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Technical Note—Preservation of Supermodularity in Parametric Optimization Problems with Nonlattice Structures

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This paper establishes a new preservation property of supermodularity in a class of two-dimensional parametric optimization problems, where the constraint sets may not be lattices. This property and its extensions unify several results in the literature and provide powerful tools to analyze a variety of operations models including a two-product coordinated pricing and inventory control problem with cross-price effects that we use as an illustrative example.

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1. Introduction

Consider the following optimization problem parameterized by a vector $\mathbf{x}$ in $\mathbf{X}$, a subset of a Euclidean space:

$$f(\mathbf{x}) = \max_{\mathbf{y}} \{g_0(\mathbf{x},\mathbf{y}) \colon (\mathbf{x},\mathbf{y}) \in \mathbf{D}_0\},$$

where $\mathbf{y}$ denotes a vector in a subset of a Euclidean space $\mathbf{Y}$, g_0 is a function defined on $\mathbf{X}\times\mathbf{Y}$, and $\mathbf{D}_0$ is a subset of $\mathbf{X}\times\mathbf{Y}$. An interesting question is under what conditions supermodularity properties can be preserved after the maximization operation. A widely used result from the literature states that under the assumption that $\mathbf{X}$ and $\mathbf{Y}$ are lattices and $\mathbf{D}_0$ is a sublattice of $\mathbf{X}\times\mathbf{Y}$, if $g_0(\mathbf{x},\mathbf{y})$ is supermodular on $\mathbf{D}_0$, then the function $f(\mathbf{x})$ is supermodular on the set of $\mathbf{x}$ for which the maximization is well defined (see Topkis 1998, Theorem 2.7.6). In addition, the optimal solution set is increasing in $\mathbf{x}$. This preservation property of supermodularity plays a key role in many Markovian decision processes to derive monotone structural results of optimal policies. Unfortunately, this property, though powerful, requires the set $\mathbf{D}_0$ to be a sublattice. Relaxing the lattice requirement has been a significant challenge. Indeed, without the lattice condition, the analysis becomes much more complicated even in some very simple settings in which supermodularity can be preserved.

The objective of this paper is to establish a new preservation property of supermodularity under optimization operations when the constraint set may not be a lattice. Specifically, consider the following optimization problem parameterized by a *two-dimensional* vector $\mathbf{x}\in\mathbf{S}=\{A\mathbf{y}\colon \mathbf{y}\in\mathbf{D}\}$:

$$f(\mathbf{x}) = \underset{\mathbf{y}}{\text{maximize}}\ \{g(\mathbf{y})\colon A\mathbf{y}=\mathbf{x},\ \mathbf{y}\in\mathbf{D}\}, \tag{1}$$

where A is a $2\times n$ matrix, $\mathbf{D}$ is a subset of $\Re^n$, and g is an n-dimensional function defined on $\mathbf{D}$. Under the assumption that A is nonnegative and $\mathbf{D}$ is a closed convex sublattice of $\Re^n$, we prove that if g is concave and supermodular, then so is f.

The significance of our result is that in problem (1), the constraint set $\{(\mathbf{x},\mathbf{y})\colon A\mathbf{y}=\mathbf{x},\ \mathbf{y}\in\mathbf{D}\}$ is not a lattice and may not be mapped to become one by a variable transformation in general. As the major contribution of this paper, we manage to identify some nonlattice structures of parametric optimization problems under which supermodularity can be preserved. Of course, some additional assumptions have to be imposed (see the discussion after the proof of our main result Theorem 1 in §2.1), and in general the optimal solution set is not monotone in the parameter vector (see Example 2 in §2.3). Though it may appear restrictive, relaxing the lattice requirement even slightly may render the preservation property invalid.

Our preservation result and its extensions unify a few results in the literature and significantly simplify the analysis of operations models in several papers. To illustrate its power, we consider several applications in §3. We specifically focus on a two-product periodic-review coordinated pricing and inventory control problem with cross-price effects. This application falls into the fast growing literature on integrated inventory and pricing models, for which we refer to Chen and Simchi-Levi (2012) for an up-to-date survey. Compared with Zhu and Thonemann (2009), Song and Xue (2007), and Ceryan et al. (2013), who analyze similar models with two substitutable products and develop structural properties of the optimal policies, our approach based on the preservation properties developed in this paper is significantly simpler and leads to some new structural results. In addition, it allows us to deal with both complementary products and substitutable products in a unified framework under more general settings. For more applications of our results, we refer to Hu (2011).

The rest of this paper is organized as follows. Section 2 mainly focuses on the theoretical part. Specifically we present the main result and some extensions and demonstrate their limitations. In §3, we present some applications of our main results followed by comments on their applicability.

2. Main Results

This section is divided into three parts. In §2.1, we show in Theorem 1 that concavity and supermodularity can be preserved in problem (1), and we provide several remarks and extensions on the main theorem. In §2.2, we consider the preservation of two closely related concepts (submodularity and $L^\natural$-concavity). In §2.3, the limitations of our results are demonstrated through several examples. Throughout this paper, we assume that the maximization in problem (1) and other related optimization problems is well defined for any $\mathbf{x} \in \mathbf{S}$. For omitted proofs and detailed discussions of our results, we refer to Hu (2011).

Before we proceed, we introduce the notations and basic concepts used in this paper. Sets are expressed by boldface capital letters (e.g., $\mathbf{D}$), matrices by regular capital letters (e.g., A), vectors by boldface lowercase letters (e.g., $\mathbf{x}$), and real numbers by regular lowercase letters (e.g., x_i). We also write $A = [a_{i,j}]$ or $\mathbf{x} = [x_i]$ sometimes to emphasize entries of A or components of $\mathbf{x}$. All vectors are column vectors, and $\mathbf{0}$, $\mathbf{e}$ are the vectors with all components 0, 1, respectively.

