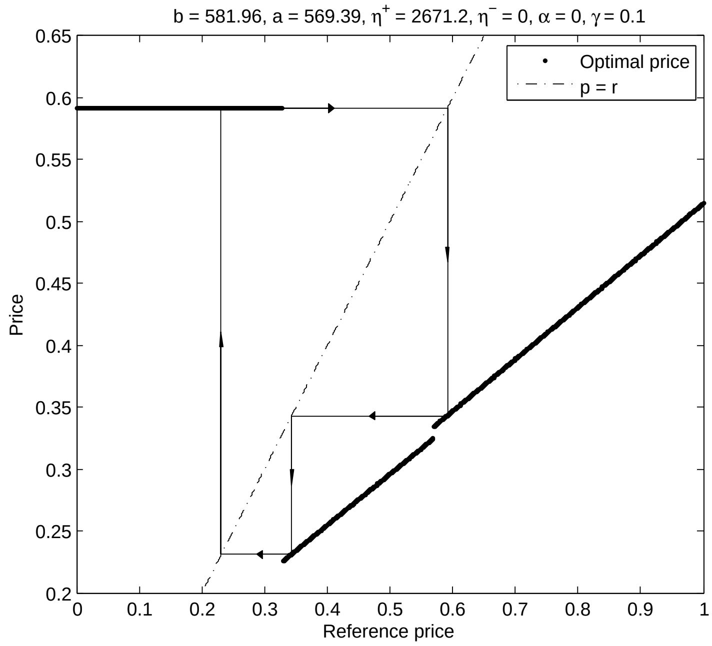

Figure 5 Optimal pricing strategy and a periodic orbit

Next, we identify conditions on parameters such that a high-low pricing strategy is optimal. Define the constant $K = \frac{a+\eta^+ - \sqrt{(a+\eta^+)^2 - \gamma(\eta^+)^2}}{\eta^+}$ and recall the constant $R = \frac{b}{a+\sqrt{(a+\eta^+)(a+\eta^-)}} = \frac{b}{a+\sqrt{a(a+\eta^+)}}$ (here, by Assumption 1, $\eta^- = 0$), which is the discontinuous point in the myopic pricing strategy, defined in (5).

Proposition 5. *Under Assumption 1, if the following inequality also holds,*

$$\frac{\eta^+ U + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} \leq R, \tag{13}$$

then a high-low pricing strategy $\{p_H, p_L\}$ is optimal, where $p_H = r^*$ and $p_L = \frac{\eta^+ r^* + b}{2(a+\eta^+)}$.

Proposition 5 above formally settles the conjecture of Popescu and Wu (2007) in the sense that it provides a verifiable condition from problem parameters that guarantees the optimality of high-low pricing strategy. One direct implication of condition (13) is that a high-low pricing strategy is always optimal if the feasible prices are not too high, i.e., U is sufficiently small. This is intuitive since if the firm's highest possible price is already low, then there is not much room for the firm to set different discount levels. Condition (13) is only a sufficient condition, a necessary and sufficient condition is hard to obtain since we have no prior knowledge on N, $R_0, ..., R_N$, nor r^*

and it is possible that $n < N$, which means it is not clear which linear piece p_L will lie in. However, Proposition 5 suggests a possibility of solving explicitly the optimal cyclic pricing strategy. That is, if we know the period is exactly n, then we are able to exploit such structures to solve for r^* and the discontinuous points $R_0, R_1, ..., R_N$ given in Proposition 3. Next, we identify conditions such that the period of the optimal cyclic pricing strategy is at most n.

PROPOSITION 6. *Let $\underline{R}_0 = R$ and for $k \geq 1$ recursively define $\underline{R}_k$ to be the unique solution of the equation*

$$\frac{\eta^+ r + b + \sum_{i=0}^{k}(\prod_{j=0}^{i} m_{k+1-j})b}{2(a+\eta^+) - m_{k+1}\eta^+} = \underline{R}_{k-1}.$$

Under Assumption 1, if the following inequality holds for some $k \geq 0$

$$\frac{\eta^+ U + \frac{1}{1-K}b}{2(a+\eta^+) - K\eta^+} \leq \underline{R}_k, \tag{14}$$

then the length of period n must satisfy $n \leq k+2$.

Note that the recursively defined constants $\underline{R}_k$ provide lower bounds for the unknown constants R_k given in Proposition 3, i.e., $\underline{R}_k \leq R_k$ for $k = 0, 1, ..., N$ (see the proof of Proposition 6 in Appendix A). Similar to Proposition 5, one practical implication here is that the cycle length or the complexity of the pricing strategy depends on the flexibility of pricing, i.e., the range of feasible prices. When U is small, there is less room for the firm to apply intricate pricing strategy that will result in a large cycle length. The advantage of condition (14) is that it can be easily verified from the problem parameters. For instance, from the parameters we used in Figure 5 and with $U = 1$, it is straightforward to compute that

$$\frac{\eta^+ U + \frac{1}{1-K}b}{2(a+\eta^+) - K\eta^+} = 0.502$$

and

$$\underline{R}_0 = 0.302, \quad \underline{R}_1 = 0.493, \quad \underline{R}_2 = 0.949.$$

Applying Proposition 6, we know that the optimal solution has a periodic orbit with period less than or equal to 4. Indeed, one can see from Figure 5 that the optimal solution has a periodic orbit with period 3.

Finally, we present how to solve for r^* and $R_0, R_1, ..., R_N$ if we know the period is exactly n. First, it is straightforward to see from our previous analysis that if the period is n then $r^* \in [R_{n-2}, R_{n-1})$. Next, let $V_0(r) = V^-$, $V_0^m(r) = 0$ and recursively for $k = 1, ..., N$ define

$$V_k(r) = \max_{p\in[0,U]} \Pi^+(r,p) + \gamma V_{k-1}(p),$$

$$V_k^m(r) = \max_{p\in[0,U]} \Pi^+(r,p) + \gamma V_{k-1}^m(p).$$

Note that the expressions for $V_k^m(r), k = 1, ..., N$ can be explicitly computed and it is not difficult to see that

$$V_k(r) = V_k^m(r) + \gamma^k V_0(r) = V_k^m(r) + \gamma^k V^-.$$

In the proof of Proposition 3, given in Appendix A, we have shown that

$$V(r) = V_{n-1}(r), \quad r \in [R_{n-2}, R_{n-1}).$$

Recall that

$$r^* = p^- = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V(p),$$

$$V^- = \max_{p\in[0,U]} p(b - ap) + \gamma V(p).$$

Thus, without loss of optimality, r^* and V^- can be explicitly solved by

$$r^* = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V_{n-1}(p) = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V_{n-1}^m(p),$$

$$V^- = \max_{p\in[0,U]} p(b - ap) + \gamma V_{n-1}(p) = \frac{1}{1-\gamma^n}[r^*(b - ar^*) + \gamma V_{n-1}^m(r^*)].$$

Once the expression for V^- is obtained, we have the explicit expressions for all $V_k(\cdot)$ for $k = 0, ..., N$. By continuity of $V(\cdot)$, for $k \geq 0$, R_k can be sequentially computed by solving the equation

$$V_k(R_k) = V_{k+1}(R_k),$$

and finally, N is obtained by $N = \sup\{k : R_k \leq U\}$.

Combined with Proposition 6, if we can find some $k \geq 0$ such that (14) holds, then we can repeat the above computations by assuming the period $n = 2, ..., k+2$. As the value function is the unique solution to the Bellman equation (4), the process we suggested is also guaranteed to yield a unique solution. We illustrate the computing process above by continuing with the example in Figure 5. By Proposition 6, the period $n \le 4$. As a start, we assume $n = 2$, then

$$r^* = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V_1^m(p) = 0.5890$$

and consequently we can solve

$$R_0 = 0.3290, \quad R_1 = 0.5691.$$

Notice that $r^* \notin [R_0, R_1)$ which leads to a contradiction with $n = 2$.

We proceed to assume $n = 3$, then

$$r^* = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V_2^m(p) = 0.5915$$

and

$$R_0 = 0.3291, \quad R_1 = 0.5692, \quad R_2 > 1.$$

Here, $r^* \in [R_1, R_2)$ is consistent with our assumption that $n = 3$. Thus, we have solved explicitly r^* and R_0, R_1.

5. Numerical Study

In this section, we first provide some empirical examples and try to understand what are the implications of real data for the gain-loss asymmetry as well as other parameters used in our model. In particular, we show that Assumption 1 generally gives a more parsimonious model while retaining most of the explanatory power of the full model. We then study the performance of several simple pricing strategies as opposed to the optimal pricing strategy. We further examine the performance of simple cyclic pricing strategies and the robustness of Proposition 4 numerically when Assumption 1 is violated.

5.1. Empirical Examples

We utilize the data set provided by Chevalier et al. (2003) of the canned tuna product category in the Bayesm Package of the R software. The data set includes volume of canned tuna sales as well as a measure of display activity, log price and log wholesale price of seven different brands over certain period of time.

