# Dynamic Stochastic Inventory Management with Reference Price Effects

Xin Chen
Department of Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, xinchen@illinois.edu

Peng Hu
School of Management, Huazhong University of Science and Technology, Wuhan, China 430074, hu_peng@hust.edu.cn

Stephen Shum
College of Business, City University of Hong Kong, Kowloon, Hong Kong, swhshum@cityu.edu.hk

Yuhan Zhang
Morgan Stanley, 1585 Broadway 3rd floor New York City NY 10036, yuhan.zhang@morganstanley.com

We analyze a periodic review stochastic inventory model in which demand depends on not only the current selling price but also a memory-based reference price. Pricing and inventory decisions are made simultaneously at the beginning of each period. Assuming all shortages are backlogged, the objective is to maximize the expected total discounted profit over either a finite horizon or an infinite horizon. In the finite horizon case, we prove that a reference price dependent base-stock policy is optimal, and we analyze the firm's optimal price and base-stock level. In the infinite horizon case, we show that the reference price converges to some steady state in the optimal trajectory and characterize this steady state.

## 1. Introduction

The impact of a price change on sales can be larger than expected. Back in 1951, the sales in new cars changed from "fairly brisk" to "very slow" after a price hike (The News and Courier 1951). A recent increase in price of Hersey's chocolate also led to concerns in the company's sales and earnings in the short run (Nicholson 2014). The reason for the customer pushback after a price hike may lie beyond commonly used static economic models (Kalyanaram and Winer 1995, Nasiry and Popescu 2011). In repeat-purchase settings, consumers often develop their own ideas of a "fair price", also referred to as the *reference price*, after observing past prices of the product. If the current selling price is lower than the reference price, consumers see it as a gain and hence are more likely to make the purchase. Otherwise, they see it as a loss and would be less inclined to make the purchase. This phenomenon is usually referred to as the *reference price effect*. Consumers are loss-averse (loss-neutral) if demand is more responsive (equally responsive) to consumers' perceived losses than (as) their perceived gains.

There have been studies on how managers can take the reference effect into consideration to make better pricing decisions. Examples include Greenleaf (1995), Kopalle et al. (1996), Fibich et al. (2003), Popescu and Wu (2007) and Nasiry and Popescu (2011). However, these papers do not consider inventory decisions. At the same time, there is an extensive body of research on joint pricing and inventory decisions under demand uncertainty (see Elmaghraby and Keskinocak 2003, Chan et al. 2004 and Chen and Simchi-Levi 2012 for reviews). Examples include Federgruen and Heching (1999), Chen and Simchi-Levi (2004a,b, 2006), Huh and Janakiraman (2008) and Song et al. (2009). While the first stream of literature typically ignores demand uncertainty, the second stream of literature often assumes that demand depends on the firm's pricing strategy only through the current price.

To our best knowledge, only a few papers have integrated reference price effects into inventory models. Chen et al. (2009) analyze a joint dynamic pricing and economic lot sizing problem with the reference price effect. They develop strongly polynomial time algorithms for a few special cases of the problem, and propose a heuristic algorithm with error bound estimates for the general case. Ahn et al. (2007) study a closely related model. They prove structural results for their model and develop closed-form solutions and heuristics for various special cases. However, both of these two papers do not consider demand uncertainty. Urban (2008) analyzes a single-period joint inventory and pricing model with both reference price effect and demand uncertainty, and finds in numerical studies that reference prices have a substantial impact on a firm's profit. Gimpl-Heersink (2008) considers a demand model similar to ours but focuses the analysis on one-period and two-period settings with loss-neutral customers.

We study the joint inventory and pricing problem of a firm facing reference price effect and loss-averse customers in a stochastic multi-period setting. After inferring the reference price and confirming the initial inventory at the beginning of each period, the firm makes pricing and ordering decisions simultaneously. Unused inventory at the end of a period is carried over to the next period, and shortages are fully backlogged. The goal is to study the firm's optimal decisions in the presence of reference price effect and uncertainty in a dynamic setting. In particular, are base-stock policy and list-price policy optimal? How do reference price and the intensity of reference price effect affect the optimal decisions, and what is the impact of customers' loss-aversion? Do the optimal decisions converge to some steady states in the long-run? This paper aims to shed light on these questions.

A technical challenge in addressing these questions is that the single-period expected revenue is in general not concave and not smooth in the selling price and the reference price. To tackle this challenge, we introduce a transformation technique to generate a modified revenue function that is concave. This allows us to prove that a reference price dependent base-stock policy is optimal and characterize the optimal price and base-stock level. For the infinite horizon model, we prove that both the reference price and the base-stock level converge and characterize the steady state.

The rest of this paper is organized as follows. We present the model in Section 2, and characterize the firm's optimal policy in Section 3. Section 4 extends our results to a more general demand model. Finally, Section 5 concludes our paper. All the proofs are delegated to the appendix. In this paper, the terms "increasing" and "decreasing" are used in a weak sense.

## 2. The Model

Consider a firm making inventory and pricing decisions over a planning horizon with $T$ periods. At the beginning of each period, the firm decides the price and the order quantity for that period. The order is received immediately and incurs a per unit cost $c$. The price in each period, $p_t$, is restricted to a bounded interval $\mathcal{P} = [p_{\min}, p_{\max}]$, and we assume that $p_{\min} \geq c$ to ensure the marginal profit to be non-negative.

The expected demand $d_t$ depends on the price $p_t$ and the reference price $r_t$ in the same period. In particular, the expected demand is given by

$$d_t = (b_t - ap_t) + \eta(r_t - p_t),$$

where $b_t > 0$ represents the market size, $a > 0$ measures the sensitivity of demand with respect to the selling price, and the difference $r_t - p_t$ denotes consumers' perceived gain when $r_t > p_t$ and loss when $r_t < p_t$. The reference price effect is given by $\eta(z) = \eta^+ \max\{0, z\} + \eta^- \min\{0, z\}$, where the non-negative parameters $\eta^-$ and $\eta^+$ measure the sensitivities of demand associated with the perceived loss and gain, respectively. While the market size $b_t$ may change over time, consumers' response to price and reference price (i.e., $a$ and $\eta^{\pm}$) are assumed to be time-independent. Our demand model is similar to the ones in Greenleaf (1995) and Nasiry and Popescu (2011). One advantage of this model is that it is much easier to calibrate when compared with more complex demand models. In addition, different parameters in the model can be easily understood by managers and practitioners. Consumers are usually called loss-averse if $\eta^- \geq \eta^+$, loss seeking if $\eta^- \leq \eta^+$ and loss neutral if $\eta^- = \eta^+$. Prospect theory (Kahneman and Tversky 1979) postulates that loss aversion behavior is more common than loss-seeking behavior. For this reason, we focus on the loss-averse case, similar to Nasiry and Popescu (2011).

The reference price in a period is formed based on the prices observed by customers in all previous periods. We adopt the exponentially smoothed adaptive expectations process (see, e.g., Mazumdar et al. 2005) in which the reference price is a linear combination of past prices. Formally, given the price and the reference price in period $t$, the reference price in period $t+1$ evolves according to

$$r_{t+1} = (1-\alpha)p_t + \alpha r_t,$$

where $0 \leq \alpha < 1$ is called the memory factor or carryover constant. This reference price evolution model is commonly used both in empirical studies (e.g., Kalyanaram and Little 1994, Greenleaf

1995) and in analytical models (e.g., Popescu and Wu 2007). To avoid the trivial case where past prices have no impact on demand, we assume that $\alpha < 1$. The initial reference price is given by $r_1 \in \mathcal{P}$, and hence all $r_t$ belong to the interval $\mathcal{P}$. To facilitate discussion, we denote $r$ and $q$ as the current reference price ($r_t$) and target reference price ($r_{t+1}$) respectively, and we express price $p_t$ and expected demand $d_t$ as functions of these reference prices, where

$$p(r,q) = \frac{q-\alpha r}{1-\alpha}, \quad \text{and} \quad d_t(r,q) = b_t - a\frac{q-\alpha r}{1-\alpha} + \frac{\eta(r-q)}{1-\alpha}.$$

Furthermore, demand in each period $t$ is stochastic and follows the additive model

$$D_t = d_t(r,q) + \varepsilon_t,$$

where $\varepsilon_t$ is a random variable with zero mean and independent across time. We assume that there exists $D_{\min} > 0$ such that $D_t \geq D_{\min}$ with non-zero probability for any $r, q \in \mathcal{P}$.

Define $m^{\pm} = \eta^{\pm}/[(1+\alpha)a]$, which measures the relative strength of the reference price effect when compared with the direct price effect. We make the following assumption.

Assumption 1. $m^+ \leq m^- \leq m^+ + \sqrt{1+2m^-}$.

Assumption 1 is not restrictive. Clearly, it is satisfied when customers are loss-neutral. When customers are loss-averse, this condition is satisfied when $m^- \leq 1+\sqrt{2}$, i.e., the reference price effect is not much stronger than the direct price effect, which is consistent with the results from many empirical studies, for example, Hardie et al. (1993).

Unsatisfied demand is fully backlogged and any excess inventory is carried over to the next period. For an inventory level $y$ at the end of period $t$, denote $h_t^0(y)$ as the associated inventory holding cost when $y > 0$ and backlogging cost when $y < 0$. The objective of the firm is to find an inventory and pricing policy to maximize the total expected discounted profit over the planning horizon. Given the initial inventory level $x$ and reference price $r$ in period $t \leq T$, the profit-to-go function $v_t^0(x,r)$ at the beginning of period $t$ satisfies the dynamic programming recursion

$$\begin{aligned} v_t^0(x,r) = \underset{s \geq x, q: p(r,q) \in \mathcal{P}}{\text{maximize}} \quad & p(r,q)d_t(r,q) - c(s-x) - \mathbb{E}h_t^0(s - d_t(r,q) - \varepsilon_t) \\ & + \gamma\, \mathbb{E}v_{t+1}^0(s - d_t(r,q) - \varepsilon_t, q), \end{aligned}$$

where $0 \leq \gamma \leq 1$ is the discount factor, and decision variables $s$ and $q$ denote the inventory order-up-to level and the target reference price respectively. Moreover, assume that the terminal value $v_{T+1}^0(x,r)$ equals $cx$. That is, any backlogged demand is satisfied and any leftover inventory incurs reimbursement with the per unit cost/value $c$ at the end of the planning horizon.