Given an $m \times n$ matrix A, denote by $A \geqslant 0$ if all its entries are nonnegative. We say A is an $\mathscr{L}_0$-matrix if $m = n$ and it has nonnegative diagonal entries and nonpositive off-diagonal entries. Given any two n-dimensional vectors $\mathbf{x} = [x_i]$ and $\mathbf{y} = [y_i]$, denote by $\mathbf{x} \leqslant \mathbf{y}$ if $x_i \leqslant y_i$ for all i, $\mathbf{x} \vee \mathbf{y} = [\max\{x_i, y_i\}]$ and $\mathbf{x} \wedge \mathbf{y} = [\min\{x_i, y_i\}]$. In addition, we call a set $\mathbf{S}$ of $\Re^n$ *convex* if $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \mathbf{S}$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{S}$ and $0 \leqslant \lambda \leqslant 1$, and a *sublattice* (of $\Re^n$) if $\mathbf{x}$, $\mathbf{y} \in \mathbf{S}$ implies that $\mathbf{x} \wedge \mathbf{y}$, $\mathbf{x} \vee \mathbf{y} \in \mathbf{S}$.

Given a function f defined on a subset $\mathbf{S}$ of $\Re^n$ (in case $\mathbf{S}$ is not specified, we implicitly assume $\mathbf{S} = \Re^n$), f is called *supermodular* if $\mathbf{S}$ is a sublattice and $f(\mathbf{x}) + f(\mathbf{y}) \leqslant f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y})$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{S}$, and *convex* if $\mathbf{S}$ is convex and $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leqslant \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$ for all $0 \leqslant \lambda \leqslant 1$ and $\mathbf{x}$, $\mathbf{y} \in \mathbf{S}$. We say f is *concave* (or *submodular*) if $-f$ is convex (or supermodular). For more discussions on these concepts, we refer to Rockafellar (1970), Topkis (1998), and Simchi-Levi et al. (2005).

2.1. Main Theorem and Its Extensions

We now state our main theoretical result in this paper.

Theorem 1. *Given any $2 \times n$ matrix $A \geqslant 0$, if $\mathbf{D}$ is a nonempty closed convex sublattice, then so is the set $\mathbf{S}$; moreover, if g is concave and supermodular on $\mathbf{D}$, then so is f on $\mathbf{S}$.*

Proof. It is straightforward to show that $\mathbf{S}$ is closed and convex. Concavity of f on $\mathbf{S}$ follows from Theorem 5.4 in Rockafellar (1970). It remains to prove that $\mathbf{S}$ forms a sublattice of $\Re^2$ and f is supermodular on $\mathbf{S}$, i.e., $\mathbf{x} \wedge \tilde{\mathbf{x}} \in \mathbf{S}$, $\mathbf{x} \vee \tilde{\mathbf{x}} \in \mathbf{S}$ and $f(\mathbf{x}) + f(\tilde{\mathbf{x}}) \leqslant f(\mathbf{x} \wedge \tilde{\mathbf{x}}) + f(\mathbf{x} \vee \tilde{\mathbf{x}})$ for any $\mathbf{x}$, $\tilde{\mathbf{x}} \in \mathbf{S}$. For this purpose, let $\mathbf{y}$ and $\tilde{\mathbf{y}}$ be the optimal solutions associated with $\mathbf{x}$ and $\tilde{\mathbf{x}}$ in problem (1), respectively. In addition, define $\mathbf{a} = A(\mathbf{y} \wedge \tilde{\mathbf{y}})$ and $\mathbf{b} = A(\mathbf{y} \vee \tilde{\mathbf{y}})$.

Observe that $\mathbf{y} \wedge \tilde{\mathbf{y}} \in \mathbf{D}$ since $\mathbf{D}$ is a sublattice, $\mathbf{a} \in \mathbf{S}$ by the definition of $\mathbf{S}$, and $\mathbf{a} \leqslant \mathbf{x} \wedge \tilde{\mathbf{x}}$ by $A \geqslant 0$. Therefore $\mathbf{x} \wedge \tilde{\mathbf{x}}$ belongs to the convex hull of $\{\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{a}\}$ (see Figure 1 for the illustration). That is, there exist $0 \leqslant \lambda$, $\mu \leqslant \lambda + \mu \leqslant 1$ such that $\mathbf{x} \wedge \tilde{\mathbf{x}} = \lambda\mathbf{x} + \mu\tilde{\mathbf{x}} + (1-\lambda-\mu)\mathbf{a}$. Moreover, since $\mathbf{a} + \mathbf{b} = A\mathbf{y} + A\tilde{\mathbf{y}} = \mathbf{x} + \tilde{\mathbf{x}} = \mathbf{x} \wedge \tilde{\mathbf{x}} + \mathbf{x} \vee \tilde{\mathbf{x}}$, we know that $\mathbf{x} \vee \tilde{\mathbf{x}} = \mu\mathbf{x} + \lambda\tilde{\mathbf{x}} + (1-\mu-\lambda)\mathbf{b}$. Because $\mathbf{S}$ is convex, this implies that $\mathbf{x} \wedge \tilde{\mathbf{x}}$, $\mathbf{x} \vee \tilde{\mathbf{x}} \in \mathbf{S}$, i.e., $\mathbf{S}$ is a sublattice of $\Re^2$.

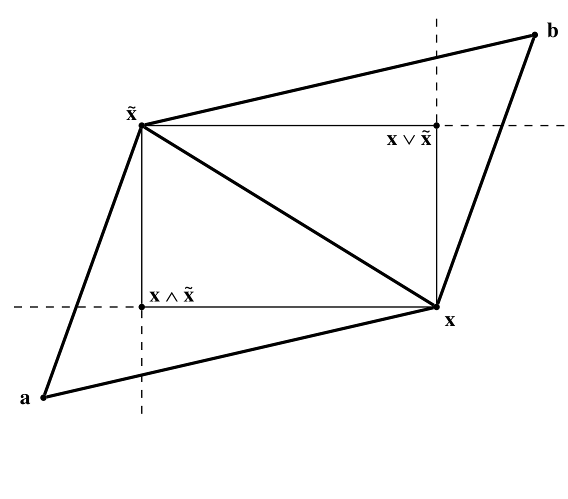
To show the supermodularity of f, notice that from its concavity, we can verify that

$$(\lambda+\mu)[f(\mathbf{x}) + f(\tilde{\mathbf{x}})] + (1-\lambda-\mu)[f(\mathbf{a}) + f(\mathbf{b})] \leqslant f(\mathbf{x} \wedge \tilde{\mathbf{x}}) + f(\mathbf{x} \vee \tilde{\mathbf{x}}).$$

In addition, the definition of f and the supermodularity of g lead to

$$f(\mathbf{x}) + f(\tilde{\mathbf{x}}) = g(\mathbf{y}) + g(\tilde{\mathbf{y}}) \leqslant g(\mathbf{y} \wedge \tilde{\mathbf{y}}) + g(\mathbf{y} \vee \tilde{\mathbf{y}}) \leqslant f(\mathbf{a}) + f(\mathbf{b}).$$

Figure 1. Relative positions of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$, $\tilde{\mathbf{x}}$, and $\mathbf{x} \wedge \tilde{\mathbf{x}}$, $\mathbf{x} \vee \tilde{\mathbf{x}}$ in the proof of Theorem 1.