The data set is extracted and aggregated from the Dominick's Finer Foods database maintained by the University of Chicago Booth School of Business at `http://research.chicagobooth.edu/marketing/databases/dominicks/index.aspx`. The original database records comprehensively the weekly store-level data of each product sold by Dominick's Finer Foods, a large supermarket chain in the Chicago area.

In addition to parameters b, a, η^+ and η^- in the demand function (2) which can be estimated using ordinary least squares (OLS), there is an additional unknown variable α to be estimated. Here, we follow the simple approach employed by Greenleaf (1995) to estimate α. The reference price was calculated using equation (1) while varying α in increments of 0.01 from 0 to 1 and our estimator $\hat{\alpha}$ is chosen to be the one that maximizes R^2.

The following table reports the estimates of the key parameters that of interest, i.e., α, η^+ and η^- for five brands. The rest two brands, belonging to larger volume category, are excluded from the analysis since the estimate for a, the price sensitivity, has a wrong sign. We also report the goodness of fit measurements, i.e., R^2 and adjusted R^2, for the comparison between the full model and the restricted model in which Assumption 1 is imposed. All the standard errors are computed via the bootstrapping procedure described in Freedman (1984). One can also refer to Appendix C for details of the procedure as well as more detailed results on the item "Star Kist 6 oz.".

In Table 1, for all five brands, $\hat{\eta}^+$ is statistically significant and indicates that the perceived discount term $\max\{r_t - p_t, 0\}$ has a large impact on the sales. On the other hand, $\hat{\eta}^-$ has the wrong sign in all but one brand and is statistically insignificant in all cases. As a result, restricting $\hat{\eta}^- = 0$ will give a more parsimonious model while retaining the explanatory power. By comparing the full model with the restricted model, we find surprisingly and uniformly across all five brands, except "Bumble Bee Solid 6.12 oz", that restricting $\hat{\alpha} = 0$ also has little effect on the goodness of fit of the model.

Table 1 Parameter Estimates and Goodness of Fit (Standard Errors in Parentheses*)

	$\hat{\alpha}$	$\hat{\eta}^+$	$\hat{\eta}^-$	R^2	Adjusted R^2
Star Kist 6 oz.					
Full Model	0	268587	-17356	0.360	0.354
	(0.064)	(34198.15)	(24011.89)		
Restricted Model	0	267124	0	0.359	0.355
Chicken of the Sea 6 oz.					
Full Model	0.33	573859	-58196	0.570	0.566
	(0.074)	(44329.31)	(39511.26)		
Restricted Model	0	502684	0	0.558	0.555
Bumble Bee Solid 6.12 oz.					
Full Model	0.99	15787	-4195	0.496	0.491
	(0.008)	(5305.44)	(4823.14)		
Restricted Model	0	7646.8	0	0.462	0.459
Bumble Bee Chunk 6.12 oz.					
Full Model	0.15	343059	-11904	0.640	0.637
	(0.066)	(18962.34)	(17330.99)		
Restricted Model	0	333538	0	0.639	0.637
Geisha 6 oz.					
Full Model	0.48	7062.1	574.0	0.545	0.541
	(0.15)	(1466.40)	(1194.65)		
Restricted Model	0	5402.0	0	0.537	0.534

* Standard errors are obtained from bootstrapping.

Our result, in agreement with that by Greenleaf (1995), shows that the coefficient of the perceived discount is greater than the perceived surcharge. As we have pointed out in Section 1, this result does not necessarily contradict with the prediction made by prospect theory (Tversky and Kahneman 1991). We consider the demand for the canned tuna in Chicago area was mainly driven by promotions. In other words, when there is no promotion, consumers' reference price tends to be below price and $\max\{r_t - p_t, 0\} = 0$. The only demand left is base demand $b - ap_t$. On the other hand, promotions will reduce price below consumers' reference price and increase the sales greatly by $\eta^+ \max\{r_t - p_t, 0\}$.

In summary, our empirical study illustrates that Assumption 1 is statistically plausible for some realistic settings and practical applications.

5.2. Performance of Simple Pricing Strategies

In this subsection, we first compare the performance of several simple pricing strategies with that of the optimal pricing strategy based on one of the empirical examples ("Star Kist 6 oz.") in Section 5.1. Note that Assumption 1 is satisfied in the "Star Kist 6 oz." case in Section 5.1. To examine what happens when Assumption 1 fails, we first design different parameter configurations that violate Assumption 1 and study the performance of simple cyclic pricing strategies under all these scenarios. Then, we numerically illustrate how the optimal pricing strategies will change when the assumption $\eta^- = 0$ is relaxed.

The complete parameter estimates for the item "Star Kist 6 oz." are provided in Appendix C and since their numerical values are quite large, for convenience we divide all the parameters in demand function by 100 without affecting the optimal solutions. The demand function is then given by

$$D(r,p) = 581.96 - 569.39p + 2671.2\max\{r-p, 0\},$$

and the reference price formulation is

$$r_{t+1} = p_t.$$

We set the price range to be $[0,1]$ which includes the price range of historical data and the initial reference price to be the average price in the data set, i.e., $r_0 = 0.8$.

Now we compare the performance of three simple pricing strategies with the optimal pricing strategy over a horizon of $T = 100$. For the rest of this section, we set the discount factor $\gamma = 0.9$. The optimal pricing strategy here is solved by using the algorithm developed in Hu (2012) for the finite horizon problem when Assumption 1 holds.

One simple strategy we consider is the constant pricing strategy or Every Day Low Price (EDLP) mentioned in Fibich et al. (2003), Popescu and Wu (2007) and Nasiry and Popescu (2011). In this case, the constant pricing strategy p_{EDLP} amounts to solving

$$\max_{p\in[0,1]} \sum_{t=0}^{100} \gamma^t p(581.96 - 569.39p + 2671.2\max\{r_t - p, 0\}),$$

$$\text{s.t.} \quad r_{t+1} = p, \quad 0 \le t \le 99, \quad r_0 = 0.8.$$

It is easy to see that the optimal solution is $p_{\mathrm{EDLP}} = 0.48$. Note that similar to the method by Fibich et al. (2003), here p_{EDLP} depends on r_0.

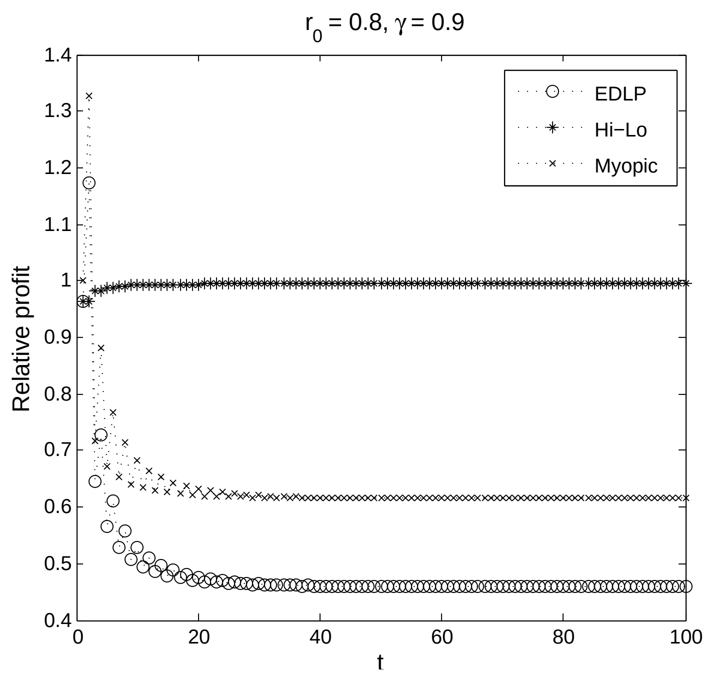

Figure 6 Profit comparison of simple pricing strategies

Another simple strategy we consider is the high-low pricing strategy. The high-low pricing strategy p_H and p_L amount to solving

$$\max_{p_1,p_2\in[0,1]} \sum_{t=0}^{100} \gamma^t p_i(581.96 - 569.39p_i + 2671.2\max\{r_t - p_i, 0\}),$$

$$\text{s.t.} \quad r_{t+1} = p_i, \quad 0 \le t \le 99, \quad r_0 = 0.8,$$

$$i = t \bmod 2, \quad 0 \le t \le 100.$$

The above problem turns out to be difficult to solve exactly. Instead, we discretize the prices and search for the optimal high-low pricing strategies. The optimal solution is solved as $p_H = 1$ and $p_L = 0.49$.

Finally, we consider the myopic pricing strategy which is explicitly solved as (6).