To facilitate discussion, introduce $v_t(x,r) = v_t^0(x,r) - cx$ and $y = s - d_t(r,q)$ as the expected leftover inventory for each period $1 \le t \le T+1$. The above problem can be reformulated as

$$v_t(x,r) = \underset{y,q}{\text{maximize}} \;\; \pi_t(r,q) + \mathbb{E}[\gamma v_{t+1}(y-\varepsilon_t, q) - h_t(y-\varepsilon_t)], \qquad (1)$$
$$\text{subject to} \;\; d_t(r,q) + y \ge x, \quad p(r,q) \in \mathcal{P},$$

where functions $\pi_t(r,q)$ and $h_t(y)$ defined below denote the expected net profit and the transformed inventory holding and backlogging cost associated with the leftover inventory $y$, respectively.

$$\pi_t(r,q) = [p(r,q) - c]d_t(r,q), \quad h_t(y) = h_t^0(y) + (1-\gamma)cy.$$

Similar to Federgruen and Heching (1999) and Chen and Simchi-Levi (2004a), we assume that $h_t(y)$ is convex, $\lim_{|y|\to\infty} h_t(y) = \infty$, and $\mathbb{E}h_t(y-\varepsilon_t)$ is well-defined for any $y$. In the following, denote $[y_t(x,r), q_t(x,r)]$ as an optimal solution to problem (1), and $s_t(x,r) = y_t(x,r) + d_t(r, q_t(x,r))$ and $p_t(x,r) = p(r, q_t(x,r))$ as the optimal inventory order-up-to level and the optimal price in period $t$, respectively. When there exist multiple optimal solutions, we always select the lexicographically smallest one for convenience.

## 3. Main results

### 3.1. Finite Horizon Model

When there is no reference price effect, problem (1) becomes a special case of the model studied by Federgruen and Heching (1999). The authors, following the standard approach, inductively show that the dynamic programming problem is a concave maximization problem, and successfully prove that the base-stock and list-price policy is optimal, i.e., an order is placed only when the initial inventory level exceeds some threshold, and a higher initial inventory level leads to a lower price. In our problem, the additional dimension in the state space due to the reference price $r$ brings a significant challenge to apply the method because $\pi_t(r,q)$ is not jointly concave in $(r,q)$. This is why Gimpl-Heersink (2008) fails to extend her theoretical results to multi-period models even for the loss-neutral case.

To circumvent the challenge resulted from the lack of joint concavity, we introduce the transformed profit-to-go function $v_t^\lambda(x,r) = v_t(x,r) - \lambda r^2$ for some $\lambda \ge 0$ (see the proof of Proposition 1 for an explicit expression of $\lambda$). From (1) we know that $v_t^\lambda(x,r)$ satisfies

$$v_t^\lambda(x,r) = \underset{y,q}{\text{maximize}} \;\; \pi_t^\lambda(r,q) + \mathbb{E}[\gamma v_{t+1}^\lambda(y-\varepsilon_t, q) - h_t(y-\varepsilon_t)], \qquad (2)$$
$$\text{subject to} \;\; d_t(r,q) + y \ge x, \quad p(r,q) \in \mathcal{P},$$

where $\pi^\lambda(r,q) = \pi_t(r,q) - \lambda r^2 + \gamma\lambda q^2$ denotes the transformed expected one-period revenue function. Observe that an optimal solution to problem (1) also solves problem (2), and vice verse.

While $\pi_t(r,q)$ itself in problem (1) may not be jointly concave, we can properly select $\lambda$ such that $\pi_t^\lambda(r,q)$ is jointly concave. Specifically, we have the following results.

Proposition 1 **(Concavity and Supermodularity of Transformed Problem)**. *There exists some $\lambda \geq 0$ such that $\pi_t^\lambda(r,q)$ is jointly concave and supermodular in $(r,q)$ for each $1 \leq t \leq T$.*

With the help of Proposition 1, we are ready to characterize $v_t(x,r)$ and the optimal ordering policy.

Theorem 1 **(Optimality of Base-stock Policy)**. *$v_t(x,r)$ is decreasing in $x$ and increasing in $r$, and $v_t^\lambda(x,r)$ is jointly concave for some $\lambda \geq 0$. Moreover, a reference price dependent base-stock policy is optimal.*

Theorem 1 shows that the profit-to-go $v_t(x,r)$ increases in the reference price $r$ and decreases in the initial inventory $x$, and the transformed profit-to-go $v_t^\lambda(x,r)$ is jointly concave even though $v_t(x,r)$ may not be. Furthermore, it also suggests that a base-stock policy is optimal, where the base-stock level depends on the reference price $r$. That is, the optimal order-up-to inventory level can be written as $s_t(x,r) = \max\{x, s_t(r)\}$ for some base-stock level $s_t(r)$.

Next we study how the optimal price $p_t(x,r)$ depends on the initial inventory level $x$. In particular, we are interested to know whether list-price policy is still optimal. A widely used approach to answer such a question is to employ Theorem 2.8.2, Topkis (1998) (see Lemma 1(c) in the Online Supplement), which gives sufficient conditions to ensure the monotonicity of optimal solutions to parametric maximization problems with supermodular objective functions. However, this approach requires the feasible set to have a certain lattice structure, which unfortunately is not the case in our problem. Instead, we apply a recently developed result of Chen et al. (2013) and Hu (2011), which allows for non-lattice structures (see Lemma 2 in the Online Supplement) and enables us to show the following proposition.

Proposition 2 **(Optimality of List-Price Policy)**. *The optimal price in period $t$, $p_t(x,r)$, is decreasing in $x$ when consumers are loss neutral (i.e., $\eta^+ = \eta^-$) or when the firm is myopic (i.e., $t = T$ or $\gamma = 0$). However, $p_t(x,r)$ is not necessarily decreasing in $x$ in general.*

Proposition 2 shows that the list-price policy is optimal when consumers are loss-neutral or the firm is myopic, but the list-price policy may not be optimal in general when consumers are loss-averse. This is illustrated by Example 1 in the Online Supplement. The intuition is that, when consumers are loss-averse, the negative impact of a price cut on future demand due to a lower future reference price can be larger than its positive impact on the current demand. Thus, when inventory level is high, the firm may raise the price to ensure that inventory not sold in the current period can be sold faster in future periods.

We now move to study how the optimal decisions depend on the reference price and the intensity of reference price effect. As mentioned above, the optimal order-up-to inventory level can be written as $s_t(x,r) = \max\{x, s_t(r)\}$ for some base-stock level $s_t(r)$. In fact, we can express $s_t(r) = y_t(r) + d_t(r, q_t(r))$, where $[y_t(r), q_t(r)]$ solves the problem below by relaxing the constraint $d_t(r,q) + y \geq x$ from (1):

$$\underset{y,q:p(r,q)\in\mathcal{P}}{\text{maximize}} \; \{\pi_t(r,q) + \mathbb{E}[\gamma v_{t+1}(y-\varepsilon_t, q) - h_t(y-\varepsilon_t)]\}\,. \tag{3}$$

The optimal price and the optimal target reference price have the form $p_t(x,r) = p_t(r)$ and $q_t(x,r) = q_t(r)$ when $x \leq s_t(r)$, according to the proof of Theorem 1. In other words, when the firm places an order, the optimal decisions are characterized by the target reference price $q_t(r)$ and order-up-to level $s_t(r)$.

The next proposition study how $q_t(r)$ and $s_t(r)$ depend on the reference price $r$.

Proposition 3 **(Impact of Reference Price)**. *For each $1 \leq t \leq T$, $q_t(r)$ is increasing in $r$, and there exist $[Q_t^-, Q_t^+] \subseteq \mathcal{P}$ such that $r < q_t(r)$ if $r < Q_t^-$, $r = q_t(r)$ if $Q_t^- < r < Q_t^+$, and $q_t(r) > r$ if $r > Q_t^+$. Moreover, $s_t(r)$ is increasing in $r$ when consumers are loss neutral; and it is increasing and then decreasing and then increasing in $r$ when the firm is myopic. In general, $s_t(r)$ does not exhibit any specific monotonic pattern.*

Proposition 3 shows two results. First, the target reference price is increasing in the initial reference price, and there exists a region $[Q_t^-, Q_t^+]$ such that the target reference price is always closer to this region than the current reference price. Thus, even if the optimal price does not necessarily increase or decrease in the reference price, the result tells us when a price markup ($p_t > r$) or a price markdown ($p_t < r$) should be adopted. In particular, a price markup (or markdown) should be implemented if the initial reference price is lower (or higher) than this region.

The second result is that, while a higher current-period reference price leads to a higher base-stock level when consumers are loss-neutral, the base-stock level is not monotone in the reference price when consumers are loss-averse even in the simple case where the firm is myopic. The reason is that, the optimal price is equal to the current-period reference price when the current-period price lies in the region $[Q_t^-, Q_t^+]$, implying that the optimal price is increasing in the current-period price. Thus, when the current-period price increases, the demand decreases and the base-stock level also decreases. This does not happen when consumers are loss-neutral because the region $[Q_t^-, Q_t^+]$ reduces to a singleton in this case.

Next we consider the impact of the intensity of reference price effect.

Proposition 4 **(Impact of Reference Price Effect)**. *For any given $r$, $q_t(r)$ decreases in both $\eta^-$ and $\eta^+$.*

Proposition 4 shows that a stronger reference price effect leads to a lower reference price. Thus, a stronger reference price effect also leads to a lower price. Even though it can lead to a lower demand in the future, this is compensated by the benefit of a higher demand in the current period and a lower level of leftover inventory. The optimal base-stock level, however, may increase or decrease with a more intensified reference price effect, as illustrated by Example 2 in the Online Supplement, because the demand may increase or decrease in $\eta^-$ or $\eta^+$ depending on whether there is a perceived loss or perceived gain.

Other than the impact of the reference price effect, it is also interesting to study the impact of demand uncertainty. As illustrated by Examples 2 and 3 in the Online Supplement the optimal price and optimal base-stock level can be non-monotone in demand uncertainty, because higher demand uncertainty increases the chances for both very high demand and very low demand.

