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Online Appendix: Preservation of Quasi-K-Concavity and Its Applications

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Appendix

To prove Theorem 1, we need the following result.

Proposition 1 *Let $\alpha(\cdot)$ be a concave function in a bounded interval $\mathcal{D} = [\underline{d}, \bar{d}]$ and $\beta(\cdot)$ be a continuous function. There exists a $d(y)$ maximizing $\alpha(d) + \beta(y - d)$ for $d \in \mathcal{D}$ such that $y - d(y)$ is an increasing function of y.*

To prove the above result, one can first replace d by a new variable $\tilde{d} = y - d$. Since $\alpha(\cdot)$ is concave and $\beta(\cdot)$ is a function of a single variable, the function $\alpha(y - \tilde{d}) + \beta(\tilde{d})$ is supermodular in $(y, \tilde{d})$. Thus, there exists a $\tilde{d}(y)$ maximizing $\alpha(y - \tilde{d}) + \beta(\tilde{d})$ such that $\tilde{d}(y)$ is increasing in y (note that $\tilde{d}(y)$ can be chosen as either the largest optimal solution for all y or the smallest optimal solution for all y). Then the above lemma holds for $d(y) = y - \tilde{d}(y)$.

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Proof of Theorem 1

Define $d(y) = \min\{d : d \in \arg\max_{d\in\mathcal{D}}[\alpha(d) + \beta(y - d)]\}$. By Proposition **??**, $y - d(y)$ is increasing. Define

$$y_0 = \sup\{y : \Gamma(y) \text{ is non-decreasing on } (-\infty, y]\}.$$

We claim that $d(y_0) \in \{\arg\max_{d\in\mathcal{D}} \alpha(d)\}$. In the sequel, we will first prove the lemma under this claim. The proof for the claim itself will be provided after that.

Note that if $\Gamma(\cdot)$ is indeed quasi-K-concave, y_0 defined above would be its largest changeover point. Therefore, to prove the lemma, it is sufficient to show that $\Gamma(y)$ is non-K-increasing for $y \geq y_0$ or for $y_0 \leq y_2 \leq y_1$

$$\Gamma(y_2) \geq \Gamma(y_1) - K.$$

Let $\xi^0 > 0$ be the largest changeover of $\beta(y)$. One should note that, if $\xi^0 = \infty$, then $y_0 = \infty$ and the lemma is clearly true. So in the following proof, ξ^0 is assumed to be finite.

We first show the right-continuity of $y - d(y)$ at $y = y_0$. Since $y - d(y)$ is increasing in y, $\lim_{y\to y_0^+} y - d(y)$ always exists (superscript "+" means taking the right limit) and it is sufficient to show that it equals $y_0 - d(y_0)$. Assume $\lim_{y\to y_0^+} y - d(y) = y_0 - \tilde{d}$ for some $\tilde{d} \geq 0$. Then by continuity of $\Gamma(\cdot)$,

$$\Gamma(y_0) = \alpha(\tilde{d}) + \beta(y_0 - \tilde{d})$$

and so $\tilde{d} \in \{\arg\max_d[\alpha(d) + \beta(y_0 - d)]\}$. Furthermore, by the monotonicity of $y - d(y)$ we have $y_0 - \tilde{d} = \lim_{y\to y_0^+} y - d(y) \geq y_0 - d(y_0)$. Hence, $\tilde{d} \leq d(y_0)$. As $d(y_0)$ is assumed to be the smallest maximizer of $[\alpha(d) + \beta(y_0 - d)]$, $\tilde{d} = d(y_0)$ and $\lim_{y\to y_0^+} y - d(y) = y_0 - d(y_0)$.

We next show by contradiction that $y_0 - d(y_0) \geq \xi^0$. Suppose $y_0 - d(y_0) < \xi^0$, by right-continuity of $y - d(y)$ at y_0 there exists a number $\eta > 0$ such that for any $y \leq y'$ in the interval $[y_0, y_0 + \eta]$, $y_0 - d(y_0) \leq y - d(y) \leq y' - d(y') \leq \xi^0$. We can show that $\Gamma(y)$ is non-decreasing in this interval $[y_0, y_0 + \eta]$ with $\eta > 0$:

$$\begin{aligned} \Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\ &\geq \alpha(d(y)) + \beta(y' - d(y)) \\ &\geq \alpha(d(y)) + \beta(y - d(y)) \\ &= \Gamma(y), \end{aligned}$$

where the first inequality follows from the fact that $d(y')$ is optimal for $\Gamma(y')$; the second inequality holds because $\beta(\cdot)$ is increasing on $(-\infty, \xi^0]$. This contradicts with the definition of y_0. Therefore, $y_0 - d(y_0) \geq \xi^0$.