We plot the relative ratio of the profit obtained up to time t to the optimal profit up to time t, for $0 \le t \le T$, under the constant, high-low, and myopic pricing strategies in Figure 6. Our result shows that even though in the first two periods constant and myopic pricing strategies obtain a profit higher than the optimal pricing strategy, their performances decay quickly. Over the whole planning horizon, the high-low pricing strategy, which achieves more than 99% of the optimal profit, is much better than both the constant and myopic pricing strategies. By checking the dynamics of the optimal pricing strategy for the infinite horizon problem, we find that the optimal pricing strategy in this particular case is indeed a high-low pricing strategy and a time horizon of $T = 100$ is long enough to exhibit the long-run behavior. Interestingly, Figure 6 shows that even myopic pricing strategy can outperform the constant pricing strategy.

With the understanding that the high-low pricing strategy performs quite well in our empirical example, we next examine the performance of the high-low pricing strategy when Assumption 1 is not satisfied. More specifically, we consider the following demand functions

$$D(r,p) = 581.96 - 569.39p + 2671.2\max\{r-p,0\} + \eta^- \min\{r-p,0\},$$

where η^- is chosen such that the ratio $\eta^-/\eta^+ \in \{0, 0.02, 0.04, ..., 0.98\}$ with $\eta^+ = 2671.2$. We also let $\alpha \in \{0, 0.2, 0.4, 0.6, 0.8\}$ in the reference price formulation (1). In total, these parameters include 250 scenarios.

Instead of solving for an exact optimal solution, we numerically approximate the optimal solution through discretization and value iterations. Both the price and reference price are discretized with a step size of 0.0005 (a total of 2000 points for each of them). The ratio of the profit attained by the high-low pricing strategy over the profit attained by the optimal pricing strategy is reported in Figure 7 for all 250 combinations.

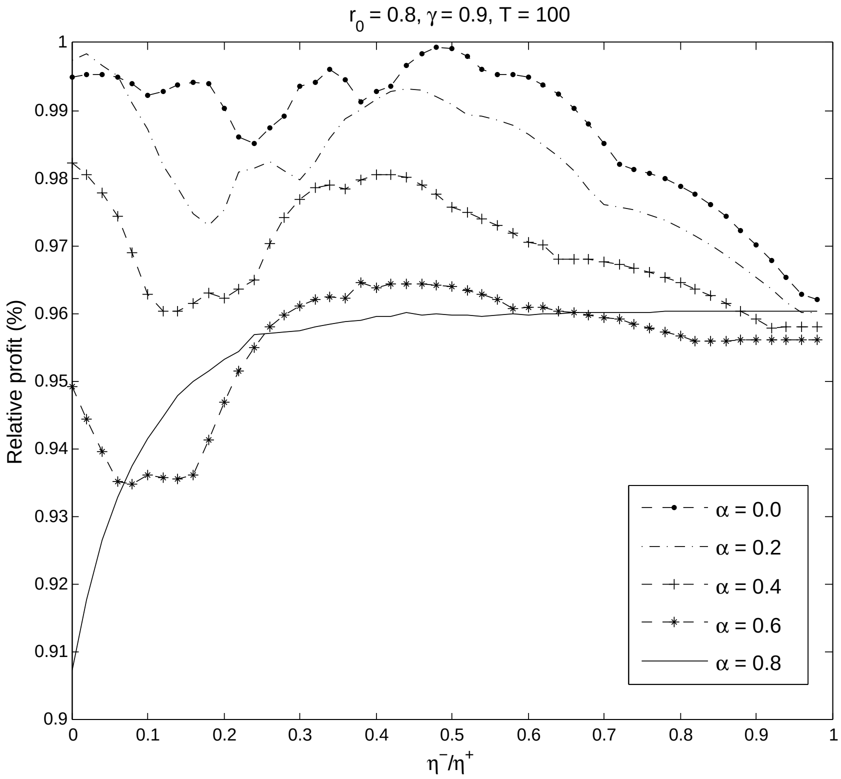

Figure 7 Relative profit under different parameter combinations

Table 2 Performance of cyclic pricing strategies

Cycle Length	1	2	3	4	5
Relative Profit	89.87%	90.74%	96.76%	98.34%	98.30%

From Figure 7 we note that there is no clear monotonic relationship between relative profit and the two parameters η^- and α. However, we observe that even in the worst case $(\alpha = 0.8, \eta^- = 0)$ the high-low pricing strategy achieves above 90% of the optimal profit. This is somewhat surprising given how complex the optimal pricing strategy in the worst case is as illustrated in Figure 3.

One might wonder in the worst scenario whether other cyclic pricing strategies with relatively short cycle will achieve better performance. We answer this question in Table 2. Note that for this particular parameter configuration, even the constant pricing strategy (cycle length is 1) can have a decent performance and a cyclic pricing strategies with cycle length of 3 or 4 can already achieve more than 95% of the optimal profit. Another interesting fact is that increasing the length of cycle, though complicating the optimization process, does not necessarily result in a better performance as indicated in the last two columns in Table 2. This is because, when the optimal pricing strategy is of cycle length 2, for instance, imposing a constraint that cycle length equals to 3 or any odd numbers will only make the resulting cyclic pricing strategies more different from the optimal.

Finally, we check to what extent Proposition 4 will still hold if the assumption $\eta^- = 0$ is relaxed. For this purpose, we fix $\alpha = 0$ and vary η^- such that the ratio $\eta^-/\eta^+ \in \{0, 0.02, 0.04, ..., 0.98\}$. When $\eta^-/\eta^+ < 0.16$, it is found that high-low pricing strategy is always optimal and the dynamics proposed in Proposition 4 holds true. For conciseness, in Figure 8, we only report the results for the values $\eta^-/\eta^+ \in \{0.16, 0.18, 0.20, 0.22, 0.24, 0.26\}$.

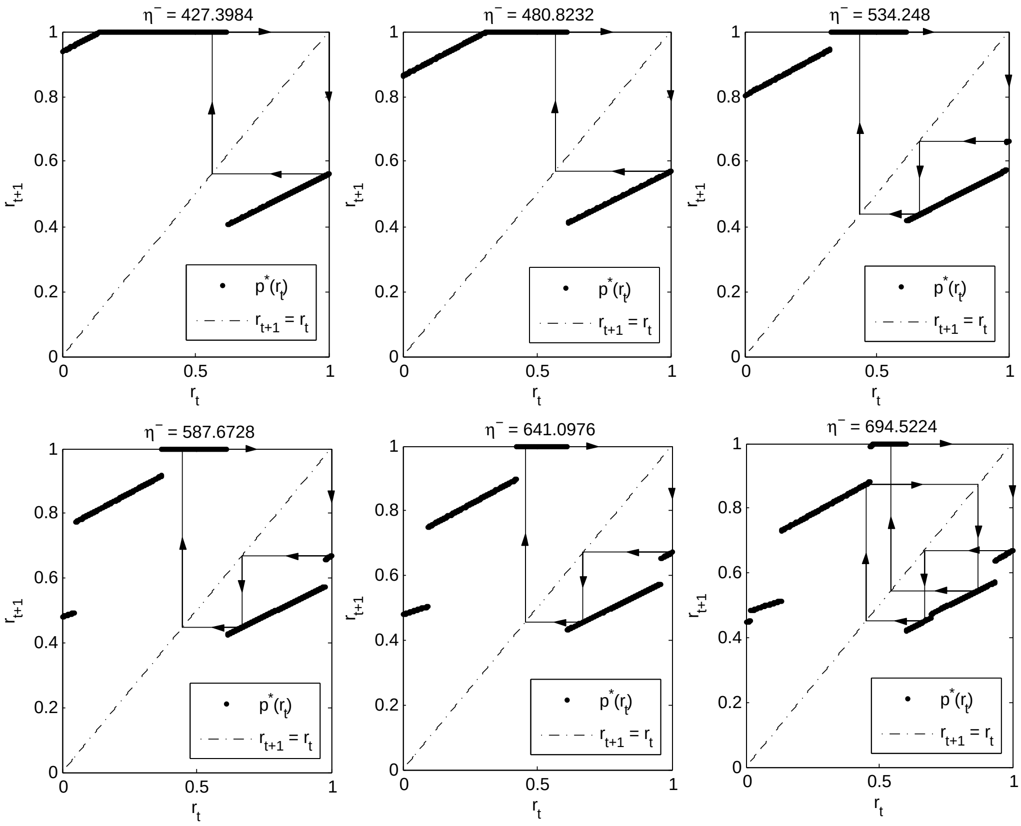

Figure 8 The optimal pricing strategies when $\alpha = 0$ and $\eta^-/\eta^+ \in \{0.16, 0.18, 0.20, 0.22, 0.24, 0.26\}$

When $\eta^-/\eta^+ = 0.16, 0.18$, similar to the cases for $\eta^-/\eta^+ < 0.16$, there is only one discontinuous point in the optimal pricing strategies and the high-low pricing strategy is optimal. For $\eta^-/\eta^+ = 0.20, 0.22, 0.24$, there are multiple discontinuous points appearing in the optimal pricing strategies, however, the periodic orbits in these figures indicate that a cyclic skimming pricing strategy with cycle length 3 is still optimal. That is, the conclusion of Proposition 4 still holds for $\eta^-/\eta^+ \leq 0.24$.