### 3.2. Infinite Horizon Model

We now consider the infinite horizon setting with a discount factor $0 < \gamma < 1$ and stationary system inputs (i.e., $h_t(x) = h(x)$, $b_t = b$, and $\varepsilon_t$ is identically distributed to $\varepsilon$ for each $t$ in the model). While the finite horizon model is more applicable to products with shorter life cycles such as consumer electronics, the infinite horizon model is more applicable to fast moving consumer goods with long life cycles such as shampoo and conditioners. Additionally, a good understanding of infinite horizon models may provide useful insights to facilitate the development of efficient heuristics for the finite horizon setting.

By a routine technique in dynamic programming, we can show that the limit of $v_t(x,r)$ as $t$ goes to infinity exists and can be characterized by the following Bellman equation:

$$\begin{aligned} v(x,r) = \underset{y,q}{\text{maximize}} \quad & \pi(r,q) + \mathbb{E}[\gamma v(y-\varepsilon,q) - h(y-\varepsilon)], \\ \text{subject to} \quad & d(r,q) + y \ge x, \quad p(r,q) \in \mathcal{P}. \end{aligned} \tag{4}$$

Similar to Theorem 1, we can prove the optimality of a reference price dependent base-stock policy for problem (4). Let $s(r)$ be the base-stock level associated with initial reference price $r$. A sequence $\{(x_t, r_t)\}$ is called a state path of problem (4) if for any $t \ge 1$, $(y_t, r_{t+1})$ solves problem (4) when $(x,r) = (x_t, r_t)$ and $x_{t+1} = y_t - \varepsilon_t$, where $\varepsilon_t$ is the realized demand uncertainty in period $t$.

We are interested in the asymptotic property of the state path of problem (4). Compared with a classical joint inventory and pricing model, Bellman equation (4) has one more state variable $r$. This added dimension of state space brings significant challenges. To circumvent the difficulty, we lay out a brief roadmap of our analysis before moving on to the details. First, we will study a simplified version of the problem where the firm is allowed to return the product to the manufacturer and obtain a full refund. We show that, in this simplified problem, the optimal ordering quantity can be uniquely determined for any fixed reference price trajectory, which allows us to characterize the optimal ordering policy first. Finally, we demonstrate how this simplified problem can be used to characterize the state path and reference price trajectory for problem (4).

When the firm is allowed to return products to the manufacturer and obtain a full refund, the inventory level after ordering and returning products is allowed to be lower than the initial inventory level. Thus, the constraint $d(r,q)+y \geq x$ in problem (4) disappears. The profit-to-go now depends only on $r$, and if denote it as $\tilde{v}(r)$, then from problem (4),

$$\tilde{v}(r) = \underset{q:p(r,q)\in\mathcal{P}}{\text{maximize}} \ \pi(r,q) - h^m + \gamma\tilde{v}(q), \tag{5}$$

where $h^m = \min_y[\mathbb{E}h(y-\varepsilon)]$. Define $y^m = \arg\min_y[\mathbb{E}h(y-\varepsilon)]$. It can be verified the target inventory level after ordering/returning is $\tilde{s}(r) = y^m + d(r, \tilde{q}(r))$, where $\tilde{q}(r)$ solves problem (5) and the safety stock $y^m$ is independent of $r$. Therefore, a state path $\{(\tilde{x}_t, \tilde{r}_t)\}$ of problem (5) satisfies $\tilde{x}_{t+1} = y^m - \varepsilon_t$ and $\tilde{r}_{t+1} = \tilde{q}(\tilde{r}_t)$ for all $t \geq 1$.

Observe that problem (5) is equivalent to the dynamic pricing problem without demand uncertainty analyzed in Popescu and Wu (2007). Their results imply the stability and convergence of $\{\tilde{r}_t\}$ in the case when returns are allowed. Specifically, $\{\tilde{r}_t\}$ *monotonically* converges to some interval $[R^-, R^+] \subseteq \mathcal{P}$ in the sense that it decreasingly (increasingly) converges to $R^+$ ($R^-$) if $\tilde{r}_1 > R^+$ ($\tilde{r}_1 < R^-$), and satisfies $\tilde{r}_t = \tilde{r}_1$ for all $t \geq 1$ if $R^- \leq \tilde{r}_1 \leq R^+$. It is appropriate to point out that $R^- = R^+$ in the loss-neutral case. Our next theorem shows that a state path of problem (4) coincides with that of problem (5) after a finite number of periods and hence the reference price of problem (4) also converges.

THEOREM 2 **(Convergence of Reference Price)**. *For any state path $\{(x_t, r_t)\}$ of problem (4), there exists some finite $\tau$ depending on the sample path $\{\varepsilon_t\}$ such that $\{(x_t, r_t) : t \geq \tau\}$ is also a state path of problem (5). Moreover, $\{r_t\}$ converges to some interval $[R^-, R^+] \subseteq \mathcal{P}$.*

While $\{r_t\}$ monotonically converges after some period $\tau$, it is not necessarily monotone when $t \leq \tau$ due to the existence of demand uncertainty. This property is different from the case when returns are allowed. However, the steady state reference price is the same as that in a model when returns are allowed. This is because the inventory level will drop to below the base-stock level after finite number of periods and will stay below the base-stock level (associated with any reference price encountered). Thus, thereafter, the system in our model evolves exactly the same as the one in the model allowing return.

Proposition 5 below further characterizes the optimal inventory policy in the long run.

Proposition 5 (**Long-Run Inventory Policy**). *Given any state path $\{(x_t, r_t)\}$, if $t$ is sufficiently large, then the expected inventory level is given by $\mathbb{E}x_t = \arg\min_y \mathbb{E}h(y-\varepsilon)$, which is independent of $r_t$, and the base-stock level is given by $s_t = \mathbb{E}x_t + d(r_t, r_{t+1})$. In addition, the base-stock level $s(r)$ is decreasing in $r \in [R^-, R^+]$.*

Proposition 5 shows that both the base-stock level and the safety stock converge in the long run. In particular, the long-run base-stock level is decreasing in the reference price and the long-run safety stock is independent of the reference price. The reason is that, the uncertainty involved in the additive demand is independent of the price and the reference price, and can be fully subsumed by inventory decisions. This is different from the case of a multiplicative demand uncertainty, as we are going to see in Section 4. Moreover, under slightly more restrictive conditions, we can show the base-stock level $s(r)$ is increasing in $r$ when $r \le R^-$ and $r \ge R^+$. (See Remark on Proposition 5 in the Online Supplement).

## 4. Extension: General Demand Uncertainty

In the previous section, we study the firm's optimal pricing and inventory decisions when the demand uncertainty follows the additive model. This section considers the case with more general demand model when the multiplicative demand uncertainty term is involved, where the realized demand is determined by

$$D_t = \xi_t d_t(r,q) + \varepsilon_t. \tag{6}$$

Similar to problem (1), given the initial inventory level $x$ and reference price $r$, the profit-to-go function $v_t(x,r)$ in period $t$ is given by

$$v_t(x,r) = \underset{d,s,q}{\text{maximize}} \quad [p(r,q)-c]d + \mathbb{E}[\gamma v_{t+1}(s-\xi_t d-\varepsilon_t, q) - h_t(s-\xi_t d - \varepsilon_t)], \tag{7a}$$

$$\text{subject to} \quad d = d_t(r,q),\ s \ge x,\ p(r,q) \in \mathcal{P}, \tag{7b}$$

where the decision variable $d$ represents the expected demand in period $t$.

In addition to the non-concavity of the objective function, another challenge of problem (7a) is that its feasible set is not convex because $d_t(r,q)$ is a nonlinear function in the loss-averse case. To circumvent the challenge, we need the following assumption for analytic tractability. Recall that $h_t^0(y) = h_t(y) - (1-\gamma)cy$ is the inventory holding and backlogging cost function.

Assumption 2. *$(p_{\min} - \gamma c)d - h_t^0(-d)$ is increasing in $d$ when $d \ge 0$.*

Assumption 2 states that a higher demand always leads to a higher profit, even in the case when the demand is backlogged and then satisfied in a future period. To understand this, consider the case when an amount $d$ of demand is backlogged in period $t$ and then satisfied in period $t+1$. This generates an additional revenue no smaller than $p_{\min}d$ and a backlogging cost of $h_t^0(-d)$ in period $t$, as well as a discounted ordering cost $\gamma cd$ in period $t+1$.

Assumption 2 implies that the objective function of problem (7a) is increasing in $d$ and hence the constraint $d = d_t(r,q)$ in (7b) can be replaced by $d \le d_t(r,q)$. With this assumption, we prove the following result in the finite horizon setting.

THEOREM 3. *Under the general demand model* (6) *and Assumption 2, a reference price dependent base-stock policy is optimal.*

For the corresponding infinite horizon counterpart, the limit of $v_t(x,r)$ as $t$ goes to infinity, denoted by $v(x,r)$, satisfies the Bellman equation:

$$\begin{aligned} v(x,r) = \underset{d,s,q}{\text{maximize}} \quad & [p(r,q)-c]d + \mathbb{E}[\gamma v(s-\xi d-\varepsilon, q) - h(s-\xi d-\varepsilon)], \\ \text{subject to} \quad & d = d(r,q),\ s \ge x,\ p(r,q) \in \mathcal{P}. \end{aligned} \tag{8}$$

To see the asymptotic property of the state path of problem (8), similar to the additive demand model, we consider the problem when returns are allowed:

$$\bar{v}(r) = \underset{q:p(r,q)\in\mathcal{P}}{\text{maximize}} \quad \pi(r,q) + \varphi(d(r,q)) + \gamma \bar{v}(q), \tag{9}$$

where $\varphi(d) = -\min_s \mathbb{E}h(s-\xi d-\varepsilon)$. Denote by $\bar{q}(r)$ the optimal solution to problem (9). Moreover, let $\bar{s}(r) = \arg\min_s \mathbb{E}h(s - \xi d(r,\bar{q}(r)) - \varepsilon)$ and $\bar{y}(r) = \bar{s}(r) - d(r,\bar{q}(r))$ be the associated optimal target inventory level and safety stock, respectively.

Proposition 6 below gives a sufficient condition on the existence and stability of steady state reference prices of problem (9). It then shows that a steady state reference price of problem (9) is also that of problem (8).