Now we focus our attention on $\xi^0 \leq y_2 - d(y_2) \leq y_1 - d(y_1)$, we verify the lemma by discussing several different cases.

If $d(y_2) \geq d(y_1)$, then $y_2 - d(y_1) \geq y_2 - d(y_2) \geq \xi^0$ and therefore

$$\begin{aligned}
\Gamma(y_2) &= \alpha(d(y_2)) + \beta(y_2 - d(y_2)) \\
&\geq \alpha(d(y_1)) + \beta(y_2 - d(y_1)) \\
&\geq \alpha(d(y_1)) + \beta(y_1 - d(y_1)) - K \\
&= \Gamma(y_1) - K,
\end{aligned}$$

where the first inequality follows from the optimality of $d(y_2)$ and the second one from the non-K-increasing of $\beta(y)$ for $y \geq \xi^0$.

If $d(y_2) < d(y_1)$, then we have the following two different cases:

Case I: $d(y_0) \geq d(y_2)$. In this case, obviously $y_2 - d(y_0) \leq y_2 - d(y_2)$.

$$\begin{aligned}
\Gamma(y_2) &= \alpha(d(y_2)) + \beta(y_2 - d(y_2)) \\
&\geq \alpha(d(y_0)) + \beta(y_2 - d(y_0)) \\
&\geq \alpha(d(y_0)) + \beta(y_1 - d(y_1)) - K \\
&\geq \alpha(d(y_1)) + \beta(y_1 - d(y_1)) - K \\
&= \Gamma(y_1) - K,
\end{aligned}$$

where the second inequality follows from $\xi^0 \leq y_0 - d(y_0) \leq y_2 - d(y_0) \leq y_2 - d(y_2) \leq y_1 - d(y_1)$ and the last one from the optimality of $d(y_0)$ for $\alpha(d)$ that we claimed.

Case II: $d(y_2) > d(y_0)$.

$$\begin{aligned}
\Gamma(y_2) &= \alpha(d(y_2)) + \beta(y_2 - d(y_2)) \\
&\geq \alpha(d(y_2)) + \beta(y_1 - d(y_1)) - K \\
&\geq \alpha(d(y_1)) + \beta(y_1 - d(y_1)) - K \\
&= \Gamma(y_1) - K,
\end{aligned}$$

where the first inequality follows from the non-K-increasing of $\beta(y)$ and the second one follows from the concavity of $\alpha(d)$ and that $d(y_0)$ is its maximizer.

The above cases cover all possibilities and we have proved the lemma under the claim that $d(y_0)$ is a maximizer of $\alpha(d)$. We now turn to prove the claim itself. Observe that $d(y_0)$ can either lie in the interior of $\mathcal{D} = [\underline{d}, \overline{d}]$ or on its boundary. We distinguish between these two cases. If $d(y_0)$ is an interior point of $\mathcal{D}$, then from the first order optimality condition,

$$\alpha'(d(y_0)) = \beta'(y_0 - d(y_0)).$$

If $\alpha'(d(y_0)) > 0$, then $\beta'(y_0 - d(y_0)) > 0$. Since $\beta(\cdot)$ is continuously differentiable, $\beta'(x) > 0$ for x in a small neighborhood of $y_0 - d(y_0)$. As $\lim_{y \to y_0^+} d(y) = d(y_0)$, one can show that there exists a small neighborhood $\mathcal{U}$ of y_0 such that for any $y', y \in \mathcal{U}$ with $y' > y > y_0$, $\beta(y' - d(y)) > \beta(y - d(y))$. Then

$$\begin{aligned}
\Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\
&\geq \alpha(d(y)) + \beta(y' - d(y)) \\
&> \alpha(d(y)) + \beta(y - d(y)) \\
&= \Gamma(y).
\end{aligned}$$

This contradicts with the definition of y_0.

If $\alpha'(d(y_0)) < 0$, then $\beta'(y_0 - d(y_0)) < 0$. There exists some $y' < y_0$ that is sufficiently close to y_0 such that $\beta(y' - d(y_0)) > \beta(y_0 - d(y_0))$. Then

$$\begin{aligned}
\Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\
&\geq \alpha(d(y_0))) + \beta(y' - d(y_0)) \\
&> \alpha(d(y_0)) + \beta(y_0 - d(y_0)) \\
&= \Gamma(y_0),
\end{aligned}$$

which also contradicts with the definition of y_0. Therefore, $\alpha'(d(y_0)) = 0$ and $d(y_0)$ is an interior maximizer for $\alpha(\cdot)$.