Combining the above two inequalities, we conclude that $f(\mathbf{x})+f(\tilde{\mathbf{x}}) \leqslant f(\mathbf{x}\wedge\tilde{\mathbf{x}})+f(\mathbf{x}\vee\tilde{\mathbf{x}})$. □

Compared with the preservation property stated by Theorem 2.7.6 in Topkis (1998), the constraint set $\{(\mathbf{x},\mathbf{y})\colon A\mathbf{y}=\mathbf{x}, \mathbf{y}\in\mathbf{D}\}$ in the parametric maximization problem (1) may not be a lattice. To compensate for this nonlattice structure, we have to restrict the parameter $\mathbf{x}$ in a two-dimensional space and the matrix A to be nonnegative, which allows us to construct a forward-bending parallelogram containing the rectangle formed by $\mathbf{x}$, $\tilde{\mathbf{x}}$, and $\mathbf{x}\wedge\tilde{\mathbf{x}}$, $\mathbf{x}\vee\tilde{\mathbf{x}}$ as illustrated in Figure 1. Another key to prove Theorem 1 is to construct a vector $\mathbf{a}$ such that $\mathbf{x}\wedge\tilde{\mathbf{x}}$ can be expressed as a convex combination of $\mathbf{x}$, $\tilde{\mathbf{x}}$, and $\mathbf{a}$. This makes it possible to show the desirable inequality $f(\mathbf{x})+f(\tilde{\mathbf{x}}) \leqslant f(\mathbf{x}\wedge\tilde{\mathbf{x}})+f(\mathbf{x}\vee\tilde{\mathbf{x}})$ by applying the concavity of f.

One may ask whether the concavity assumption can be weakened for the preservation of supermodularity. The answer is negative in general as we illustrate later in Example 2. Nevertheless, it can be weaker if A in problem (1) has specific forms. Consider the special case that one row of the matrix A has at most one nonzero entry. Then in the proof of Theorem 1 we can further verify that either $\tilde{\mathbf{x}}$ or $\mathbf{x}$ lies on the line determined by $\mathbf{x}\wedge\tilde{\mathbf{x}}$ and $\mathbf{a}$. Hence the concavity of f along either all vertical or all horizontal lines is sufficient for the preservation of supermodularity. Such a result is summarized as the corollary below.

Corollary 1. *Consider the optimization problem parameterized by $[x,t]\in\mathbf{S}$:*

$$f(x,t)=\underset{\mathbf{y}}{\text{maximize}}\ \{g(\mathbf{y},t)\colon \mathbf{a}'\mathbf{y}+bt=x, [\mathbf{y},t]\in\mathbf{D}\},$$

where $\mathbf{S}=\{[\mathbf{a}'\mathbf{y}+bt,t]\colon [\mathbf{y},t]\in\mathbf{D}\}$, $\mathbf{a}\geqslant\mathbf{0}$ and $b\geqslant 0$. Denote $\mathbf{D}(t)=\{\mathbf{y}\colon [\mathbf{y},t]\in\mathbf{D}\}$ and $\mathbf{S}(t)=\{x\colon [x,t]\in\mathbf{S}\}$. If $\mathbf{D}(t)$ is convex for any t and $\mathbf{D}$ forms a sublattice, then $\mathbf{S}(t)$ is convex for any t and $\mathbf{S}$ forms a sublattice. Moreover, if $g(\mathbf{y},t)$ is supermodular on $\mathbf{D}$ and concave in $\mathbf{y}\in\mathbf{D}(t)$ for any t, then $f(x,t)$ is supermodular on $\mathbf{S}$ and concave in $x\in\mathbf{S}(t)$ for any t.

As another special case, when $n=2$ and A is nonsingular, Theorem 1 implies that if $g(\mathbf{x})$ is concave and supermodular on $\mathbf{D}$, then so is the function $g(A^{-1}\mathbf{x})$ in $\mathbf{x}\in A(\mathbf{D})$. Notice that A^{-1} is an $\mathscr{L}_0$-matrix. In general, the following result provides some sufficient conditions on the preservation of supermodularity under linear variable transformations.

Corollary 2. *Given any 2×2 $\mathscr{L}_0$ matrix P, if $\mathbf{D}$ is a convex sublattice in $\Re^2$, then so is the set $\mathbf{S}=\{\mathbf{x}\colon P\mathbf{x}\in\mathbf{D}\}$; moreover, if a function g on $\mathbf{D}$ is component-wise concave and supermodular, then so is $g(P\mathbf{x})$ on $\mathbf{S}$.*

Our first remark shows that Theorem 1 can be extended to cases in which more than two linear constraints may be present as long as the parameter $\mathbf{x}$ is two dimensional in the parametric maximization problem.

Remark 1. If the equality constraint in problem (1) is replaced by $A\mathbf{y}=B\mathbf{x}$ for some matrices A and B such that $B'A\geqslant 0$ and $B'B$ is an $\mathscr{L}_0$-matrix, then the domain of f becomes $\mathbf{S}=\{\mathbf{x}\colon A\mathbf{y}=B\mathbf{x}$ for some $\mathbf{y}\in\mathbf{D}\}$. If g is concave and supermodular on $\mathbf{D}$, then so is f on $\mathbf{S}$.

It is worth mentioning that the optimal solution sets of problem (1) may not possess certain monotonicity properties (see Example 2 in §2.3). However, the remark below shows that if the equality $A\mathbf{y}=\mathbf{x}$ in problem (1) is replaced by $A\mathbf{y}\leqslant\mathbf{x}$, then the corresponding optimal solution set has a certain monotonicity property.