PROPOSITION 6. *(a) If demand uncertainty is multiplicative, $h(y) = h^+ \max\{y,0\} + h^- \max\{-y,0\}$ for some $h^\pm \ge 0$, and $p_{\min} \ge c + \min_y \mathbb{E}[h(y-\xi)]$, then any reference price trajectory of problem* (9) *monotonically converges to some interval $[R^-,R^+]$. Furthermore, when $r \in [R^-,R^+]$, the base-stock level $\bar{s}(r)$ is decreasing in $r$, and the safety stock $\bar{y}(r)$ can be expressed by $(b-ar)\bar{y}^m$ for $\bar{y}^m = \arg\min_y \mathbb{E}h(y-\xi+\mathbb{E}\xi)$.*

*(b) Given a steady state reference price $r$ of problem* (9) *and any $x \le \bar{s}(r)$, if we start with the state $(x,r)$ in problem* (8)*, then it is optimal to have the price equal to $r$ and order up to $\bar{s}(r)$ at every period. That is, $r$ is also a steady state reference price of problem* (8).

Unlike the case with additive demand uncertainty, the safety stock depends on the reference price when multiplicative uncertainty is involved. In particular, it can be interpreted as the safety stock in a newsvendor model with demand $(b-ar)\xi$. When $\xi$ is continuously distributed with cumulative distribution function $F(\cdot)$, the safety stock is given by $(b-ar)[F^{-1}(\frac{h^-}{h^-+h^+})-\mathbb{E}\xi]$. Thus, while the safety stock can be increasing or decreasing in $r$ depending on whether the base-stock level is below or above the average demand, the base-stock level is always decreasing in $r$ for $r\in[R^-,R^+]$.

Notice that $\min_y \mathbb{E}[h(y-\xi)]$ in Proposition 6(a) can be interpreted as a measure of variability of the random variable $\xi$, and the condition $p_{\min}\geq c+\min_y \mathbb{E}[h(y-\xi)]$ basically requires the variability of $\xi$ to be small. For example, when $\xi$ is uniformly distributed over the interval $[1-\kappa, 1+\kappa]$ for some $0\leq\kappa\leq 1$, and $h(x)=h_0|x|$ for some $h_0\geq 0$, this condition reduces to $\kappa h_0\leq 2(p_{\min}-c)$.

When consumers are loss-neutral, the statements in Proposition 6 hold with $R^+=R^-$ even without the conditions stated in the proposition. However, when consumers are loss-averse, it remains an open question whether the state path of problem (8) will converge to a steady state if the conditions do not hold. Unlike the additive demand case, we are not able to prove that the inventory level will remain always below the base-stock level after a certain period and a state path of problem (8) will coincide with that of problem (9) eventually. Nevertheless, similar to the additive demand case, we do observe from our numerical study that an order will be placed at every period after a finite number of periods and thus the reference price trajectory converges to a steady state in the interval $[R^-,R^+]$(See Section 4.3.2, Zhang 2010). In addition, the steady state reference price is affected by demand uncertainties and inventory-related cost when multiplicative uncertainty is involved(See Section 4.3.3, Hu 2011), which is quite different from the additive demand case.

## 5. Discussions

We study a joint inventory and pricing model taking into account the reference price effect. Despite the technical challenges resulted from the increase in the dimension of the dynamic program (due to the reference price effect) and the non-smooth demand function (due to loss aversion), we develop a transformation technique which allows us to characterize the optimal pricing and inventory policy of the firm. We also study the state path of our model in the infinite horizon counterpart. We prove that, after a finite number of periods, the state path coincides with the one when products can be returned for a full refund. We focus on the additive demand model, and show that some results can be extended to the case with a more general demand model. Our main results for the additive demand model are summarized in Table 1.

The transformation technique developed in this paper can be extended to other demand models, and also to the case when the market consists of different segments with different memory factors and different sensitivities to price and reference price. However, the conditions to ensure joint concavity depend heavily on the form of the underlying demand model and are usually less transparent.

**Table 1** Summary of Main Results

| | Base-stock policy | List-Price policy | Impact of reference price | Impact of reference price effect | Long-run reference price & base-stock level |
|---|---|---|---|---|---|
| Loss-neutral case | Optimal | Optimal | $q_t(r)$ increases; and $s_t(r)$ increases | $q_t(r)$ decrease; and $s_t(r)$ is not monotone in general | Converge |
| Loss-averse case | | Not optimal | $q_t(r)$ increases; and $s_t(r)$ is not monotone | | |

This paper should only be taken as an initial attempt at studying inventory and pricing models with reference effects. Two extensions are particularly note-worthy. First, our discussion is based on the assumption that unsatisfied demand is fully backlogged. Whether similar results hold in the lost-sales case remains to be determined. Second, besides the impact of the reference price, other types of reference effects also affect a firm's decisions (see, for example, Yang et al. 2014a, Yang et al. 2014b and the references therein). When considering joint pricing and inventory problems, the impact of a reference fill rate is also worth investigating. This line of inquiry is pursued by Liu and van Ryzin (2011) for the case of no demand uncertainty. When there is demand uncertainty, the investigation becomes challenging because the reference fill-rate is a function of demand. Another technical challenge is that the reference fill-rate is a kinked function of demand and the inventory level. Thus, this problem deserves a separate study.

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## Appendix: Proofs

### Proof of Proposition 1

Because both $(p-c)(b_t-ap)$ and $\eta(z)$ are concave, and $\eta(z)$ is increasing, it can be verified that $\pi_t(p+z,p+\alpha z)=(p-c)[b_t-ap+\eta(z)]$ is component-wise concave and supermodular in $p$ and $z$. Since $p=\frac{q-\alpha r}{1-\alpha}$ and $z=\frac{r-q}{1-\alpha}$, by Lemma 2(b), $\pi_t(r,q)$ is component-wise concave and supermodular. Therefore $\pi_t^\lambda(r,q)=\pi_t(r,q)-\lambda r^2+\gamma\lambda q^2$ is also supermodular.

We next prove that under a less restrictive condition than Assumption 1,

$$m^- - m^+ \le \sqrt{1+2m^-}+\sqrt{1+2m^+}, \tag{10}$$

$\pi_t^\lambda(r,q)$ is jointly concave when $\lambda=\frac{a(1+\alpha)}{2(1-\alpha)}(1+m^- - \sqrt{1+2m^-})$, i.e.,

$$\lambda=\frac{a(1+\alpha)}{2(1-\alpha)}\left[1+\frac{\eta^-}{(1+\alpha)a}-\sqrt{1+\frac{2\eta^-}{(1+\alpha)a}}\right]. \tag{11}$$

First observe that $\eta(z) = \min\{\eta^+ z, \eta^- z\}$ by $\eta^+ \leq \eta^-$. Thus, we can express $\pi_t^\lambda(p+z, p+\alpha z) = \min\left\{\bar{\pi}_t^-(p,z), \bar{\pi}_t^+(p,z)\right\}$, where

$$\bar{\pi}_t^\pm(p,z) = (p-c)[b_t - ap + \eta^\pm z] - \lambda(p+z)^2 + \gamma\lambda(p+\alpha z)^2.$$

Note that both $\bar{\pi}_t^\pm(p,z)$ are quadratic and correspond to the Hessian matrices

$$\nabla \bar{\pi}_t^\pm(p,z) = \begin{bmatrix} -2[a+\lambda(1-\gamma)], & \eta^\pm - 2\lambda(1-\gamma\alpha) \\ \eta^\pm - 2\lambda(1-\gamma\alpha), & -2\lambda(1-\gamma\alpha^2) \end{bmatrix},$$

where the diagonal entries are non-positive by $\lambda \geq 0$ and $0 \leq \gamma, \alpha \leq 1$. Furthermore, their determinants can be expressed by $\Delta(\lambda; \eta^\pm, \gamma)$, where

$$\Delta(\lambda;\eta,\gamma) = -4\gamma(1-\alpha)^2\lambda^2 + 4\left[(1-\gamma\alpha^2)a + (1-\gamma\alpha)\eta\right]\lambda - \eta^2.$$

It suffices to prove $\Delta(\lambda;\eta^\pm,\gamma) \geq 0$ for some $\lambda \geq 0$, which immediately ensures that both $\bar{\pi}_t^\pm(p,z)$ are jointly concave in $(r,q)$, and hence so is their minimum $\pi_t^\lambda(r,q)$. Observe that $\Delta(\lambda;\eta,\gamma)$ is increasing in $\gamma$. We shall prove the stronger yet simpler result that both $\Delta(\lambda;\eta^\pm,1) \geq 0$ for some $\lambda \geq 0$, where

$$\Delta(\lambda;\eta,1) = -4[(1-\alpha)\lambda]^2 + 4\left[(1+\alpha)a + \eta\right](1-\alpha)\lambda - \eta^2.$$

Note that $\Delta(\lambda;\eta,1)$ is concave and quadratic in $\lambda$; moreover, it corresponds to the roots

$$\lambda^\pm(\eta) = \tfrac{a(1+\alpha)}{2(1-\alpha)}\left[1 + \tfrac{\eta}{(1+\alpha)a} \pm \sqrt{1 + \tfrac{2\eta}{(1+\alpha)a}}\,\right].$$

From the above expression, it is straightforward to verify that $\lambda^-(\eta^-) \geq \lambda^-(\eta^+)$, and $\lambda^-(\eta^-) \leq \lambda^+(\eta^+)$ if and only if inequality (10) holds. Thus, if select $\lambda = \lambda^-(\eta^-)$, i.e., the one given in (11), then we conclude $\Delta(\lambda;\eta^-,1) = 0$ and $\Delta(\lambda;\eta^+,1) \geq 0$. □

### Proof of Theorem 1

The monotonicity of $v_t(x,r)$ in term of $x$ is obvious because for problem (1), its objective is independent of $x$ and the feasible set shrinks as $x$ increases. For the monotonicity of $v_t(x,r)$ in term of $r$, denote by $d_t^0(p,r) = b_t - ap + \eta(r-p)$ and reformulate problem (1) as

$$\begin{aligned} v_t(x,r) = \underset{y,p}{\text{maximize}} \quad & (p-c)d_t^0(p,r) + \mathbb{E}[\gamma v_{t+1}(y-\varepsilon_t, \alpha r + (1-\alpha)p) - h_t(y-\varepsilon_t)], \\ \text{subject to} \quad & d_t^0(p,r) + y \geq x, \quad p \in \mathcal{P}. \end{aligned}$$

For this problem, because $p \geq p_{\min} \geq c$ and $d_t^0(p,r)$ is increasing in $r$, its objective increases and its feasible set expands as $r$ increases. Therefore $v_t(x,r)$ is increasing in $r$.