We next consider the case where $d(y_0)$ is on the boundary of $\mathcal{D}$. Consider first $d(y_0) = \underline{d}$. From the first order optimality condition, we have that

$$\alpha'(\underline{d}) - \beta'(y_0 - \underline{d}) \leq 0.$$

We need to show that $\underline{d}$ is a maximizer of $\alpha(d)$ in $\mathcal{D}$. Suppose this is not true, then as $\alpha(d)$ is differentiable and concave, $\alpha'(\underline{d}) > 0$ and therefore $\beta'(y_0 - \underline{d}) > 0$. By an argument similar to the one used in the previous two paragraphs, we can show that for $y' \geq y > y_0$ with y' sufficiently close to y_0,

$$\begin{aligned}
\Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\
&\geq \alpha(d(y)) + \beta(y' - d(y)) \\
&\geq \alpha(d(y)) + \beta(y - d(y)) \\
&= \Gamma(y),
\end{aligned}$$

where the last inequality follows from the fact that $\beta'(y_0 - \underline{d}) > 0$ and the continuity of $\beta(\cdot)$. This contradicts with the definition of $\Gamma(y_0)$. Therefore, $\underline{d}$ is a maximizer of $\alpha(d)$. The case that $d(y_0) = \overline{d}$ can be similarly proven. Thus we have proved our claim that $d(y_0) \in \{\arg\min_{d\in\mathcal{D}} \alpha(d)\}$. This concludes the proof of Theorem 1.

Proof of Theorem 2

Since part (b) has been implicitly proven and used in Chen and Simchi-Levi [?], we focus on part (a).

Let $d(y) \in \arg\max_{d\in\mathcal{D}}[\alpha(d) + \beta(y - d)]$ (its existence is guaranteed by the continuity of $\alpha(\cdot)$ and $\beta(\cdot)$) and d^* be a maximizer of $\alpha(\cdot)$ in $\mathcal{D}$. First note that if $\beta(\cdot)$ does not have a finite maximizer, then it must be monotone. In this case, we can show that $\Gamma(\cdot)$ is also monotone. We only prove the case in which $\beta(\cdot)$ is increasing (the case in which $\beta(\cdot)$ is decreasing can be proven similarly). Let $y_1 \leq y_2$. Then

$$\Gamma(y_1) = \alpha(d(y_1)) + \beta(y_1 - d(y_1)) \leq \alpha(d(y_1)) + \beta(y_2 - d(y_1)) \leq \alpha(d(y_2)) + \beta(y_2 - d(y_2)) = \Gamma(y_2),$$

where the first inequality holds since $\beta(\cdot)$ is increasing and the remaining equalities and inequality follow from the definition of $d(y)$.

We now assume that $\beta(\cdot)$ has a finite maximizer, denoted as x^*. It is not hard to show that $y^* = x^* + d^*$ is a maximizer of the function $\Gamma(\cdot)$. We can show that $\Gamma(\cdot)$ is increasing in $(-\infty, y^*]$ and decreasing in $[y^*, \infty)$. Let $y < y^*$ and $\eta = y^* - y$. To this end, we prove that $\Gamma(\cdot)$ is increasing in a neighborhood of y. Since $y < y^*$, either $d^* - d(y)$ or $y^* - d^* - (y - d(y))$ must be no less than $\eta/2$. We focus on the case with $d^* - d(y) \geq \eta/2$ (Note that the other case is symmetric). In this case, for any $y' \in [y, y + \eta/2]$, let $d = d(y) + y' - y$. Then $d(y) \leq d \leq d^*$, $y' - d = y - d(y)$, and therefore

$$\Gamma(y) = \alpha(d(y)) + \beta(y - d(y)) \leq \alpha(d) + \beta(y' - d) \leq \Gamma(y'),$$

where the first inequality holds since $\alpha(d)$ is increasing for $d \leq d^*$. We next prove that $\Gamma(y') \leq \Gamma(y)$ for any $y' \in [y - \eta/2, y]$. For a given $y' \in [y - \eta/2, y]$, if $y' - d(y') \leq y^* - d^*$, then

$$\Gamma(y') = \alpha(d(y')) + \beta(y' - d(y')) \leq \alpha(d(y')) + \beta(y - d(y')) \leq \Gamma(y),$$

where the first inequality holds since $\beta(\cdot)$ is increasing in $(-\infty, y^* - d^*]$; if $y' - d(y') \geq y^* - d^*$, denoting $d = d(y') + (y - y')$, we have that $d(y') \leq d \leq d^*$, $y - d = y' - d(y')$ and

$$\Gamma(y') = \alpha(d(y')) + \beta(y' - d(y')) \leq \alpha(d) + \beta(y - d) \leq \Gamma(y),$$

where the first inequality holds since $\alpha(d)$ is increasing for $d \leq d^*$. Thus, $\Gamma(\cdot)$ is increasing in $(-\infty, y^*]$. Similarly, we can prove that $\Gamma(\cdot)$ is decreasing in $[y^*, \infty)$.