Remark 2. Consider the following problem:

$$f(\mathbf{x})=\underset{\mathbf{y}}{\text{maximize}}\ \{g(\mathbf{y})\colon A\mathbf{y}\leqslant\mathbf{x},\ \mathbf{y}\in\mathbf{D}\}, \tag{2}$$

which is obtained by replacing the equality constraints $A\mathbf{y}=\mathbf{x}$ in (1) by the inequality constraints. By adding some nonnegative slack variables, we can establish the supermodularity of f by Theorem 1. Denote $\mathbf{Y}_1$ and $\mathbf{Y}_2$ as the optimal solution sets of problem (2) associated with $\mathbf{x}_1$ and $\mathbf{x}_2$ with $\mathbf{x}_1\leqslant\mathbf{x}_2\in\mathbf{S}$, respectively, then $\mathbf{Y}_2$ is greater than $\mathbf{Y}_1$ in the so-called *$\mathscr{C}$-flexible set order* (Quah 2007). Moreover, for any $\mathbf{y}_1\in\mathbf{Y}_1$ (or $\mathbf{y}_2\in\mathbf{Y}_2$) there exists $\mathbf{y}_2\in\mathbf{Y}_2$ (or $\mathbf{y}_1\in\mathbf{Y}_1$) such that $\mathbf{y}_1\leqslant\mathbf{y}_2$ by Proposition 3, Quah (2007).

2.2. Preservation of Submodularity and $L^\natural$-Concavity

Next we consider two other preservation properties for the special case of problem (1):

$$f(\mathbf{x})=\underset{\mathbf{y}_1,\ldots,\mathbf{y}_N}{\text{maximize}}\ \left\{\sum_{n=1}^{N} f_n(\mathbf{y}_n)\colon \sum_{n=1}^{N}\mathbf{y}_n=\mathbf{x}, \mathbf{y}_n\in\mathbf{S}_n, \forall n\right\}, \tag{3}$$

where $\mathbf{S}_n$ are subsets of $\Re^2$, and f is defined on $\mathbf{S}=\{\sum_{n=1}^{N}\mathbf{y}_n\colon \mathbf{y}_n\in\mathbf{S}_n, \forall n\}$. In the following we illustrate how our results on supermodularity translate to the preservation of submodularity and $L^\natural$-concavity. For this purpose, we need the following result, which is an immediate corollary of Theorem 1.

Corollary 3. *Suppose P is a nonsingular 2×2 matrix. For problem (3), if all $P^{-1}(\mathbf{S}_n)$ are convex sublattices of $\Re^2$, and $f_n(P\mathbf{x})$ are concave and supermodular on $P^{-1}(\mathbf{S}_n)$, then $P^{-1}(\mathbf{S})$ forms a convex sublattice of $\Re^2$, and $f(P\mathbf{x})$ is concave and supermodular on $P^{-1}(\mathbf{S})$.*

When P is the identity matrix, Corollary 3 states the preservation of concavity and supermodularity in problem (3). In general, we may have some flexibility to choose the matrix P depending on applications. Three interesting instances of P are listed:

$$J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad J_1=\begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix},\quad J_2=\begin{bmatrix}-1 & 0\\ -1 & 1\end{bmatrix}.$$

Note that $J(J\mathbf{x})=J_1(J_1\mathbf{x})=J_2(J_2\mathbf{x})=\mathbf{x}$.

The linear transformation J maps a vector $[x_1, x_2]$ to $[x_1, -x_2]$. Geometrically, $J(\mathbf{S})$ is the reflection of the set $\mathbf{S}$ at the horizontal axis. Interestingly the transformation shows a simple but useful relation between two-dimensional submodular functions and supermodular functions, whose proof follows directly from the definitions of supermodularity and submodularity and thus is omitted.

Lemma 1. *When both* $\mathbf{S}$ *and* $J(\mathbf{S})$ *are sublattices of* $\Re^2$, *a two-dimensional function* $f(\mathbf{x})$ *is supermodular on* $\mathbf{S}$ *if and only if* $f(J\mathbf{x})$ *is submodular on* $J(\mathbf{S})$.

Note that if $\mathbf{S}$ has the form $\{[x_1, x_2]: l_i \leqslant x_i \leqslant u_i, i = 1, 2\}$, then $J(\mathbf{S}) = \{[x_1, x_2]: l_1 \leqslant x_1 \leqslant u_1, -u_2 \leqslant x_2 \leqslant -l_2\}$. In this case, both $\mathbf{S}$ and $J(\mathbf{S})$ are sublattices.

Lemma 1 allows us to convert a statement on supermodularity to a related statement on submodularity. For example, together with Corollary 2, it implies that for any 2×2 nonnegative matrix B, if g is component-wise concave and submodular on $\Re^2$ then so is $g(B\mathbf{x})$ in $\mathbf{x}$.

The other two transformations, J_1 and J_2, map a vector $[x_1, x_2]$ to $[x_1 - x_2, -x_2]$ and $[-x_1, x_2 - x_1]$, respectively. We will use them in the following to analyze a concept called *$L^\natural$-concavity*.

Definition 1. An n-dimensional concave function f defined on $\mathbf{S}$ is called $L^\natural$-concave if the set $\mathbf{S}_+ = \{(\mathbf{x}, \xi): \mathbf{x} - \xi\mathbf{e} \in \mathbf{S}, \xi \geqslant 0\}$ forms a sublattice of $\Re^{n+1}$ and $\psi(\mathbf{x}, \xi) = f(\mathbf{x} - \xi\mathbf{e})$ is supermodular on $\mathbf{S}_+$.

Note that in a two-dimensional space, a set $\mathbf{S}$ of the following form,

$$\mathbf{S} = \{[x_1, x_2] \in \Re^2: l_1 \leqslant x_1 \leqslant u_1,\ l_2 \leqslant x_2 \leqslant u_2,\ l_0 \leqslant x_1 - x_2 \leqslant u_0\}, \tag{4}$$

satisfies the condition in the definition of $L^\natural$-concavity.

The concept of $L^\natural$-concavity is closely related to concavity and supermodularity. Indeed, a $L^\natural$-concave function is both concave and supermodular. The concept is fundamental in discrete convex analysis (see, e.g., Murota 2003), and finds applications in inventory models (see, e.g., Zipkin 2008, Chen et al. 2012a).

We next turn to the preservation of $L^\natural$-concavity on problem (3). It is appropriate to point out that such a result cannot be implied from Theorem 1 directly because the $L^\natural$-concavity of a two-dimensional function is defined through the supermodularity of a three-dimensional auxiliary function, and Theorem 1 only works when the parameter vectors are two-dimensional in general (see Example 2 in the next subsection). Interestingly, the following lemma provides a characterization for a class of two-dimensional $L^\natural$-concave functions through the supermodularity of three auxiliary two-dimensional functions.