However, the last figure shows that another 2% increment in the ratio will result in a very different optimal pricing strategy.

6. Conclusion

In this paper we analyzed a dynamic pricing problem in a market with gain-seeking consumers. In this model, demand depends on both current selling price and reference price, where the latter evolves according to an exponentially smoothing process of past prices.

We showed that even employing the myopic pricing strategy can result in a very complicated dynamics of reference prices. We identified conditions that lead to simple pricing dynamics, for example, high-low pricing, cyclic skimming pricing or cyclic penetrating pricing.

Realizing the complexity of the problem, we restricted ourselves to an empirically validated special case and proved that a cyclic skimming pricing strategy is optimal over an infinite horizon. We further provided conditions on the upper bound of the cycle lengths. Although our characterization of the optimal pricing strategy built on a piece-wise linear demand model, both Proposition 3 and Proposition 4 can be extended to general nonlinear demand functions proposed in Popescu and Wu (2007) by imposing an assumption similar to Assumption 1. We relegated the extension to Appendix B.

Our work is only a start in exploring the effects of gain-seeking behavior/phenomenon on dynamic pricing problems. It would be very interesting to both categorize and characterize the possible patterns of the optimal pricing strategy under the general case. As one may see from our analysis of the special case, the structure of the optimal solution has intimate connections with the dynamics of the optimal solution. Thus, it is important to understand how the structure and the dynamics of the optimal solution interact with each other under more general settings.

Finally, it would be interesting to study the impact of gain-seeking reference price effects on the joint pricing and inventory decisions. Pricing and inventory integration has received much attentions in the past few years (see, for example, Chen and Simchi-Levi 2004a,b, 2006, 2012). Recently, Chen et al. (2013) incorporated reference price effect into coordinated pricing and inventory models. However, their model focuses only on loss-averse consumers.

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Appendix A: Proofs of Lemmas and Propositions

Proof of Lemma 1

Let us first consider the unconstrained problem: $\max_p \Pi(r,p)$. Recall that

$$\begin{aligned}\Pi(r,p) &= \max\{\Pi^+(r,p), \Pi^-(r,p)\}\\ &= \max_{\eta^- \le \eta \le \eta^+} p(b - ap + \eta(r-p)),\end{aligned}$$

where the last equality is due to the linearity of the function $p(b - ap + \eta(r-p))$ in η.

We can then rewrite $\max_p \Pi(r,p)$ as

$$\max_{\eta^- \le \eta \le \eta^+} \max_p -(a+\eta)p^2 + \eta rp + bp.$$

Clearly, the inner maximization problem has a unique optimal solution $p(r,\eta) = \frac{\eta r + b}{2(a+\eta)}$ with the optimal objective value $v(r,\eta) = \frac{(\eta r+b)^2}{4(a+\eta)}$. Note further that $\frac{\partial^2 v}{\partial r \partial \eta} = \frac{\eta r(a+\eta)+a(b+\eta r)}{2(a+\eta)^2} > 0$, which indicates that $v(r,\eta)$ is supermodular. Since either $\eta = \eta^+$ or $\eta = \eta^-$, the supermodularity of $v(r,\eta)$ implies the existence of R such that for $r \le R$, $v(r,\eta^-) \ge v(r,\eta^+)$ and for $r \ge R$, $v(r,\eta^+) \ge v(r,\eta^-)$. At R, $v(r,\eta^-) = v(r,\eta^+)$ and thus R can be computed explicitly as $R = \frac{b}{a+\sqrt{(a+\eta^+)(a+\eta^-)}}$. As a result, the unconstrained solution is exactly what presents in (6).

Observe that when $r > R$, we have $p^m(r) = \frac{\eta^+ r + b}{2(a+\eta^+)} \le r$. To see this, suppose $p^m(r) > r$, then

$$\Pi(r, p^m(r)) = p^m(r)(b - ap^m(r) + \eta^-(r - p^m(r))) > p^m(r)(b - ap^m(r) + \eta^+(r - p^m(r))) = v(r,\eta^+),$$

which contradicts with the fact that $\Pi(r,p^m) = v(r,\eta^+)$ for $r > R$. Therefore, when $r > R$, $p^m(r)$ can never violate the upper bound U. On the other hand, by a similar argument for $r \le R$, $p^m(r) = \frac{\eta^- r + b}{2(a+\eta^-)} \ge r$. Moreover, $p^m(r)$ is increasing in r and is equal to U at R_U. Thus, if $R \le R_U$, then $\frac{\eta^- r + b}{2(a+\eta^-)} \le U$ on $[0,R]$ and the unconstrained solution specified above is optimal.

However, if $R > R_U$, then for $r \le R$

$$v(r,\eta^-) = \begin{cases} \dfrac{(\eta^- r + b)^2}{4(a+\eta^-)}, & r \le R_U, \\ -(a+\eta^-)U^2 + \eta^- rU + bU, & R_U \le r \le R, \end{cases}$$

and we need to compare again between $v(r,\eta^-)$ and $v(r,\eta^+)$ on $[R_U, R]$. It is straightforward to show that R' is the unique positive solution of $v(r,\eta^-) = v(r,\eta^+)$ and when $r < R'$, $v(r,\eta^-) > v(r,\eta^+)$, when $r > R'$, $v(r,\eta^-) < v(r,\eta^+)$. Therefore, we arrive at the second form of $p^m(r)$.

Proof of Proposition 1

Suppose that a periodic orbit of period 2: $\{r_0, r_1\}$ exists, where $r_0, r_1 \in [0, U]$ and $r_0 \neq r_1$. First we observe either $r_0 \leq R, r_1 > R$ holds or $r_0 > R, r_1 \leq R$ holds. To see this, suppose that, on the contrary, $r_i \leq R$ for $i = 0, 1$, then $p^m(r_i) \geq r_i$ by the definition of R, which further implies $r_{1-i} = \alpha r_i + (1-\alpha)p^m(r_i) \geq r_i$. Therefore, $r_{1-i} = r_i$ for $i = 0, 1$, leads to contradiction with $r_0 \neq r_1$. Similarly, it is also impossible that $r_i \geq R$ for $i = 0, 1$. In the following we assume, without loss of generality, that $r_0 \leq R, r_1 > R$. Then, r_0, r_1 satisfy the following equations:

$$r_1 = \alpha r_0 + (1-\alpha)\frac{\eta^- r_0 + b}{2(a+\eta^-)},$$
$$r_0 = \alpha r_1 + (1-\alpha)\frac{\eta^+ r_1 + b}{2(a+\eta^+)}.$$

Thus, r_0, r_1 can be explicitly solved as

$$r_0 = \frac{[(1+\alpha)\eta^+ + 2a + 2\eta^- + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-},$$
$$r_1 = \frac{[(1+\alpha)\eta^- + 2a + 2\eta^+ + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-}.$$

We have shown that if a periodic orbit of period 2 exists, then it must be given by the unique solution identified above and has to satisfy $r_0 \leq R$ and $r_1 > R$. On the other hand, given r_0 and r_1 specified by above expressions with $r_0 \leq R, r_1 > R$. Then, by our construction, $r_1 = \alpha r_0 + (1-\alpha)p^m(r_0)$ and $r_0 = \alpha r_1 + (1-\alpha)p^m(r_1)$, which implies $\{r_0, r_1\}$ is a periodic orbit of period 2. Overall, a periodic orbit of period 2 exists if and only if

$$\frac{[(1+\alpha)\eta^+ + 2a + 2\eta^- + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-} \leq R,$$
$$\frac{[(1+\alpha)\eta^- + 2a + 2\eta^+ + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-} > R.$$

With cumbersome algebraic manipulations, the above two inequalities can be simplified to

$$4(1-\alpha^2)a^2 + 4(1-\alpha-\alpha^2)a\eta^+ + 4a\eta^- - (1+\alpha)^2(\eta^+)^2 + 4\eta^+\eta^- \geq 0,$$
$$4(1-\alpha^2)a^2 + 4(1-\alpha-\alpha^2)a\eta^- + 4a\eta^+ - (1+\alpha)^2(\eta^-)^2 + 4\eta^+\eta^- > 0.$$

Observe that the second inequality is naturally satisfied when the first inequality is satisfied. Thus, we have arrived at condition (8).