We now prove the joint concavity of $v_t^\lambda(x,r)$ inductively. Suppose that it is true in period $t+1$, which is trivial when $t = T$ by $v_{T+1}^\lambda(x,r) = -\lambda r^2$. In period $t$, by Proposition 1, convexity of $h_t$ and the inductive assumption, the objective function of problem (2) is jointly concave in $(r,q,y)$. In addition, because $d_t(r,q)$ is concave and $p(r,q)$ is linear, the feasible set of problem (2) is convex in $(r,q,y,x)$. That is, (2) is a parametric concave maximization problem, implying that $v_t^\lambda(x,r)$ is concave. Furthermore, because problem (1) has the same optimal solutions as problem (2), it immediately follows the optimality of the inventory order-up-to level $s_t(x,r)=\max\{x,s_t(r)\}$ with $[s_t(r),q_t(r)]$ solving problem (3). In addition, when $x\le s_t(r)$, an order is placed and the the target reference price $q_t(x,r)=q_t(r)$. □

### Proof of Proposition 2

When consumers are loss-neutral, denote by $\eta^\pm=\eta$ and reformulate problem (1) as

$$v_t(x,r)=\underset{s\ge x,p\in\mathcal{P}}{\text{maximize}}\ \ \{(p-c)[(b_t+\eta r)-(a+\eta)p]-\mathbb{E}h_t(s+(a+\eta)p-(b_t+\eta r)-\varepsilon_t)\}$$
$$+\{\gamma\mathbb{E}v_{t+1}(s-(b_t+\eta r)+(a+\eta)p-\varepsilon_t,\alpha r+(1-\alpha)p)\},\qquad(12)$$

For the objective function of problem (12), its first bracketed term is submodular in $s$ and $p$ by convexity of $h_t(x)$ and Theorem 2.2.6, Simchi-Levi et al. (2014). Thus, $p_t(x,r)$ is decreasing in $x$ by Lemma 1(c) if we can further prove its second bracketed term is also submodular in $s$ and $p$. Observe that by letting $A=\frac{a+\eta}{1-\alpha}$, we can express

$$v_{t+1}(s+(a+\eta)p,(1-\alpha)p)=v_{t+1}(A(1-\alpha)p+s,(1-\alpha)p).$$

From Lemma 1(a), it suffices to show the function $v_{t+1}(Ar-x,r)$ is supermodular in $(x,r)$.

In fact by similar to the proof of Theorem 1, we can inductively verify a stronger result that the transformed profit-to-go function $\hat{v}_t(Ar-x,r)=v_t(Ar-x,r)-\hat\lambda r^2$ is concave and supermodular in $(x,r)$ for $\hat\lambda=\frac{\eta(\alpha a+\eta)}{2(1-\alpha)(a+\eta)}$. Recall that the expected demand in period $t$ is given by $d=b_t-(a+\eta)p+\eta r$. From the reference price evolution model $q=\alpha r+(1-\alpha)p$, price $p=p_t(d,q)$ and reference price $r=r_t(d,q)$ in the period $t$ can be reformulated by functions of $d$ and $q$ as below:

$$p_t(d,q)=\frac{-\alpha(d-b_t)+\eta q}{\alpha a+\eta},\quad\text{and}\quad r_t(d,q)=\frac{(1-\alpha)(d-b_t)+(a+\eta)q}{\alpha a+\eta}.$$

Therefore the expected profit can be expressed by $\hat\pi_t(d,q)=[p_t(d,q)-c]\,d-\hat\lambda[r_t(d,q)]^2+\gamma\hat\lambda q^2$. Furthermore, it can be verified that $\partial^2_{dq}\hat\pi_t(d,q)=0$, and

$$\partial^2_d\hat\pi_t(d,q)=-\frac{2\alpha(a+\eta)+(1-\alpha)\eta}{(a+\eta)(\alpha a+\eta)},\quad\text{and}\quad\partial^2_q\hat\pi_t(d,q)=-\eta\left(\frac{1-\gamma}{1-\alpha}+\frac{a}{\alpha a+\eta}+\frac{\gamma a}{a+\eta}\right).$$

That is, $\hat\pi_t(d,q)$ is concave and supermodular. Suppose $\hat v_t(Ar-x,r)$ is concave and supermodular in $(x,r)$, which is straightforward when $t=T+1$ by $\hat\lambda\ge 0$. By the definition of $\hat v_t(x,r)$.

$$\hat v_t(Ar-x,r)=\underset{d,q}{\text{maximize}}\ \ \hat\pi_t(d,q)+\mathbb{E}[\gamma\hat v_{t+1}(y-\varepsilon_t,q)-h_t(y-\varepsilon_t)],\qquad(13a)$$
$$\text{subject to}\ \ r_t(d,q)=r,\ \ d+y\ge Ar-x,\ \ p_t(d,q)\in\mathcal{P}.\qquad(13b)$$

Replacing the decision variable $y$ by $Aq-y$ in the above problem,

$$\begin{aligned}\hat{v}_t(Ar-x,r) = \underset{d,q}{\text{maximize}} \;\; & \hat{\pi}_t(d,q) + \mathbb{E}[\gamma\mathbb{E}\hat{v}_{t+1}(Aq-y-\varepsilon_t,q) - h_t(Aq-y-\varepsilon_t)] \\ \text{subject to} \;\; & r_t(d,q)=r,\; Aq+d-y\geq Ar-x,\; p_t(d,q)\in\mathcal{P},\end{aligned}$$

where by $r_t(d,q)=r$, the constraint $Aq+d-y\geq Ar-x$ is equivalent to $x\geq A[r_t(d,q)-q]-d+y=\frac{(1-\alpha)a}{\alpha a+\eta}d+y+\frac{a+\eta}{\alpha a+\eta}(aq-b)$. Therefore by introducing some slack variable $\delta$, we have that

$$\begin{aligned}\hat{v}_t(Ar-x,r) = \underset{d,q,\delta}{\text{maximize}} \;\; & \hat{\pi}_t(d,q) + \mathbb{E}[\gamma\hat{v}_{t+1}(Aq-y-\varepsilon_t,q) - h_t(Aq-y-\varepsilon_t)], \\ \text{subject to} \;\; & r_t(d,q)=r, \quad \tfrac{(1-\alpha)a}{\alpha a+\eta}d+y+\tfrac{a+\eta}{\alpha a+\eta}aq+\delta=x, \\ & p_t(d,q)\in\mathcal{P}, \quad \delta\geq-\tfrac{a+\eta}{\alpha a+\eta}b.\end{aligned}$$

Since $\hat{v}_{t+1}(Aq-y,q)$ is concave and supermodular in terms of $q$ and $y$ by the inductive assumption, and so is $-\mathbb{E}h_t(Aq-y-\varepsilon_t)$ by convexity of $h_t(x)$, it follows from Lemma 2(a) that $\hat{v}_t(Ar-x,r)$ as a function of $x$ and $r$ is concave and supermodular.

When the firm is myopic, i.e., $\gamma=0$ or $t=T$, observe that $p=p_t(x,r)$ solves the problem

$$v_t(x,r) = \underset{p\in\mathcal{P}}{\text{maximize}} \;\; \{(p-c)[b_t-ap+\eta(r-p)] - \mathbb{E}h_t(x-[b_t-ap+\eta(r-p)]-\varepsilon_t)\}.$$

Since the expected demand $d=b_t-ap+\eta(r-p)$ is strictly decreasing in $p$, we can rewrite the price $p=p_t^0(r,d)$ as a function of $r$ and $d$, where $p_t^0(r,d)$ is strictly decreasing in $d$. In addition, the above problem becomes

$$v_t(x,r) = \underset{d}{\text{maximize}} \;\; \left\{[p_t^0(r,d)-c]d - \mathbb{E}h_t(x-d-\varepsilon_t)\right\},$$

where objective function is supermodular in terms of $x$ and $d$ by convexity of $h_t$ and Proposition 2.2.5 and Theorem 2.2.6 in Simchi-Levi et al. (2014). Thus, its optimal solution, denoted by $d_t^0(x,r)$, is increasing in $x$, implying that $p_t(x,r)=p_t^0(r,d_t^0(x,r))$ is decreasing in $x$.

In general, Example 1 in the Online Supplement shows $p_t(x,r)$ can be increasing in $x$. □

### Proof of Proposition 3

For the monotonicity of $q_t(r)$, note that it solves the problem

$$\underset{q:p(r,q)\in\mathcal{P}}{\text{maximize}} \;\; \pi_t(r,q) + \max_y \mathbb{E}\left[\gamma v_{t+1}(y-\varepsilon_t,q) - h_t(y-\varepsilon_t)\right],$$

where its objective function is supermodular in $(r,q)$ by Proposition 1. Moreover, since $p(r,q)$ is linear, and it is decreasing in $r$ and increasing in $q$, the set $\{(r,q):p(r,q)\in\mathcal{P}\}$ forms a sublattice by Example 2.2.7, Topkis (1998). Therefore $q_t(r)$ is increasing by Lemma 1(c).