Lemma 2. *Given* $\mathbf{S} \subset \Re^2$ *of the form* (4), *a function* f *on* $\mathbf{S}$ *is* $L^\natural$*-concave if and only if* $f(\mathbf{x})$, $f(J_1\mathbf{x})$, *and* $f(J_2\mathbf{x})$ *are supermodular on* $\mathbf{S}$, $J_1(\mathbf{S})$, *and* $J_2(\mathbf{S})$, *respectively.*

From Corollary 3 we can immediately conclude the following result, whose proof is omitted.

Corollary 4. *Assume in problem* (3) *that all* $\mathbf{S}_n$ *are convex sublattices of* $\Re^2$.

(a) *If all* f_n *are concave and supermodular on* $\mathbf{S}_n$, *then so is* f *on the convex sublattice* $\mathbf{S}$.

(b) *If all* $J(\mathbf{S}_n)$ *are sublattices of* $\Re^2$, *and all* f_n *are concave and submodular on* $\mathbf{S}_n$, *then* f *is concave and submodular on the convex sublattice* $\mathbf{S}$.

(c) *If* $\mathbf{S}_n$ *is of the form* (4) *and all* f_n *are* $L^\natural$*-concave on* $\mathbf{S}_n$, *then* f *is* $L^\natural$*-concave on* $\mathbf{S}$.

Notice that in Corollary 4(b) the condition on $J(\mathbf{S}_n)$ is indispensable. Actually it may fail if $J(\mathbf{S}_n)$ are not sublattices. As an example, consider the problem below for all $x_1 \geqslant 0$ and $x_2 \geqslant 0$:

$$f(x_1, x_2) = \text{maximize}\{y_1: y_1 + z_1 = x_1,\ y_2 + z_2 = x_2,\ 0 \leqslant y_1 \leqslant y_2,\ z_1,\ z_2 \geqslant 0\},$$

where the objective function is linear and the set $\mathbf{S}_1 = \{[y_1, y_2]: 0 \leqslant y_1 \leqslant y_2\}$ forms a convex sublattice. Solving the problem gives us $f(x_1, x_2) = \min\{x_1, x_2\}$, which is supermodular as is consistent with Corollary 4(a). However, we cannot apply Corollary 4(b) because $J(\mathbf{S}_1)$ is not a sublattice. In fact, f is not submodular because $f(0, 0) + f(1, 1) = 0 + 1 > 0 + 0 = f(0, 1) + f(1, 0)$.

2.3. Examples

Next we demonstrate the limitations of our results through a few examples. Specifically, these examples show that the conditions in Theorem 1 cannot be discarded in general.

Example 1 (Linear Programs). Consider the linear programming problem

$$f(\mathbf{x}) = \underset{\mathbf{y},\mathbf{z}}{\text{maximize}}\ \{\mathbf{p}'\mathbf{y}: A\mathbf{y} + \mathbf{z} = \mathbf{x},\ \mathbf{y} \geqslant \mathbf{0},\ \mathbf{z} \geqslant \mathbf{0}\},$$

where

$$\mathbf{p} = [-1,\ 0,\ 0,\ -1], \quad A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -2 & -1 & 2 & 1 \end{bmatrix}.$$

One can show that $f(x_1, x_2) = \min\{0, x_1 + x_2, 2x_1 + x_2, 3x_1 + 2x_2\}$, which is not supermodular since

$$f(0,0) + f(1,-1) = 0 + 0 > -2 + 0 = f(0,-1) + f(1,0).$$

This implies that Theorem 1 may fail without the condition $A \geqslant 0$ even for linear programming problems.

Example 2 (Quadratic Programs). Suppose P, Q are $n \times n$ symmetric matrices such that $P + Q$ is negative definite. Define $g(\mathbf{y}, \mathbf{z}) = \frac{1}{2}\mathbf{y}'P\mathbf{y} + \frac{1}{2}\mathbf{z}'Q\mathbf{z}$ and for all $\mathbf{x} \in \Re^n$,

$$f(\mathbf{x}) = \underset{\mathbf{y},\mathbf{z}}{\text{maximize}}\ \{g(\mathbf{y}, \mathbf{z}): \mathbf{y} + \mathbf{z} = \mathbf{x}\}.$$

Calculation shows that $\mathbf{y}(\mathbf{x}) = (P+Q)^{-1}Q\mathbf{x}$ solves the problem, and f is quadratic associated with the Hessian matrix $\nabla^2 f(\mathbf{x}) = P(P+Q)^{-1}Q$. When $n=2$, we further have

$$\nabla^2 f(\mathbf{x}) = P(P+Q)^{-1}Q = \frac{|Q|}{|P+Q|}P + \frac{|P|}{|P+Q|}Q,$$

where $|M|$ denotes the determinant of a matrix M.

We make several observations on Example 2. First, it is a special case of problem (1) when $n=2$. If g is concave and supermodular, then both $-P$ and $-Q$ are positive semidefinite $\mathscr{L}_0$-matrices. From the expression of $\nabla^2 f(\mathbf{x})$, we know that f is concave and supermodular, too. This result is consistent with the statement of Theorem 1; however, it does not seem to follow directly from Theorem 2.7.6 in Topkis (1998) since the constraint set does not form a sublattice. We can reformulate the example to be an unconstrained optimization problem by eliminating the decision variable $\mathbf{z}$ as $f(\mathbf{x}) = \text{maximize}_{\mathbf{y}} g(\mathbf{y}, \mathbf{x}-\mathbf{y})$. Unfortunately, one can verify that in general the objective function $g(\mathbf{y}, \mathbf{x}-\mathbf{y})$ is not supermodular in $(\mathbf{x}, \mathbf{y})$ even when g itself is supermodular.