Proof of Proposition 2

By translating R to the origin and scaling $\frac{\eta^- R + b}{2(a+\eta^-)} - \frac{\eta^+ R + b}{2(a+\eta^+)}$ to 1, then $\alpha r + (1-\alpha)p^m(r)$ is equivalent to the following discontinuous map studied in Rajpathak et al. (2012).

$$f(r) = \begin{cases} Ar + \mu, & r \leq 0, \\ Br + \mu - 1, & r > 0. \end{cases}$$

Denote $\{r_0, r_1, ..., r_{n-1}\}$ be the periodic orbit (if exists) of the above system. Then it is easy to see that $r_0 < r_1 < ... < r_{n-1}$ ($r_0 > r_1 > ... > r_{n-1}$) if and only if $r_0 < r_1 < ... < r_{n-2} \leq 0$ and $r_{n-1} > 0$ ($r_0 > r_1 > ... > r_{n-2} > 0$ and $r_{n-1} \leq 0$). If we denote each point $r_t, 0 \leq t < n$, to be $\mathcal{L}$ if $r_t \leq 0$ and $\mathcal{R}$ if $r_t > 0$, then the periodic orbit can be coded as $\{\underbrace{\mathcal{L}, \mathcal{L}, ..., \mathcal{L}}_{n-1}, \mathcal{R}\}$ ($\{\underbrace{\mathcal{R}, \mathcal{R}, ..., \mathcal{R}}_{n-1}, \mathcal{L}\}$), which is exactly the so-called *prime pattern* in Rajpathak et al. (2012). Applying Theorem 1 in Rajpathak et al. (2012), we arrive at conditions (9) and (10) that guarantee each type of periodic solution with period n respectively.

Proof of Lemma 2

We first prove that $V(r)$ is increasing and convex in r. To see this, define the following value iteration for $i \geq 0$:

$$V_{i+1}(r) = \max_{p\in[0,U]} \Pi(r,p) + \gamma V_i(\alpha r + (1-\alpha)p),$$

with $V_0(r) = 0$. We inductively show that $V_i(r)$ is increasing and convex in r for all $i \geq 0$. Clearly, $V_0(r)$ trivially has the property. For $i > 0$, suppose $V_i(r)$ is increasing and convex in r. Since $\Pi(r,p)$ is increasing in r, we immediately have $V_{i+1}(r)$ is also increasing in r.

To see convexity of $V_{i+1}(r)$, it is sufficient to show that $\Pi(r,p)$ is convex in r. Indeed, recall that $\Pi(r,p) = \max\{\Pi^+(r,p), \Pi^-(r,p)\}$, where $\Pi^{\pm}(r,p) = p[b - ap + \eta^{\pm}(r-p)]$. As both $\Pi^+(r,p)$ and $\Pi^-(r,p)$ are convex in r, $\Pi(r,p)$ is also convex in r. By Proposition 2.1.15 in Simchi-Levi et al. (2014), $V_{i+1}(r)$ is also convex. By Theorem 4.6 in Stokey et al. (1989), $\lim_{i\to\infty} V_i(r) = V(r)$. Thus, $V(r)$ is both increasing and convex in r.

Since $V(r)$ is increasing and convex in r while both $\Pi^+(r,p)$ and $\Pi^-(r,p)$ are increasing and convex in r, by applying again Proposition 2.1.15 in Simchi-Levi et al. (2014) to problems (12a) and (12b) respectively, we have $V^+(r)$ and $V^-(r)$ are also increasing and convex.

On the other hand, as $\frac{\partial^2\Pi^+(r,p)}{\partial r\partial p} = \eta^+ > 0$ and $\frac{\partial^2\Pi^-(r,p)}{\partial r\partial p} = \eta^- > 0$, $\Pi^+(r,p)$ and $\Pi^-(r,p)$ are supermodular in r and p. Moreover, since $V(\cdot)$ is convex, by Theorem 2.2.6 in Simchi-Levi et al. (2014), $V(\alpha r + (1-\alpha)p)$ is also supermodular. As a result, $p^+(r)$ and $p^-(r)$ are increasing in r.

Proof of Lemma 3

Clearly, as $V^-(r) = V^-$ is constant and $V^+(r)$ is increasing, there are three cases. If $V^+(r) \geq V^-$ for all $r \in [0,U]$, then $R_0 = 0$. If $V^+(r) \leq V^-$ for all $r \in [0,U]$, then $R_0 = U$. If neither of the above cases is true, then we must have $V^+(0) < V^-$ and $V^+(U) > V^-$. As $V^+(r)$ is continuous and increasing, there must exists $0 < R_0 < U$, such that if $r \leq R_0$, then $V(r) = V^-$ and $p^*(r) = p^-$ and if $r > R_0$, then $V(r) = V^+(r)$ and $p^*(r) = p^+(r)$.

Next, we show that whenever $p^*(r) = p^-$, then $p^*(r) = p^- > r$ and whenever $p^*(r) = p^+(r)$, then $p^*(r) = p^+(r) < r$. Recall from Section 2 that $p^*(r) \neq r$ for any $r \in [0, U]$. Now suppose $p^*(r) = p^+(r)$, but $p^*(r) > r$, then $V^+(r) \geq V^-$. However,

$$V^+(r) = \Pi^+(r, p^+(r)) + \gamma V(p^+(r)) < \Pi^-(r, p^+(r)) + \gamma V(p^+(r)) \leq \Pi^-(r, p^-) + \gamma V(p^-) = V^-,$$

a contradiction. The same claim can be made when $p^*(r) = p^-$.

Finally, if $R_0 = 0$ or $R_0 = U$, then $p^*(0) = p^+(0) < 0$ or $p^*(U) = p^-(U) > U$, both lead to a contradiction. So we must have $0 < R_0 < U$.

Proof of Proposition 3

We first introduce some notations to formalize the idea discussed in Section 4. Let $V_0(r) = V^-$, then there exists $R_1 = \sup\{r \in [R_0, U] : p^+(r) < R_0\} \in (R_0, U]$ such that for $r \in (R_0, R_1)$, $p^+(r) = p_1(r) = \frac{\eta^+ r + b}{2(a+\eta^+)}$ and

$$V^+(r) = V_1(r) = \max_{p \in [0,U]} \Pi^+(r, p) + \gamma V_0(p) = \frac{(\eta^+ r + b)^2}{4(a+\eta^+)} + \gamma V^-.$$

If $R_1 = U$, then our proposition holds with $N = 0$.

More generally, for $k \geq 2$, if $R_{k-1} < U$, define $R_k = \sup\{r \in [R_0, U] : p^+(r) < R_{k-1}\} \in (R_0, U]$. Note that $p^+(R_{k-1}) < R_{k-1}$ by Lemma 3 and it follows $R_k > R_{k-1}$. We further define

$$\begin{aligned} V_k(r) &= \max_{p \in [0,U]} \Pi^+(r, p) + \gamma V_{k-1}(p), \\ p_k(r) &= \arg\max_{p \in [0,U]} \Pi^+(r, p) + \gamma V_{k-1}(p). \end{aligned} \tag{15}$$

By our construction, on $[R_{k-1}, R_k)$, $R_{k-2} \leq p^+(r) < R_{k-1}$. Thus, if $V(r) = V^+(r) = V_{k-1}(r)$ on $[R_{k-2}, R_{k-1})$, then $V(p^+(r)) = V_{k-1}(p^+(r))$ on $[R_{k-1}, R_k)$ and there is no loss of optimality by replacing problem (12a) with problem (15) above. Therefore, we have inductively shown that $V(r) = V^+(r) = V_k(r)$ and $p^*(r) = p^+(r) = p_k(r)$ on $[R_{k-1}, R_k)$.

Next, we inductively show that $V_k(r)$ has the following parametric form:

$$V_k(r) = \frac{1}{2} A_k r^2 + B_k r + C_k,$$

with $A_k \leq \eta^+$. We have already shown the base case for $k = 1$, with

$$\begin{aligned} A_1 &= \frac{(\eta^+)^2}{2(a+\eta^+)} = \frac{\eta^+ m_2}{\gamma} \leq \eta^+, \\ B_1 &= \frac{\eta^+ b}{2(a+\eta^+)} = \frac{m_2 b}{\gamma}. \end{aligned}$$

With inductive hypothesis,

$$\begin{aligned} V_{k+1}(r) &= \max_{p \in [0,U]} \{\Pi^+(r, p) + \gamma V_k(p)\} \\ &= \max_{p \in [0,U]} \{-(a + \eta^+ - \frac{1}{2}\gamma A_k)p^2 + (b + \eta^+ r + \gamma B_k)p + \gamma C_k\}. \end{aligned}$$

By Lemma 3, $p_{k+1}(r) < U$. Combined with $A_k \le \eta^+$, we know above problem is a concave maximization problem with an interior solution. Thus, the optimal solution can be derived from first order condition:

$$p_{k+1}(r) = \frac{b + \eta^+ r + \gamma B_k}{2(a+\eta^+) - \gamma A_k}.$$

We can then express

$$\begin{aligned}V_{k+1}(r) &= -(a+\eta^+ - \frac{1}{2}\gamma A_k)p_{k+1}(r)^2 + (b+\eta^+ r + \gamma B_k)p_{k+1}(r) + \gamma C_k \\ &= \frac{1}{2}\frac{(\eta^+)^2}{2(a+\eta^+) - \gamma A_k} r^2 + \frac{(b+\gamma B_k)\eta^+}{2(a+\eta^+) - \gamma A_k} r + \frac{1}{2}\frac{(b+\gamma B_k)^2}{2(a+\eta^+ - \gamma A_k)} + \gamma C_k.\end{aligned}$$

Thus,

$$\begin{aligned}A_{k+1} &= \frac{(\eta^+)^2}{2(a+\eta^+) - \gamma A_k} \le \frac{(\eta^+)^2}{2(a+\eta^+) - \gamma \eta^+} \le \eta^+, \\ B_{k+1} &= \frac{(b+\gamma B_k)\eta^+}{2(a+\eta^+) - \gamma A_k}.\end{aligned}$$

Combined with the expressions for A_1, B_1, it is not difficult to see that

$$\begin{aligned}A_k &= \frac{\eta^+}{\gamma} m_{k+1}, \\ B_k &= \frac{1}{\gamma}\sum_{i=0}^{n}\prod_{j=0}^{i} m_{k+1-j} b.\end{aligned}$$

Consequently, we can compute

$$p_k(r) = \frac{\eta^+ r + b + \sum_{i=0}^{k-1}(\prod_{j=0}^{i} m_{k-j})b}{2(a+\eta^+) - m_k \eta^+}.$$

If $R_k = U$, the above analysis has shown that our proposition holds with $N = k-1$.