For the existences of $Q_t^{\pm}$, introduce $z_t(r) = \frac{r-q_t(r)}{1-\alpha}$. If $z_t(r)$ is increasing in $r$, then we can select $Q_t^- = \sup\{r \in \mathcal{P} : z_t(r) < 0\}$ and $Q_t^+ = \inf\{r \in \mathcal{P} : z_t(r) > 0\}$, where $Q_t^- = p_{\min}$ when $z_t(p_{\min}) \geq 0$, and $Q_t^+ = p_{\max}$ when $z_t(p_{\max}) \leq 0$. Thus, it remains to show that the monotonicity of $z_t(r)$ in $r$. For this purpose, notice that $z_t(r)$ solves the problem:

$$\underset{z:r-z\in\mathcal{P}}{\text{maximize}} \ \left\{\bar{\pi}_t^{\lambda}(r,z) + u_t^{\lambda}(r-(1-\alpha)z)\right\},$$

where $\bar{\pi}_t^{\lambda}(r,z) = \pi_t^{\lambda}(r, r-(1-\alpha)z)$ and $u_t^{\lambda}(q) = \max_y \mathbb{E}\left[-h_t(y-\varepsilon_t) + \gamma v_{t+1}^{\lambda}(y-\varepsilon_t, q)\right]$. It can be verified that the partial derivative of $\bar{\pi}_t^{\lambda}(r,z)$ in $r$ satisfies

$$\partial_r \bar{\pi}_t^{\lambda}(r,z) = \eta(z) + 2[a - \gamma\lambda(1-\alpha)]z + C_0(r),$$

where $C_0(r)$ is some term depending only on $r$. When $\gamma = 0$, it is straightforward to see from the above expression that $\partial_r \bar{\pi}_t^{\lambda}(r,z)$ is increasing in $z$. When $\gamma > 0$, by substituting the expression of $\lambda$ provided in (11), we can verify that it is increasing in $z$ when Assumption 1 holds, implying $\bar{\pi}_t^{\lambda}(r,z)$ is supermodular in $(r,z)$. Furthermore, because $v_{t+1}^{\lambda}(x,r)$ is jointly concave by Theorem 1, $u^{\lambda}(q)$ is concave and hence $u_t^{\lambda}(r-(1-\alpha)z)$ is supermodular in $(r,z)$ by Theorem 2.2.6, Simchi-Levi et al. (2014). Because $\{(z,r) : r - z \in \mathcal{P}\}$ forms a sublattice in $(r,z)$ by Example 2.2.7 in Topkis (1998), we conclude from Lemma 1(c) that $z_t(r)$ is increasing in $r$.

We now characterize the reference price dependent base-stock level $s_t(r)$. When consumers are loss-neutral, following the notations in problem (13), consider the transformed profit-to-go function $\hat{v}_t(x,r) = v_t(x,r) - \hat{\lambda}r^2$. Observe that we can express $\hat{v}_t(x,r) = \max_{s\geq x} \hat{u}_t(s,r)$, where

$$\hat{u}_t(s,r) = \underset{d,q}{\text{maximize}} \ \ \hat{\pi}_t(d,q) + \mathbb{E}[\gamma\hat{v}_{t+1}(y-\varepsilon_t, q) - h_t(y-\varepsilon_t)], \tag{14a}$$

$$\text{subject to} \ \ r_t(d,q) = r, \ d + y = s, \ p_t(d,q) \in \mathcal{P}. \tag{14b}$$

To see the monotonicity of $s_t(r)$ in $r$, suppose $\hat{v}_{t+1}(x,r)$ is jointly concave and supermodular in $(x,r)$, which is trivial for $t = T$ by $\hat{\lambda} \geq 0$. It together with the concavity and supermodularity of $\hat{\pi}_t(d,q)$ ensures that the objective function in (14a) is jointly concave and supermodularity in $(d,q,y)$. Therefore $\hat{u}_t(s,r)$ in (14) is concave and supermodular by Lemma 2(a). By lemma 1(c), $s_t(r)$ is increasing in $r$ and $\hat{v}_t(x,r)$ is also jointly concave and supermodular. Repeating this process for all $t = T, T-1, \cdots, 1$, we conclude that $s_t(r)$ is increasing in $r$ for each $t$.

When the firm is myopic, i.e., $\gamma = 0$ or $t = T$, observe that $[s_t(r), q_t(r)]$ solves the problem

$$\underset{s,q:p(r,q)\in\mathcal{P}}{\text{maximize}} \ \{[p(r,q) - c]d_t(r,q) - \mathbb{E}h_t(s - d_t(r,q) - \varepsilon_t)\}.$$

By the definition of $Q_t^-$ and $Q_t^+$, it leads no loss of optimality to let $d_t(r,q) = b_t - a\frac{q-\alpha r}{1-\alpha} + \frac{\eta^-(r-q)}{1-\alpha}$ when $r < Q_t^-$, and $d_t(r,q) = b_t - a\frac{q-\alpha r}{1-\alpha} + \frac{\eta^+(r-q)}{1-\alpha}$ when $r > Q_t^+$. Similar to the loss-neutral case discussed previously, we can prove that $s_t(r)$ is increasing in $r$ when $r < Q_t^-$ and $r > Q_t^+$, respectively. When $Q_t^- < r < Q_t^+$, $q_t(r) = r$ and hence $s_t(r)$ solves the problem

$$\underset{s}{\text{maximize}} \ \{(r-c)(b_t - ar) - \mathbb{E}h_t(s + ar - b_t - \varepsilon_t)\}.$$

Since the objective function is submodular in $(s, r)$ by convexity of $h_t$, we conclude that $s_t(r)$ is decreasing in $r \in (Q_t^-, Q_t^+)$ by Lemma 1(a,c). □

### Proof of Proposition 4

Since $q_t(r) = \alpha r + (1-\alpha)p_t(r)$, it suffices to prove the monotonicity of $p_t(r)$ in $\eta^\pm$. For this purpose, notice that $p_t(r)$ solves the problem

$$\underset{p}{\text{maximize}} \ \left\{\pi_t^0(p,r) + u_t(\alpha r + (1-\alpha)p) : p \in [p_{\min}, p_{\max}]\right\}, \tag{15}$$

where $\pi_t^0(p,r) = (p-c)[b_t - ap + \eta^+ \max\{0, r-p\} + \eta^- \min\{0, r-p\}]$, and

$$u_t(q) = \max_y \mathbb{E}\left[-h_t(y - \varepsilon_t) + \gamma v_{t+1}^\lambda(y - \varepsilon_t, q)\right].$$

If we can prove $\pi_t^0(p,r)$ is submodular in $(p, \eta^-, \eta^+)$, then $p_t(r)$ is decreasing in $\eta^-$ and $\eta^+$ by Lemma 1(c). However, one can easily verify that $\pi_t^0(p,r)$ is not submodular in $(p, \eta^-, \eta^+)$ in general. It means we cannot apply Lemma 1(c) to problem (15) directly.

To conquer this difficulty, we next show it leads no loss of optimality to restrict $p$ in a subset of $[p_{\min}, p_{\max}]$ in problem (15), on which $\pi_t^0(p,r)$ is submodular in $(p, \eta^-, \eta^+)$. Observe that $v_{t+1}(x,q)$ and hence $u_t(q)$ are increasing in $r$ by Theorem 1. Furthermore, if $p < \frac{1}{2}(r+c)$, then $p < r$ by $r \geq c$. It implies that

$$\partial_p \pi_t^0(p,r) = (b_t + ac - 2ap) + \eta^+(r + c - 2p) > b_t - ar,$$

where $b_t - ar \geq 0$ because we assumed $d_t(p,r) \geq 0$ for any $p, r \in [p_{\min}, p_{\max}]$. Thus, the objective function of problem (15) is increasing in $p$ when $p < \frac{1}{2}(r+c)$, implying $p_t(r)$ also solves the problem

$$\underset{p}{\text{maximize}} \ \left\{\pi_t^0(p,r) + u_t(\alpha r + (1-\alpha)p) : \tfrac{1}{2}(r+c) \leq p \leq p_{\max}\right\}.$$

Reformulate $\pi_t^0(p,r)$ as below:

$$\pi_t^0(p,r) = (p-c)(b_t - ap) + \eta^- \min\{0, (p-c)(r-p)\} + \eta^+ \max\{0, (p-c)(r-p)\}.$$

When $p \geq \frac{1}{2}(r+c)$, it is submodular in $(p, \eta^-, \eta^+)$ because $(p-c)(r-p)$ is decreasing in $p$. Thus, $p_t(r)$ is decreasing in $\eta^-$ and $\eta^+$ by Lemma 1(c). □

## Proof of Theorem 2

Given any state path $\{(x_t, r_t) : t \geq 1\}$ of problem (4), suppose an order is place in period $\tau$, where $\tau$ is finite with probability 1 because the base-stock policy is optimal, and the probability of $D_t \geq D_{\min}$ is non-zero for some $D_{\min} > 0$ as assumed. Construct $\{(\tilde{x}_t, \tilde{r}_t) : t \geq 1\}$ such that $(\tilde{x}_t, \tilde{r}_t) = (x_t, r_t)$ when $t \leq \tau$, and $(\tilde{x}_t, \tilde{r}_t) = (y^m - \varepsilon_{t-1}, \tilde{q}(r_{t-1}))$ when $t > \tau$. That is, $\{(\tilde{x}_t, \tilde{r}_t) : t \geq 1\}$ is a state path of problem (5) after period $\tau$.

Because $\varepsilon_t$ are identically distributed and $d(r, q) + \varepsilon_t$ is non-negative for any $r, q \in \mathcal{P}$, $\tilde{x}_t = y^m - \varepsilon_{t-1} < y^m + d(\tilde{r}_t, \tilde{r}_{t+1}) = \tilde{s}(r_t)$ for any $t > \tau$. This inequality indicates that the initial inventory level in period $t + 1$ is always below the target inventory level after period $\tau$ for problem (5). Therefore orders are placed in all periods $t > \tau$ for the state path $\{(\tilde{x}_t, \tilde{r}_t) : t > \tau\}$ of problem (5). The definition of $v(x, r)$ implies that $\tilde{v}(\tilde{r}_t) \leq v(\tilde{x}_t, \tilde{r}_t)$ for any $t > \tau$. On the other hand, by the definitions of $\tilde{v}(r)$ and $v(x, r)$, we also know that $\tilde{v}(r) \geq v(x, r)$ for any feasible $(x, r)$. In summary, $v(\tilde{x}_t, \tilde{r}_t) = \tilde{v}(\tilde{r}_\tau)$ for any $t > \tau$, implying that $\{(\tilde{x}_t, \tilde{r}_t) : t > \tau\}$ is also a state path of problem (5). Moreover, because $\{\tilde{r}_t\}$ converges to some interval $[R^-, R^+]$ by Theorem 4 of Popescu and Wu (2007), we know so is the sequence $\{r_t\}$. □

## Proof of Proposition 5

In the proof of Theorem 2, we in fact proved $x_t = \tilde{x}_t$ and $r_t = \tilde{r}_t$ for sufficiently large $t$, where $\{(\tilde{x}_t, \tilde{r}_t\}$ is a state path of problem (5). Therefore $\mathbb{E}x_t = \mathbb{E}\tilde{x}_t = y_m$ and $s_t = \tilde{s} = y_m + d(r_t, r_{t+1})$, where recall that $y^m = \arg\min_y \mathbb{E}h(y - \varepsilon)$.