Second, we cannot weaken the concavity assumption on g in Theorem 1 to component-wise concavity. Consider the instance with P and Q:

$$P = \begin{bmatrix} -9 & 4 \\ 4 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 4 \\ 4 & -9 \end{bmatrix}.$$

In this instance, g is supermodular, component-wise concave but not jointly concave because $g(\lambda\mathbf{y}, \lambda\mathbf{z}) = 22\lambda^2$ when $(\mathbf{y}, \mathbf{z}) = [0, 2, 8, 4]$. However, f is not supermodular. In fact, the Hessian of $f(\mathbf{x})$ is given as

$$\nabla^2 f(\mathbf{x}) = \frac{7}{18}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Third, Theorem 1 does not hold in higher dimensional spaces. Consider the instance with P, Q and the related Hessian matrix of f:

$$P = \begin{bmatrix} -11 & 8 & 0 \\ 8 & -16 & 5 \\ 0 & 5 & -10 \end{bmatrix}, \quad Q = \begin{bmatrix} -7 & 4 & 0 \\ 4 & -14 & 5 \\ 0 & 5 & -9 \end{bmatrix},$$
$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} -4.25 & 2.79 & -0.01 \\ 2.79 & -7.27 & 2.49 \\ -0.01 & 2.49 & -4.73 \end{bmatrix}.$$

In this instance, g is $L^\natural$-concave; however, f is neither supermodular nor submodular.

Finally, the optimal solution may not be monotone or may not have a clear monotonicity pattern even in cases in which we do have monotonicity. To see this, consider the instances with P, Q and their related optimal solutions $\mathbf{y}(\mathbf{x}) = [y_1(x_1, x_2), y_2(x_1, x_2)]$:

$$P = \begin{bmatrix} -6 & 5 \\ 5 & -6 \end{bmatrix}, \quad Q = \begin{bmatrix} -5 & 2 \\ 2 & -1 \end{bmatrix},$$
$$\mathbf{y}(\mathbf{x}) = \frac{1}{28}\begin{bmatrix} 21 & -7 \\ 13 & -3 \end{bmatrix}\mathbf{x};$$
$$P = \begin{bmatrix} -6 & 3 \\ 3 & -4 \end{bmatrix}, \quad Q = \begin{bmatrix} -3 & 2 \\ 2 & -6 \end{bmatrix},$$
$$\mathbf{y}(\mathbf{x}) = \frac{1}{65}\begin{bmatrix} 20 & 10 \\ -3 & 44 \end{bmatrix}\mathbf{x}.$$

In the first instance, $y_i(x_1, x_2)$ are increasing in x_1 and decreasing in x_2 for both $i = 1, 2$. In the second instance, $y_1(x_1, x_2)$ is increasing in both x_1 and x_2, and $y_2(x_1, x_2)$ is increasing in x_2 and decreasing in x_1.

To end this section, we briefly point out several interesting issues that need further research. First, extending our results to higher dimensional spaces is interesting and challenging. As we commented on Example 2, Theorem 1 could fail when the parameter vector $\mathbf{x}$ in (1) lies in a three-dimensional space. Therefore a natural question is under what conditions the preservation of supermodularity in problem (1) holds when we have more than two parameters. The second question is whether we can say anything about the structure of the optimal solution to problem (1). As we notice in the same example, the optimal solution may fail to be monotone in general. It would be interesting to identify conditions under which the optimal solution is monotone.

3. Applications

To demonstrate the power of our theoretical results, we present several applications. Specifically, we illustrate in §3.1 that our preservation properties include several results in the literature as special cases. In §3.2, we use a two-product periodic-review coordinated pricing and inventory control problem with cross-price effects as an illustrative operations model.

Theorem 1 and related results find applications to several other operations models (for more details on these models, we refer to Hu 2011). For example, we use Theorem 1 to simplify the proofs of a two-stage coordinated dynamic pricing and inventory control problem of Yang (2004) and an inventory model analyzed by Chao et al. (2008) in which a self-financing retailer sells a single product with the operational decisions limited by its cash level. The preservation property of $L^\natural$-concavity in Corollary 4 can be employed to address a capacitated production model with remanufacturing by Gong and Chao (2013). Our results are also used to derive monotone comparative statics in a stochastic inventory model with reference price effects in Chen et al. (2012b).

3.1. Results in the Literature

Zipkin (2003) considers the linear programming problem

$$f(\mathbf{x}) = \underset{\mathbf{y}}{\text{maximize}} \ \{\mathbf{p}'\mathbf{y}\colon A\mathbf{y} \leqslant \mathbf{x},\ \mathbf{0} \leqslant \mathbf{y} \leqslant \mathbf{u}\},$$

where $\mathbf{p}$ and $\mathbf{u}$ are two given n-dimensional vectors and A is a $2\times n$ matrix. Using an intricate geometrical argument, he shows that $f(\mathbf{x})$ is supermodular when $\mathbf{x}\geqslant 0$ if $A\geqslant 0$. Notice that this result is immediately implied by Remark 2 of Theorem 1. Zipkin (2003) also proves that if the following maximization problem

$$f(\mathbf{x})=\underset{\mathbf{y}}{\text{maximize}}\ \{\mathbf{p}'\mathbf{y}\colon A\mathbf{y}\leqslant\mathbf{x},\ C\mathbf{y}\leqslant\mathbf{0},\ \mathbf{y}\geqslant\mathbf{0}\}$$

is well defined for all $\mathbf{x}\geqslant\mathbf{0}$ in a two-dimensional space for a matrix C with a proper size, then $f(\mathbf{x})$ is supermodular over $\mathbf{x}\geqslant 0$ for arbitrary matrices A. Unfortunately, our result does not cover this case as it is for a very specific linear program. Indeed, as one can find in Example 5.2.3 of Hu (2011), f may fail to be supermodular for $\mathbf{x}\geqslant\mathbf{0}$ when one replaces the linear objective function $\mathbf{p}'\mathbf{y}$ by a concave and supermodular quadratic function.

Chao et al. (2009) address the following optimization problem:

$$f_0(x,t)=\underset{y}{\text{maximize}}\ \{g_0(y,t)\colon x\leqslant y\leqslant t+x\} \tag{5}$$

when they develop and analyze dynamic capacity expansion models. As the key technical tool in their analysis, Chao et al. (2009) prove that if $g_0(y,t)$ is submodular and concave in x for any given t, then so is $f_0(x,t)$. Again this result follows directly from ours. Indeed, if we let $g(y,t)=g_0(-y,t)$ and $f(x,t)=f_0(-x,t)$, then problem (5) can be rewritten as

$$f(x,t)=\underset{y_1,\,y_2}{\text{maximize}}\ \{g(y_1,t)\colon y_1+y_2=x,\ 0\leqslant y_2\leqslant t\}.$$

By Lemma 1 and Corollary 1, $f(x,t)$ is supermodular and concave in x for any given t. Thus, $f_0(y,t)$ is submodular and concave in x for any given t.