Finally, we show that the construction of the sequence R_k cannot continue forever. In other words, there exists $N \ge 0$ such that $R_{N+1} > U$. We prove by contradiction. Suppose for any $k \ge 1$, $R_k \le U$. Then the following relation between R_k and R_{k+1} must hold for any $k \ge 1$: for $r \in [R_k, R_{k+1}]$, $p^+(r) \le R_k$ and for $r \in [R_{k+1}, U]$, $p^+(r) \ge R_k$. Since R_k is a bounded increasing sequence, there exists $\bar{R} \le U$ such that

$$\lim_{k\to\infty} R_k = \bar{R}.$$

From the above relation, we conclude that for $r \in [R_{k+1}, \bar{R}]$, $p^+(r) \ge R_k$. In particular, $p^+(\bar{R}) \ge R_k$ for any $k \ge 1$. Taking limits on both sides, we obtain

$$p^+(\bar{R}) \ge \bar{R},$$

leading to a contradiction with Lemma 3 which states $p^*(r) = p^+(r) < r$ for any $r \in [R, U]$.

Proof of Proposition 4

From Lemma 3, we know that $R_0 < r^* \le U$. Thus, there exists $2 \le n \le N+2$ such that $r^* \in [R_{n-2}, R_{n-1}]$. Let $p_1^*(r^*) = p^+(r^*)$, then $p_1^*(r^*) \in [R_{n-3}, R_{n-2}]$ and $p_1^*(r^*) < r^*$. Inductively, for $1 \le i \le n-2$ if $p_i^*(r^*) \in [R_{n-2-i}, R_{n-1-i}]$, then let $p_{i+1}^*(r^*) = p^+(p_i^*(r^*))$ and it follows that $p_{i+1}^*(r^*) \in [R_{n-3-i}, R_{n-2-i}]$ if $i \le n-3$ or $p_{i+1}^*(r^*) \in [0, R_0]$ if $i = n-2$. Furthermore, it holds $p_{i+1}^*(r^*) < p_i^*(r^*)$. Finally, let $p_n^*(r^*) = p^-(p_{n-1}^*(r^*)) = p^-$. Then $p_n^*(r^*) = r^*$. We have constructed the periodic orbit $\{r^*, p_1^*(r^*), ..., p_{n-1}^*(r^*)\}$ with the property $r^* > p_1^*(r^*) > ... > p_{n-1}^*(r^*)$.

Next we show that start from any initial reference price $r_0 \in [0, U]$, the optimal reference price path r_t^* converges to the periodic orbit in at most $N+2$ periods. Clearly, for any $r_0 \in [0, R_0]$, $r_1^* = p^-(r_0) = r^*$, and r_t^* will then follow the periodic orbit. For $0 \le i \le N$ and any $r_0 \in [R_i, R_{i+1}]$, it follows that $r_1^* = p^+(r_0) \in [R_{i-1}, R_i]$ if $i \ge 1$ or $r_1^* = p^+(r_0) \in [0, R_0]$ if $i = 0$. Again, inductively, for $1 \le t \le i$ it holds that $r_t^* = p^+(r_{t-1}^*) \in [R_{i-t}, R_{i-t+1}]$. Finally, $r_{i+1}^* \in [0, R_0]$ and $r_{i+2}^* = p^-(r_{i+1}^*) = r^*$ and the reference price path from then on follows the periodic orbit. In the worst case when $r_0 \in [R_N, R_{N+1}]$, we know that $r_{N+2}^* = r^*$.

Proof of Proposition 5

The basic idea is to find the condition such that $p^+(r) \le R_0$ for any $r \in [R_0, U]$. If this holds, then for any $r_0 \in [R_0, U]$ it follows that $r_1^* = p^+(r_0) \in [0, R_0]$, $r_2^* = p^-(r_1^*) = p_H = r^*$ and $r_3^* = p^+(r_2^*) = p_L$, which shows the existence of the high-low pricing strategy $\{p_H, p_L\}$.

However, both R_0 and the number of linear pieces as well as the "jumping" points of $p^+(r)$ are unknown. Instead, we strive to find an upper bound: $\bar{p}(r)$ on $p^+(r)$ and a lower bound $\underline{R}_0$ on R_0 such that condition (13) guarantees $\bar{p}(r) \le \underline{R}_0$ for all $r \in [R_0, U]$, which then implies $p^+(r) \le \bar{p}(r) \le \underline{R}_0 \le R_0$ for all $r \in [R_0, U]$.

First, we claim that the constant K is an upper bound on m_k for $k \ge 1$. Clearly, for $k = 1$, it holds $m_1 = 0 \le K$. Suppose for $k \ge 1$, we have $m_k \le K$. Then, we want to show that

$$m_{k+1} = \frac{\gamma\eta^+}{2(a+\eta^+) - m_k\eta^+} \le \frac{\gamma\eta^+}{2(a+\eta^+) - K\eta^+} \le K.$$

The second inequality above is equivalent to:

$$\eta^+ K^2 - 2(a+\eta^+)K + \gamma\eta^+ \le 0,$$

and it is straightforward to see that K indeed satisfies the inequality above. Note that here $K < 1$. Using the bound on m_k and the expression of $p^*(r)$ in Proposition 3, we can then bound $p^+(r)$ as

$$p^+(r) \le \frac{\eta^+ r + b + \sum_{i=0}^{\infty} K^{i+1} b}{2(a+\eta^+) - K\eta^+} = \frac{\eta^+ r + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} = \bar{p}(r)$$

Next, we claim that $\underline{R}_0 = R = \frac{b}{a+\sqrt{a(a+\eta^+)}}$ provides a lower bound for R_0. It is sufficient to show that $p^*(r) \geq p^m(r)$ for any $r \in [0, R)$ because if $R > R_0$, then on (R_0, R) we have $p^*(r) = p^+(r) < r < p^m(r)$, leading to a contradiction. Indeed, if $p^*(r) < p^m(r)$ for some $r \in (0, R)$, then as $p^m(r)$ maximizes $\Pi(r,p)$ and multiple solutions occur only when $r = R$, we must have

$$\Pi(r, p^*(r)) < \Pi(r, p^m(r)).$$

Since $V(r)$ is increasing in r, it follows $V(p^*(r)) < V(p^m(r))$, which implies $p^*(r)$ is not optimal, leading to a contradiction.

Finally, as $\bar{p}(r)$ is also increasing, condition (13) is derived by simply requiring

$$\bar{p}(U) = \frac{\eta^+ U + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} \leq \underline{R}_0 = R = \frac{b}{a + \sqrt{a(a+\eta^+)}},$$

to ensure $\bar{p}(r) \leq \underline{R}$ for all $r \in [R_0, U]$.

Proof of Proposition 6

The idea is similar to the proof of Proposition 5. We claim that $\underline{R}_k \leq R_k$. The base case for $k = 0$ is shown in Proposition 5. Suppose $\underline{R}_{k-1} \leq R_{k-1}$. Following the proof of Proposition 3, denote

$$p_{k+1}(r) = \frac{\eta^+ r + b + \sum_{i=0}^{k} (\prod_{j=0}^{i} m_{k+1-j}) b}{2(a+\eta^+) - m_{k+1}\eta^+}.$$

By our construction, as $p^+(r) = p_{k+1}(r)$ on $[R_k, R_{k+1}]$ and

$$R_k = \sup\{r \in [R_0, U] : p^+(r) = p_{k+1}(r) < R_{k-1}\},$$

while for any $r < \underline{R}_k$,

$$p_{k+1}(r) < \underline{R}_{k-1} \leq R_{k-1}.$$

Thus, $\underline{R}_k \leq R_k$.