For the monotonicity of $s(r)$ in $r \in [R^-, R^+]$, note that $s(r) = y^m + d(r, \tilde{q}(r))$, where $\tilde{q}(r)$ solves problem (5). When $r \in [R^-, R^+]$, $\tilde{q}(r) = r$ and hence $s(r) = y^m + d(r, r) = y^m + b - ar$, which is clearly decreasing in $r$. □

## Proof of Theorem 3

The monotonicity of $v_t(x, r)$ can be proved similar to Theorem 1. For the concavity of $v_t^\lambda(x, r)$, reformulate problem (7a) as below:

$$
\begin{aligned}
v_t^\lambda(x, r) = \underset{s,q}{\text{maximize}} \quad & \bar{\pi}_t^\lambda(r, q) + w_t(s, d) + \gamma \mathbb{E} v_{t+1}^\lambda(s - \xi_t d - \varepsilon_t, q), \\
\text{subject to} \quad & d = d_t(r, q), \quad s \geq x, \quad p(r, q) \in \mathcal{P},
\end{aligned}
\tag{16}
$$

where $\bar{\pi}_t^\lambda(r, q) = [p(r, q) - p_{\min}] d_t(r, q) + \lambda(\gamma q^2 - r^2)$ and $w_t(s, d) = (p_{\min} - c)d - \mathbb{E}h_t(s - \xi_t d - \varepsilon_t)$. It should be pointed out that the joint concavity and supermodularity of $\pi_t^\lambda(r, q)$ can also be verified from Proposition 1 if we replace all $c$ by $p_{\min}$ in its proof.

We now inductively show the concavity of $v_t^\lambda(x, r)$. Suppose $v_{t+1}^\lambda(x, r)$ is concave, which is trivial when $t = T + 1$. Then in period $t$, $w_t(s, d)$ is increasing in $d$ by its definition and Assumption 2.

Moreover, because $v_{t+1}(x,r)$ is decreasing in $x$, by $\xi_t \geq 0$, the objective function of (16) is increasing in $d$. It leads no loss of optimality to change the constraint $d = d_t(r,p)$ in (16) to $d \leq d_t(r,q)$, i.e.,

$$v_t^\lambda(x,r) = \underset{s,q}{\text{maximize}} \;\; \pi_t^\lambda(r,q) + w_t(s,d) + \gamma \mathbb{E} v_{t+1}^\lambda(s - \xi_t d - \varepsilon_t, q),$$
$$\text{subject to} \;\; d \leq d_t(r,q), \quad s \geq x, \quad p(r,q) \in \mathcal{P}.$$

Note that the feasible set of the above problem is now convex because $d_t(r,q)$ is a concave function. Since $\pi_t^\lambda(r,q)$ is jointly concave, and so is $w_t(s,d)$ by convexity of $h_t$, we can conclude that $v_t^\lambda(x,r)$ is also jointly concave; moreover, a reference price dependent base-stock policy is optimal. □

**Proof of Proposition 6**

(a) Let $\bar{h}^m = \min_y \mathbb{E} h(y-\xi)$. By given conditions on $h(x)$, we can express

$$\varphi(d) = \underset{s}{\text{maximize}} \; [-\mathbb{E} h(s/d - \xi)] d = -\bar{h}^m d,$$

Moreover, the objective function of problem (9) can be reformulated by

$$[p(r,q) - c - \bar{h}^m] d(r,q) + \gamma \bar{v}(q).$$

Similar to Proposition 1, it is supermodular in $(r,q)$ when $p(r,q) \geq c + \bar{h}^m$. By following a similar argument in Section 5.2, Popescu and Wu (2007), we can conclude the convergence of any reference price trajectory to some interval $[R^-, R^+]$, where $R^\pm$ are the steady state reference price associated with the problem with the reference price effects $\eta^\pm(r-p)$, respectively.

When $r \in [R^-, R^+]$, $\bar{s}(r)$ solves the problem $\max_s [-\mathbb{E} h(s - (b - ar) - \varepsilon)]$, where its objective function is submodular in $(s,r)$ by convexity of $h$. Thus, $\bar{s}(r)$ is decreasing in $r \in [R^-, R^+]$ by Lemma 1(a,c). In addition, the safety stock $\bar{y}(r)$ solves the problem $\max_y [-\mathbb{E} h(y + (1 - \xi)(b - ar))]$, as well as the problem $\max_y [-\mathbb{E}(y/(b-ar) + 1 - \xi)$ by $b - ar = d(r,r) \geq 0$. Hence $\bar{y}(r) = \bar{y}^m (b - ar)$ by the definition of $\bar{y}^m$.

(b) For problem (9), if $x < \bar{s}(r)$, by definitions of steady state reference price and target inventory level, then the inventory level is raised to $\bar{s}(r)$ and the selling price is $r$. It leads to the initial inventory level $\tilde{x} = \bar{s}(r) - [\xi d(r,r) + \varepsilon]$ and reference price $r$ in next period, where clearly $\tilde{x} \leq \bar{s}(r)$. Therefore starting from $(x,r)$, an order is placed in each period. Similar to the proof of Theorem 2 for the additive demand model, the corresponding state path is also a state path of problem (8). Therefore $r$ is also a steady state reference price of problem (8). □

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## Online Supplement

### Remark on Proposition 5

Proposition 5 in Section 3.2 shows that the long-run base-stock level $s(r)$ is decreasing in $r \in [R^-, R^+]$. In fact, we can show it is increasing when $r \le R^-$ or $r \ge R^+$ under the following condition:

$$\tfrac{\alpha(1+2m^+)}{\alpha+(1+\alpha)m^+} + m^- \le m^+ + \sqrt{1+2m^-}, \tag{17}$$

where recall that $m^\pm = \eta^\pm/[(1+\alpha)a]$. This condition is satisfied, for example, when $\alpha = 0$ (i.e., customers use the price in the previous period as the reference price) or customers are loss-neutral.

To show that $s(r)$ is increasing when $r \le R^-$ or $r \ge R^+$ under this condition, first recall that we already proved $s(r) = y^m + d(r, \tilde{q}(r))$, where $\tilde{q}(r)$ solves the problem (5). It suffices to verify $d(r, \tilde{q}(r))$ is increasing in $r$ when $r > R^+$ or $r < R^-$.

Let $\tilde{v}^\lambda(r) = \tilde{v}(r) - \lambda r^2$ with $\lambda$ given in (11). Then $\tilde{q}(r)$ also solves the problem

$$\tilde{v}^\lambda(r) = \underset{q:p(r,q)\in\mathcal{P}}{\text{maximize}} \left\{[p(r,q)-c]d(r,q) - \lambda r^2 + \gamma\lambda q^2 - h^m\right\} + \gamma \tilde{v}^\lambda(q).$$

First we show that $\tilde{v}^\lambda(r)$ is concave. Define $\tilde{v}_0^\lambda(r) = 0$ and for any $n \ge 0$,

$$\tilde{v}_{n+1}^\lambda(r) = \underset{q:p(r,q)\in\mathcal{P}}{\text{maximize}} \; [p(r,q)-c]d(r,q) - \lambda r^2 + \gamma\lambda q^2 - h^m + \gamma \tilde{v}_n^\lambda(q).$$

Notice that $\tilde{v}^\lambda(r) = \lim_{n\to\infty} \tilde{v}_n^\lambda(r)$. Similar to the proof of Theorem 1, we can inductively show that all $\tilde{v}_n^\lambda(r)$ are concave, implying that $\tilde{v}^\lambda(r)$ is concave, too.

1. When $r > R^+$, $\tilde{q}(r) < r$ by and hence $\tilde{q}(r)$ solves the problem

$$\underset{q:p(r,q)\in\mathcal{P}}{\text{maximize}} \; [p(r,q)-c]d^+(r,q) - \lambda r^2 + \gamma\lambda q^2 - h^m + \gamma\tilde{v}^\lambda(q),$$

where $d^+(r,q) = b - a\frac{q-\alpha r}{1-\alpha} + \eta^+\frac{r-q}{1-\alpha}$. Define $q = q^+(r,d)$ as the inverse function of $d = d^+(r,q)$ in $q$ for any given $r$, and $p_+(r,d) = p(r, q^+(r,d))$. It can be shown

$$q^+(r,d) = \tfrac{1-\alpha}{a+\eta^+}b - \tfrac{1-\alpha}{a+\eta^+}d + \tfrac{\alpha a+\eta^+}{a+\eta^+}r, \qquad p^+(r,d) = \tfrac{1}{a+\eta^+}b - \tfrac{1}{a+\eta^+}d + \tfrac{\eta^+}{a+\eta^+}r.$$

Then $d(r, \tilde{q}(r))$ solves the problem

$$\underset{d:p_+(r,d)\in\mathcal{P}}{\text{maximize}} \; \tilde{\pi}_+^\lambda(r,d) - h^m + \gamma\tilde{v}^\lambda(q_+(r,d)), \tag{18}$$

where $\tilde{\pi}_+^\lambda(r,d) = [p_+(r,d) - c]d - \lambda r^2 + \gamma\lambda[q^+(r,d)]^2$ is a clearly quadratic function. Moreover, we can be verified that its mixed second derivative $\frac{\eta^+}{a+\eta^+} - 2\gamma\lambda\frac{(1-\alpha)(\alpha a+\eta^+)}{(a+\eta^+)^2} \ge 0$ by (17) and the definition of $\lambda$ given in (11). Therefore $\tilde{\pi}_+^\lambda(r,d)$ is supermodular in $(r,d)$. In addition, since $\tilde{v}^\lambda(q)$ is concave as proved, and the linear function $q_+(r,d)$ is increasing in $r$ and decreasing in $d$, we know that $\tilde{v}^\lambda(q_+(r,d))$ is also supermodular in $(r,d)$. Thus, the objective function of (18) is supermodular in $(r,d)$. Furthermore, because $p^+(r,d)$ increasing in $r$ and decreasing in $d$, the feasible set of problem (18) forms a sublattice in $(r,d)$ by Example 2.2.7 in Topkis (1998). By Lemma 1(c), the optimal solution $d(r,\tilde{q}(r))$ to problem (18), as well as $s(r)=y^m+d(r,\tilde{q}(r))$, is increasing in $r$ when $r<R^-$.