3.2. Coordinated Pricing and Inventory Control Model with Cross-Price Effects

Consider a retailer who decides the ordering quantities and prices of two products over a finite planning horizon with T periods. At the beginning of each period, the retailer observes the initial inventory levels x_i and then simultaneously decides the selling prices $p_i\in[l_i,u_i]$ and the order-up-to-levels y_i for products $i=1,2$. The demand of product i during a period is given by $d_i(p_1,p_2)+\varepsilon_{i,t}$, where $\varepsilon_{i,t}$ is a random variable with expected value 0, and $d_i(p_1,p_2)$ is the expected demand of product i depending on the prices of both products. Denote $\mathbf{x}=[x_1,x_2]$, $\mathbf{y}=[y_1,y_2]$, $\mathbf{p}=[p_1,p_2]$, $\mathbf{l}=[l_1,l_2]$, $\mathbf{u}=[u_1,u_2]$, $\varepsilon_t=[\varepsilon_{1,t},\varepsilon_{2,t}]$, and $\mathbf{d}_t^{\varepsilon}=\mathbf{d}(\mathbf{p})+\varepsilon_t$. We assume that random vectors ε_t are independent across time, there is no lead time for delivery, unsatisfied demand is backlogged, and unused inventory is carried over to the next period.

As is common in the literature, the expected demand $\mathbf{d}(\mathbf{p})$ is assumed to be linear as $\mathbf{d}(\mathbf{p})=\mathbf{b}-A\mathbf{p}$, where $\mathbf{b}\geqslant\mathbf{0}$ and $A=[a_{i,j}]$ denotes the price sensitivity coefficient matrix. For product i, coefficients $a_{i,i}$ and $a_{i,j}$ denote its own price sensitivity and the cross-price sensitivity to the other product j ($j\neq i$), respectively. The coefficient $a_{i,i}$ is assumed to be nonnegative, i.e., the demand of a product is decreasing in its own price. Depending on the nature of products, we consider two cases: (a) the two products are *complements*, i.e., an increase in the price of one product will decrease the demanded amount of the other product, or equivalently $a_{i,j}\geqslant 0$; (b) the two products are *substitutes*, i.e., an increase in the price of one product will increase the demanded amount of the other product, or equivalently $a_{i,j}\leqslant 0$. In both situations, we assume that the price change of one product has a stronger effect on its own demand than on the other product's demand, i.e., $a_{i,i}>|a_{i,j}|$. Note that A is positive definite under the above assumptions.

It will be convenient to use the expected demands instead of prices as the decision variables. Denote the expected demand vector as $\mathbf{d}$ and the corresponding price vector as $\mathbf{p}(\mathbf{d})=A^{-1}(\mathbf{b}-\mathbf{d})$. Then the expected one-period revenue is $\mathbf{d}'\mathbf{p}(\mathbf{d})$. The ordering cost is proportional to the ordering quantity specified by $\mathbf{c}'\mathbf{z}$ for an ordering quantity vector $\mathbf{z}=[z_1,z_2]$, where $\mathbf{z}$ is nonnegative and bounded above by a capacity vector $\mathbf{k}$. For an amount $\mathbf{x}=[x_1,x_2]$ of inventory carried over from one period to the next, an inventory holding and backorder cost $h(\mathbf{x})=h_1(x_1)+h_2(x_2)$ is incurred, where $h_i(x_i)$ represents the inventory holding cost when $x_i>0$ and the shortage penalty cost when $x_i<0$. We assume that $h_i(x)$ is convex for $i=1,2$. It is worth mentioning that results in this section remain valid when h and system inputs $\mathbf{c}$, $\mathbf{l}$, $\mathbf{u}$, $\mathbf{b}$, $\mathbf{k}$ are time dependent. For simplicity, we focus on the time-independent case.

The objective of the retailer is to find an ordering and pricing policy so as to maximize its expected total profit over the planning horizon. Let $v_t(\mathbf{x})$ be the profit-to-go function of period t starting with an inventory level $\mathbf{x}$, where we assume that $v_{T+1}(\mathbf{x})=0$, i.e., there is no value or cost for either unused inventory or backlogged demand at the end of planning horizon. The dynamic program can be formulated as

$$\begin{aligned} v_t(\mathbf{x})=\underset{\mathbf{y},\,\mathbf{d}}{\text{maximize}}\ &\{\mathbf{d}'\mathbf{p}(\mathbf{d})-\mathbf{c}'\mathbf{z}+g_t(\mathbf{y}-\mathbf{d})\}\\ \text{subject to}\ &\mathbf{y}=\mathbf{x}+\mathbf{z},\quad \mathbf{0}\leqslant\mathbf{z}\leqslant\mathbf{k},\\ &\mathbf{l}\leqslant A^{-1}(\mathbf{b}-\mathbf{d})\leqslant\mathbf{u}, \end{aligned} \tag{6}$$

where $g_t(\mathbf{x})=\mathbb{E}[v_{t+1}(\mathbf{x}-\varepsilon_t)-h(\mathbf{x}-\varepsilon_t)]$, and $\mathbb{E}$ is the expectation operator corresponding to ε_t. Let $\tilde{\mathbf{l}}=A^{-1}\mathbf{b}-\mathbf{u}$, $\tilde{\mathbf{u}}=A^{-1}\mathbf{b}-\mathbf{l}$, $\bar{\mathbf{z}}=-\mathbf{z}$ and P be the identity matrix. We can equivalently formulate problem (6) as

$$\begin{aligned} v_t(P\mathbf{x})=\underset{\mathbf{y}}{\text{maximize}}\ &\{f_t(P\mathbf{y})-\mathbf{c}'P\mathbf{y}\}+\mathbf{c}'P\mathbf{x},\\ \text{subject to}\ &\mathbf{x}=\mathbf{y}+\bar{\mathbf{z}},\quad -\mathbf{k}\leqslant P\bar{\mathbf{z}}\leqslant\mathbf{0}, \end{aligned} \tag{7a}$$

$$\begin{aligned} f_t(P\mathbf{y})=\underset{\mathbf{d}}{\text{maximize}}\ &\{\mathbf{d}'P'\mathbf{p}(P\mathbf{d})+g_t(P\tilde{\mathbf{x}})\},\\ \text{subject to}\ &\mathbf{y}=\mathbf{d}+\tilde{\mathbf{x}},\quad \tilde{\mathbf{l}}\leqslant A^{-1}P\mathbf{d}\leqslant\tilde{\mathbf{u}}, \end{aligned} \tag{7b}$$

where $f_t(\mathbf{y})$ denotes the maximal expected total profit after raising inventory level up to $\mathbf{y}$.