Condition (14) then implies, $p_2^*(r^*) \in [R_i, R_{i+1}]$ for $0 \leq i \leq k-1$. Therefore, by counting the maps from $[R_i, R_{i+1}]$ to $[R_{i-1}, R_i]$, it is easy to see that the length of period is at most $k+2$.

Appendix B: General Reference-Dependent Demand Model

In this section, we extend the analytical results obtained in Section 4 to the general reference-dependent demand model proposed by Popescu and Wu (2007). Specifically, the reference-dependent demand is assumed to be of the form

$$D(r,p) = D(p,p) + R(r-p, r).$$

The following captures gain-seeking reference price effect.

Assumption 2. *The kinked reference effect is given by*

$$R(x,r) = \mathbf{1}_{\{x\geq 0\}} R_G(x,r) + \mathbf{1}_{\{x\leq 0\}} R_L(x,r) = \max\{R_G(x,r), R_L(x,r)\}, \tag{16}$$

where $R_G(x,r)$ and $R_L(x,r)$ are twice differentiable, increasing in x with $R_G(x,r) - R_L(x,r)$ single crossing in x.

Based on (16), the profit can be written as

$$\Pi(r,p) = \max\{\Pi_G(r,p), \Pi_L(r,p)\},$$

where $\Pi_G(r,p) = pD(p,p) + pR_G(r-p,r)$ and $\Pi_L(r,p) = pD(p,p) + pR_L(r-p,r)$. We further make the following assumption on $\Pi_G(r,p)$.

Assumption 3. $\Pi_G(r,p)$ *is twice differentiable, strongly concave in p and supermodular in (r,p). Moreover, there exists $0 \leq \Theta < -\frac{\partial^2 \Pi_G(r,p)}{\partial p^2}$ such that*

$$\frac{\partial^2 \Pi_G(r,p)}{\partial r^2}\frac{\partial^2 \Pi_G(r,p)}{\partial p^2} + \gamma\Theta\frac{\partial^2 \Pi_G(r,p)}{\partial r^2} - \Theta\frac{\partial^2 \Pi_G(r,p)}{\partial p^2} \geq (\frac{\partial^2 \Pi_G(r,p)}{\partial r \partial p})^2 + \gamma\Theta^2. \tag{17}$$

uniformly holds for any (r,p).

Note that when $\Theta = 0$, condition (17) is nothing but the condition for joint concavity of $\Pi_G(r,p)$. Furthermore, when $\Pi_G(r,p) = p(b - ap + \eta^+(r-p))$ condition (17) is naturally satisfied by letting $\Theta = a + \eta^+$. Next, we state the counterpart of Assumption 1.

Assumption 4. *Consumers only remember the most recent price (i.e. $\alpha = 0$) and the demand is insensitive to the perceived surcharge (i.e. $R_L(r-p,r) = 0$).*

Similar to the proof of Proposition 3, we can define

$$V_0(r) = \max_{p\in[0,U]} \Pi_L(r,p) + \gamma V(p),$$
$$p_0(r) = \arg\max_{p\in[0,U]} \Pi_L(r,p) + \gamma V(p).$$

and for $k \geq 1$

$$V_k(r) = \max_{p\in[0,U]} \Pi_G(r,p) + \gamma V_{k-1}(p),$$
$$p_k(r) = \arg\max_{p\in[0,U]} \Pi_G(r,p) + \gamma V_{k-1}(p).$$

With above assumptions, we know that $p_0(r)$ is constant and $p_k(r)$ is increasing in r for $k \geq 1$. Moreover, we have the following result which is critical for our extension to the general demand model.

PROPOSITION 7. *For any $k \geq 0$, $\Pi_G(r,p) + \gamma V_k(p)$ is concave in p and $V_k(\cdot)$ is twice differentiable. Consequently, $p_{k+1}(r)$ is the unique solution of*

$$\frac{\partial \Pi_G(r,p)}{\partial p} + \gamma \frac{dV_k(p)}{dp} = 0.$$

Proof. The proof is to inductively show that $V_k(\cdot)$ is twice differentiable and $\frac{d^2V_k(r)}{dr^2} < -\frac{\partial^2\Pi_G(r,p)}{\partial p^2}$ for any $0 \leq k$ and r. Clearly, $V_0(r)$ is a constant and thus twice differentiable with $\frac{d^2V_0(r)}{dr^2} = 0 \leq \Theta < -\frac{\partial^2\Pi_G(r,p)}{\partial p^2}$.

Suppose $V_k(\cdot)$ is twice differentiable and $\frac{d^2V_k(r)}{dr^2} \leq \Theta < -\frac{\partial^2\Pi_G(r,p)}{\partial p^2}$, we show that V_{k+1} is also twice differentiable with $\frac{d^2V_{k+1}(r)}{dr^2} \leq \Theta$. Strong concavity of $\Pi_G(r,p)$ ensures that $V_{k+1}(r)$ is twice differentiable (Santos 1991). Using the envelope theorem, it follows that

$$\frac{dV_{k+1}(r)}{dr} = \frac{\partial \Pi_G(r,p)}{\partial r}|_{p=p^*(r)} = \frac{\partial \Pi_G(r,p^*(r))}{\partial r}.$$

Note that our inductive hypothesis ensures that $\Pi_G(r,p) + \gamma V_k(p)$ is concave and $p^*(r)$ can be characterized by the first order condition. By the implicit function theorem, we further have

$$\begin{aligned}\frac{d^2V_{k+1}(r)}{dr^2} &= \frac{\partial^2\Pi_G(r,p^*(r))}{\partial r^2} - \frac{(\frac{\partial^2\Pi_G(r,p^*(r))}{\partial r\partial p})^2}{\frac{\partial^2\Pi_G(r,p^*(r))}{\partial p^2} + \gamma\frac{d^2V_k(p)}{dp^2}} \\ &\leq \frac{\partial^2\Pi_G(r,p^*(r))}{\partial r^2} - \frac{(\frac{\partial^2\Pi_G(r,p^*(r))}{\partial r\partial p})^2}{\frac{\partial^2\Pi_G(r,p^*(r))}{\partial p^2} + \gamma\Theta} \\ &\leq \Theta,\end{aligned}$$

where the first inequality follows from our inductive assumption and the second inequality from (17). □

Applying Proposition 7 in the proof of Proposition 3, we can derive the following extension of Proposition 3.

PROPOSITION 8. *There exist an integer $N \geq 0$ and $0 < R_0 < R_1 < ... < R_N < U = R_{N+1}$ such that*

$$p^*(r) = \begin{cases} p_0(r), & 0 \leq r \leq R_0, \\ p_1(r), & R_0 < r < R_1, \\ p_{k+1}(r), & R_k \leq r < R_{k+1}, \quad k = 1, ..., N. \end{cases}$$

The proof of Proposition 8 is along the same line as the proof of Proposition 3, except we lose an explicit form of $p_k(r)$ here. However, as $p_k(r)$ is monotonically increasing in r for each $k \geq 1$, Proposition 4 can be directly generalized here with $r^* = p_0(r)$. That is, in the long run, a cyclic skimming pricing strategy is optimal.

Appendix C: Empirical Example on Star Kist 6 oz.

In this section, we briefly explain the bootstrap procedure used to estimate the standard errors of $\hat{\alpha}$ and the parameters in the linear regression. We then present more detailed results using the brand Star Kist 6 oz. as an example.

Given the data $\{(D_t, p_t), 1 \le t \le T\}$, where D_t and p_t are respectively the demand and price at period t, our procedure for the estimation of parameters as well as their standard errors are described below.

Step 1: Obtain the estimate $\hat{\alpha}$ by a grid search of $[0,1]$ that maximizes R^2. Based on $\hat{\alpha}$, the estimates of linear coefficients $\hat{b}, \hat{a}, \hat{\eta}^+$ and $\hat{\eta}^-$ as well as residuals $\{\hat{e}_t, 1 \le t \le T\}$ are obtained through OLS.

Step 2: Obtain $N = 100$ bootstrap samples of residuals by re-sampling from the empirical distributions of $\{\hat{e}_t, 1 \le t \le T\}$. For $1 \le i \le N$, denotes the i-th bootstrap sample of residuals by $\{\hat{e}_t^i, 1 \le t \le T\}$.

Step 3: Compute the bootstrap samples of the data by

$$D_t^i = \hat{b} - \hat{a}p_t + \hat{\eta}^+ \max\{r_t - p_t, 0\} + \hat{\eta}^- \min\{r_t - p_t, 0\} + \hat{e}_t^i,$$

where $r_{t+1} = \hat{\alpha} r_t + (1 - \hat{\alpha}) p_t$. For $1 \le i \le N$, the i-th bootstrap sample of the data is then $\{(D_t^i, p_t), 1 \le t \le T\}$.

Step 4: For each i, $1 \le i \le N$, repeat Step 1 on the bootstrap sample $\{(D_t^i, p_t), 1 \le t \le T\}$ and obtain the estimates $\hat{\alpha}_i, \hat{b}_i, \hat{a}_i, \hat{\eta}_i^+$ and $\hat{\eta}_i^-$. Then the estimate of the standard error of, for instance, $\hat{\alpha}$ is obtained by computing the standard deviation of $\{\hat{\alpha}_i, 1 \le i \le N\}$.