2. When $r<R^-$, by a similar argument we can prove that $s(r)$ in increasing in $r$ if

$$m^- + \tfrac{\alpha(1+2m^-)}{\alpha+(1+\alpha)m^-} \le m^- + \sqrt{1+2m^-},$$

which holds automatically because $\tfrac{\alpha(1+2m^-)}{\alpha+(1+\alpha)m^-} \le 1 \le \sqrt{1+2m^-}$.

**Numerical Examples**

Here, we provide three examples. The first example shows that the optimal price $p_t(x,r)$ may not always be decreasing in the initial inventory level $x$, the second one shows that the base-stock level $s_t(r)$ may be not monotone in the intensity of reference price effect or the magnitude of demand uncertainty, and the third one shows the optimal price is not monotone in the magnitude of demand uncertainty.

Example 1. Consider a 20-period instance with $b=10, a=2, \eta^+=0.2, \eta^-=1.2, \alpha=0.4, \gamma=0.8, c=0, \mathcal{P}=[0,2.5]$, and $\varepsilon_t$ identically and uniformly distributed on $[-0.9,0.9]$. In this case, the optimal price $p_t(x,r)$ increases in $x$ when $x=7.83$.

Example 2. Suppose that $p_{\min}=c=0, p_{\max}=a=1, b_t=3, \eta(x)=\beta x$ for some $0\le\beta\le1$, $h_t(z)=h^+\max\{z,0\}+h^-\max\{-z,0\}$ for some $h^\pm>0$, and the demand uncertainty $\varepsilon_t$ is uniformly distributed on the interval $[-\kappa,\kappa]$ for some $0<\kappa<1$. Then the demand in period $t$ is

$$d_t^0(p,r)+\varepsilon_t=[3-(1+\beta)p+\beta r]+\varepsilon \ge 1-\kappa>0, \quad \forall\, 0\le p,r\le 1.$$

Moreover, we can verify that $\mathbb{E}h(y-\varepsilon_t)$ is continuously differentiable with the derivative as follows:

$$\partial_y \mathbb{E}h(y-\varepsilon_t)=\tfrac{1}{2}(h^+-h^-)+\tfrac{1}{2}(h^++h^-)\max\left[-1,\min\left(\kappa^{-1}y,1\right)\right]. \tag{19}$$

Observe that $\mathbb{E}h_t(y-\varepsilon_t)$ achieves its minimum at $y^m=\frac{h^--h^+}{h^-+h^+}\kappa$.

In the last period $t=T$, the problem becomes

$$\begin{aligned} v_t(x,r)= \underset{p,y}{\text{maximize}}\ \ & p[3-(1+\beta)p+\beta r]-\mathbb{E}h_t(y-\varepsilon_t),\\ \text{subject to}\ \ & [3-(1+\beta)p+\beta r]+y\ge x, 0\le p\le 1.\end{aligned}$$

By Theorem 1, the base-stock level $s_t(r)=y^m+[3-(1+\beta)p^0(r)+\beta r]$, where

$$p^0(r)=\underset{p:0\le p\le 1}{\arg\max}\,\{p[3-(1+\beta)p+\beta r]\}=\min\left\{\tfrac{3+\beta r}{2(1+\beta)},1\right\}.$$

By substituting the expressions of $y^m$ and $p^0(r)$,

$$s_t(r) = \tfrac{h^- - h^+}{h^- + h^+}\kappa + \max\left[2 - \beta(1-r), \tfrac{1}{2}(3+\beta r)\right].$$

From the above equation, we have the following observations:

1. $s_t(r)$ is increasing in $\kappa$ if $h^- \geq h^+$; otherwise it is decreasing in $\kappa$.
2. $s_t(r)$ is decreasing in $\beta$ if $0 \leq \beta \leq \frac{1}{2-r}$; otherwise it is increasing in $\beta$.

EXAMPLE 3. Consider the instance specified in Example 2 with $\beta = 1$, $h^- = 1$ and $h^+ = 3$, where the base-stock level becomes $s_t(r) = \frac{1}{2}(3 + r - \kappa)$. For any $x > s_t(r)$,

$$\begin{aligned} v_t(x,r) = \underset{y,p}{\text{maximize}} \;\; & (3 - 2p + r)p - \mathbb{E}h(y - \varepsilon), \\ \text{subject to} \;\; & (3 - 2p + r) + y = x, 0 \leq p \leq 1. \end{aligned}$$

Eliminating the decision variable $p$ by $p = \frac{1}{2}[(y - x) + (3 + r)]$ gives us

$$\begin{aligned} v_t(x,r) = \underset{y}{\text{maximize}} \;\; & \tfrac{1}{2}(x-y)[(y-x) + (3+r)] - \mathbb{E}h(y - \varepsilon_t), \\ \text{subject to} \;\; & x - r - 3 \leq y \leq x - r - 1. \end{aligned}$$

Denote $y_t(x,r)$ as the optimal solution and $f(y)$ be the objective function of the above problem. Observe that $s_t(r) < \frac{1}{2}(r+3)$ and

$$f'(y) = (x - y) - \tfrac{1}{2}(r+5) - 2\max\left[-1, \min\left(\kappa^{-1}y, 1\right)\right].$$

In the following, we show the close form of $y_t(x,r)$ when $\frac{1}{2}(r+3) \leq x \leq r+3$. For this purpose, observe that for any $\frac{1}{2}(r+3) \leq x \leq r+3$, by $0 < \kappa < 1$ and $0 \leq r \leq 1$,

$$f'(\kappa) = x - \kappa - \tfrac{1}{2}(r+3) - 3 < 0 < x + \kappa - \tfrac{1}{2}(r+3) + 1 = f'(-\kappa).$$

In addition, by $x - r - 3 \leq 0$ and $x - r - 1 \geq \frac{1}{2}(1-r) \geq 0$,

$$\begin{aligned} f'(x-r-3) &= \tfrac{1}{2}(r+1) - 2\max\left\{-1, \min\left[\kappa^{-1}(x-r-3), 1\right]\right\} \geq \tfrac{1}{2}(r+1) > 0, \\ f'(x-r-1) &= \tfrac{1}{2}(r-3) - 2\max\left\{-1, \min\left[\kappa^{-1}(x-r-1), 1\right]\right\} \leq \tfrac{1}{2}(r-3) < 0. \end{aligned}$$

Therefore some $y \in [\max(-\kappa, x - r - 3), \min(\kappa, x - r - 1)]$ achieves the maximum of $f(y)$. That is, $y_t(x,r)$ is in fact the global maximizer of $f(y)$ and can be determined by solving the equation $f'(y) = (x - y) - \frac{1}{2}(3+r) - 1 - 2\kappa^{-1}y = 0$. That is,

$$y_t(x,r) = [2x - (r+5)]/(2 + 4\kappa^{-1}).$$

Obviously $y_t(x,r)$, as well as $p_t(x,r) = \frac{1}{2}[y_t(x,r) - x + (r+3)]$, is decreasing in $\kappa$ for $\frac{1}{2}(r+3) \leq x \leq \frac{1}{2}(r+5)$ and increasing in $\kappa$ for $\frac{1}{2}(r+5) \leq x \leq r+3$.

### Notes on Supermodular and Submodular Functions

Many results in this paper rely on properties of supermodular and submodular functions. We refer to Topkis (1998) for more discussions on these functions. For convenience, some properties used in this paper are summarized below. For simplicity, denote $\arg\max_{\mathcal{X}} f(x)$ as the lexicographically smallest maximizer of $f(x)$ over $\mathcal{X}$, and assume it is well-defined.

LEMMA 1. *(a) If a 2-dimensional function $f(x,y)$ is supermodular, then so is $-f(-x,y)$.*

*(b) If $f(x)$ is increasing and convex, then $xf(y+z)$ is supermodular in terms of $x,y$ and $z$.*

*(c) Given a sublattice $\mathcal{S}$ of $\Re^{n+m}$ and a supermodular function $f(x,y)$, define*

$$g(x) = \underset{y:(x,y)\in\mathcal{S}}{\text{maximize}} \; f(x,y), \quad y(x) = \arg\max_{y:(x,y)\in\mathcal{S}} f(x,y),$$

*for any $x \in \mathcal{S}_x = \{x \in \Re^n : (x,y) \in \mathcal{S}$ for some $y \in \Re^m\}$. Then $\mathcal{S}_x$ forms a sublattice of $\Re^n$, $g(x)$ is supermodular on $\mathcal{S}_x$, and $y(x)$ is increasing in $x \in \mathcal{S}_x$.*

In Lemma 1, part (a) can be easily verified from the definition of supermodularity, part (b) is a corollary of Proposition 2.2.5 in Simchi-Levi et al. (2014), and part (c) follows from Theorem 2.7.1 and Theorem 2.8.2 in Topkis (1998), respectively.

Lemma 2 below is recently developed by Chen et al. (2013). It establishes a preservation property of supermodularity under optimization operations when the constraint set may not be a sublattice, and a sufficient condition to preserve supermodularity under linear transformation.

LEMMA 2. *(a) Given a closed convex sublattice $\mathcal{S} \subseteq \Re^n$, a concave and supermodular function $g$ defined on $\mathcal{S}$, and a non-negative $2\times n$ matrix $A$, if the following function is well-defined on $\mathcal{S}_x = \{Ay : y \in \mathcal{S}\}$:*

$$f(x) = \underset{y}{\text{maximize}} \;\; \{g(y) : Ay = x, y \in \mathcal{S}\},$$

*then $\mathcal{S}_x$ forms a closed convex sublattice of $\Re^2$, and $f(x)$ is concave and supermodular on $\mathcal{S}_x$.*

*(b) Given a 2-dimensional function $g$, if it is component-wise concave and supermodular, then so is the function $g(a_{11}x_1 - a_{12}x_2, a_{22}x_2 - a_{21}x_1)$ for any $a_{ij} \geq 0$.*