Note that in the above procedure, we estimate the standard errors of the linear coefficients b, a, η^+ and η^- through Step 4 instead of directly using the standard errors from OLS because α is unknown and in OLS, replacing α with $\hat{\alpha}$ tends to result in underestimated standard errors.

In complement to Table 1, in Table 3, we present all the parameter estimates, including b and a, for the full model. In addition to the comparison between the full model and the restricted model, we also include the basic linear demand model (referred to as the *basic model*), which does not incorporate reference price effects, for comparison.

One can see that the adjusted R^2 reduced almost 50% by omitting the reference price effect in the basic model. However, for the restricted model, both R^2 and adjusted R^2 are almost the same as the unrestricted model and significantly higher than the basic model.

Finally, we attach a histogram of the estimates of α on the bootstrap samples, i.e., $\{\alpha_i, 1 \le i \le N\}$, in Figure 9. Note that over 70% of the bootstrap estimates are within 5% around 0, which indicates that our estimation of α is quite accurate.

Table 3 Results for Star Kist 6 oz. (Standard Errors in Parentheses)

$\hat{\alpha} = 0$ (0.064)*	Full Model	Basic Model	Restricted Model
Intercept ($\hat{b}$)	59661	150544	58196
	(17142.68)*	(14720)	(16401)
Price ($\hat{a}$)	59291	161303	56939
	(20802.71)*	(18159)	(19638)
Perceived discount ($\hat{\eta}^{+}$)	268587		267124
	(34198.15)*		(28472)
Perceived surcharge ($\hat{\eta}^{-}$)	-17356		
	(24011.89)*		
R^2	0.360	0.190	0.359
Adjusted R^2	0.354	0.188	0.355

* Standard errors are obtained from bootstrapping.

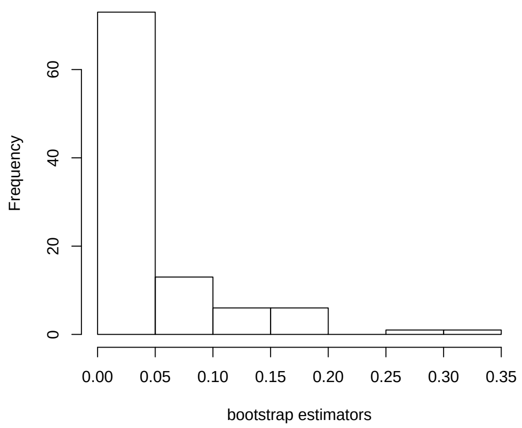

Figure 9 Histogram of bootstrap estimators of α

References

Aflaki, Sam, Ioana Popescu. 2013. Managing retention in service relationships. *Management Science* **60**(2) 415–433.

Ahn, H., M. Gumus, P. Kaminsky. 2007. Pricing and manufacturing decisions when demand is a function of prices in multiple periods. *Operations Research* **55**(6) 1039–1057.

Bell, D. R., J. M. Lattin. 2000. Looking for loss aversion in scanner panel data: The confounding effect of price response heterogeneity. *Marketing Science* **19**(2) 185–200.

Besbes, O., I. Lobel. 2012. Intertemporal price discrimination: Structure and computation of optimal policies. *Working Paper* .

Briesch, R.A., L. Krishnamurthi, T. Mazumdar, S.P Raj. 1997. A comparative analysis of reference price models. *Journal of Consumer Research* **24**(2) 202–214.

Chen, X., P. Hu, S. Shum, Y. Zhang. 2013. Stochastic inventory model with reference price effects. *Working Paper* .

Chen, X., D. Simchi-Levi. 2004a. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research* **52**(6) 887–896.

Chen, X., D. Simchi-Levi. 2004b. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research* **29**(3) 698–723.

Chen, X., D. Simchi-Levi. 2006. Coordinating inventory control and pricing strategies: The continuous review model. *Operations Research Letters* **34**(3) 323–332.

Chen, X., D. Simchi-Levi. 2012. Pricing and inventory management. *The Oxford handbook of pricing management. Oxford University Press, Oxford* .

Chevalier, J. A., A. K. Kashyap, P. E. Rossi. 2003. Why don't prices rise during periods of peak demand? evidence from scanner data. *American Economic Review* **93**(1) 15–37.

Conlisk, J., E. Gerstner, J. Sobel. 1984. Cyclic pricing by a durable goods monopolist. *The Quarterly Journal of Economics* **99** 489–505.

Fibich, G., A. Gavious, O. Lowengart. 2003. Explicit solutions of optimization models and differential games with nonsmooth (asymmetric) reference-price effects. *Operations Research* **51**(5) 721–734.

Freedman, David. 1984. On bootstrapping two-stage least-squares estimates in stationary linear models. *The Annals of Statistics* **12**(3) 827–842.

Geng, Q., C. Wu, K. Li. 2010. Pricing and promotion frequency in the presence of reference price effects in supply chains. *California Journal of Operations Management* **8**(1) 74–82.

Greenleaf, E.A. 1995. The impact of reference price effects on the profitability of price promotions. *Marketing Science* **14**(1) 82–82.

Hardie, B.G.S., E.J. Johnson, P.S. Fader. 1993. Modeling loss aversion and reference dependence effects on brand choice. *Marketing Science* **12**(4) 378–394.

Hirsch, M. W., S. Smale, R. L. Devaney. 2004. *Differential Equations, Dynamical Systems and An Introduction to Chaos*. Elsevier/Academic Press.

Hu, P. 2012. Coordinated pricing and inventory management. Ph.D. thesis, University of Illinois at Urbana-Champaign.

Jain, P., S. Banerjee. 2003. Border-collision bifurcations in one-dimensional discontinuous maps. *Int. J. Bifur. Chaos* **13** 3341–3351.

Kallio, Markku, Merja Halme. 2009. Redefining loss averse and gain seeking consumer price behavior based on demand response. *Working Paper* .

Kalyanaram, G., J.D.C. Little. 1994. An empirical analysis of latitude of price acceptance in consumer package goods. *Journal of Consumer Research* **21**(3) 408–418.

Kopalle, P.K., R.S. Winer. 1996. A dynamic model of reference price and expected quality. *Marketing Letters* **7**(1) 41–52.

Kopalle, Praveen K., P.K. Kannan, Lin Bao Boldt, Neeraj Arora. 2012. The impact of household level heterogeneity in reference price effects on optimal retailer pricing policies. *Journal of Retailing* **88**(1) 102–114.

Kopalle, Praveen K., Ambar G. Rao, Joo L. Assuno. 1996. Asymmetric reference price effects and dynamic pricing policies. *Marketing Science* **15**(1) 60–85.

Krishnamurthi, L., T. Mazumdar, S. P. Raj. 1992. Asymmetric response to price in consumer brand choice and purchase quantity decisions. *The Journal of Consumer Research* **19**(3) 387–400.

Liu, Y., W. L. Cooper. 2014. Optimal dynamic pricing with patient customers. *Working Paper* .

Mayhew, G. E., R. S. Winer. 1992. An empirical analysis of internal and external reference prices using scanner data. *The Journal of Consumer Research* **19**(1) 62–70.

Mazumdar, T., S. P. Raj, I. Sinha. 2005. Reference price research: Review and propositions. *Journal of Marketing* **69**(4) 84–102.

Nasiry, J., I. Popescu. 2011. Dynamic pricing with loss-averse consumers and peak-end anchoring. *Operations Research* **59**(6) 1361–1368.

Popescu, I., Y. Wu. 2007. Dynamic pricing strategies with reference effects. *Operations Research* **55**(3) 413–429.

Rajpathak, B., H. K. Pillai, S. Bandyopadhyay. 2012. Analysis of stable periodic orbits in the one dimensional linear piecewise-smooth discontinuous map. *Chaos: An Interdisciplinary Journal of Nonlinear Science* **22**(3) 033126.

Raman, K., F.M. Bass. 2002. A general test of reference price theory in the presence of threshold effects. *Tijdschrift Voor Economie En Management* **47**(2) 205–226.

Santos, M. 1991. Smoothness of the policy function in discrete time economic models. *Econometrica* **59**(5) 1365–1382.

Sharkovsky, A.N., L.O. Chua. 1993. Chaos in some 1-d discontinuous maps that appear in the analysis of electrical circuits. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on* **40**(10) 722–731.

Simchi-Levi, D., X. Chen, J. Bramel. 2014. *The Logic of Logistics: Theory, Algorithms, and Applications for Logistics and Supply Chain Management, Third Edition*. Springer.

Stokey, N. L., R. E. Lucas, E. C. Prescott. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press.

Tversky, A., D. Kahneman. 1991. Loss aversion in riskless choice: A reference-dependent model. *The Quarterly Journal of Economics* **106** 1039–1061.