Let $\mathbf{y}(\mathbf{x}) = [y_1(x_1, x_2), y_2(x_1, x_2)]$ be the optimal order-up-to inventory level solving problem (7a) (if there are multiple optimal solutions, then let $\mathbf{y}(\mathbf{x})$ be the largest one in terms of lexicographic order). We have the following proposition on $\mathbf{y}(\mathbf{x})$ and the profit-to-go functions v_t and f_t.

Proposition 1. *The functions $v_t(\mathbf{x})$ and $f_t(\mathbf{y})$ are concave in each period. Furthermore,*

(a) *in the complementary product case, $v_t(\mathbf{x})$ and $f_t(\mathbf{y})$ are supermodular, $v_t(A\mathbf{x})$ and $f_t(A\mathbf{y})$ are submodular, and $y_i(x_1, x_2)$ is increasing in both x_1 and x_2 for $i = 1, 2$;*

(b) *in the substitutable product case, $v_t(\mathbf{x})$ and $f_t(\mathbf{y})$ are submodular, $v_t(A\mathbf{x})$ and $f_t(A\mathbf{y})$ are supermodular, and $y_i(x_1, x_2)$ is increasing in x_i and decreasing in x_j for i, $j = 1, 2$ and $i \neq j$.*

Proof. The assumptions on the matrix A immediately imply the concavity of $\mathbf{d}'\mathbf{p}(\mathbf{d})$. Moreover, it is not hard to show that $\mathbf{d}'\mathbf{p}(\mathbf{d})$ is supermodular (submodular) and $\mathbf{d}'A'\mathbf{p}(A\mathbf{d})$ is submodular (supermodular) in the complementary (substitutable) product case.

Because $h(\mathbf{y})$ is convex, it is straightforward to verify inductively the concavity of v_t and f_t, as well as the existence of $\mathbf{y}(\mathbf{x})$, by Theorem 5.4 in Rockafellar (1970). We next verify the remaining part by selecting proper matrices P in problem (7). For this purpose, let $\mathscr{L}$ be the collection of all 2×2 matrices $L = [\ell_{i,j}]$ such that $\ell_{i,1}\ell_{i,2} \leqslant 0$ for $i = 1, 2$. Notice that $P \in \mathscr{L}$ and $A^{-1}P \in \mathscr{L}$ imply that the sets $\{\mathbf{z}: \mathbf{0} \leqslant P\mathbf{z} \leqslant \mathbf{k}\}$ and $\{\mathbf{d}: \tilde{\mathbf{l}} \leqslant A^{-1}P\mathbf{d} \leqslant \tilde{\mathbf{u}}\}$ are sublattices (Example 2.2.7, Topkis 1998). Moreover, $-h(P\mathbf{x})$ is supermodular in $\mathbf{x}$ by Lemma 2.6.2 in Topkis (1998).

(a) Let P be the identity matrix and AJ in problem (7), respectively. It is straightforward to show that P, $A^{-1}P \in \mathscr{L}$ and $\mathbf{d}'P'\mathbf{p}(P\mathbf{d})$ is supermodular. Since $v_{T+1} = 0$, we can employ Theorem 1 to prove inductively that in all periods $v_t(P\mathbf{x})$ and $f_t(P\mathbf{y})$ are supermodular. That is, $v_t(\mathbf{x})$, $f_t(\mathbf{y})$, $v_t(AJ\mathbf{x})$, and $f_t(AJ\mathbf{y})$ are supermodular. Hence $v_t(A\mathbf{x})$ and $f_t(A\mathbf{y})$ are submodular by Lemma 1. In addition, $\mathbf{y}(\mathbf{x})$ is increasing in $\mathbf{x}$ by Theorem 2.8.1 in Topkis (1998).

(b) Let $P = J$ and A in problem (7), respectively. Similarly to part (a) of this proof, we can prove inductively that $v_t(J\mathbf{x})$, $f_t(J\mathbf{y})$, $v_t(A\mathbf{x})$, and $f_t(A\mathbf{y})$ are supermodular by Theorem 1, and $v_t(\mathbf{x})$ and $f_t(\mathbf{y})$ are submodular by Lemma 1. In addition, notice that the optimal solution to problem (7) for $P = J$ can be expressed as $J\mathbf{y}(J\mathbf{x}) = [y_1(x_1, -x_2), -y_2(x_1, -x_2)]$. Since $J\mathbf{y}(J\mathbf{x})$ is increasing in $\mathbf{x}$ by Theorem 2.8.1 in Topkis (1998), we have that $y_i(x_1, x_2)$ is increasing in x_i and decreasing in x_j for $i \neq j$. $\square$

From the monotonicity results on $\mathbf{y}(\mathbf{x})$, it is not surprising to see that the optimal order-up-to-levels are increasing in the initial inventory levels of both products in the complementary product case, and is increasing in its own initial inventory level while decreasing in the other product's initial inventory level in the substitutable product case. Moreover, the results on $v_t(A\mathbf{x})$ imply that the function $v_t(\mathbf{b} - A\mathbf{p})$ in $\mathbf{p}$ is submodular (supermodular) in the complementary (substitutable) product case. Notice that if one ignores the uncertainty term in the demand model, for a given initial inventory level $\mathbf{x}$ in some period, the vector $\mathbf{p}$ satisfying $\mathbf{b} - A\mathbf{p} = \mathbf{x}$ can be regarded as the clearance prices to deplete the on-hand inventory. Our results illustrate that the clearance prices are substitutable (complementary) in the complementary (substitutable) product case in the sense that the higher the clearance price of one product, the smaller (the larger) the marginal profit of the clearance price of the other product.

A simpler version of our model was analyzed by Zhu and Thonemann (2009), which deals with only the substitutable product case without the constraint $\mathbf{z} \leqslant \mathbf{k}$. Song and Xue (2007) consider a more general setting with more than two substitutable products and derive structural results of the optimal order-up-to levels similar to Zhu and Thonemann (2009) for the two-product case. Ceryan et al. (2013) extend Zhu and Thonemann (2009) by introducing the constraint $\mathbf{z} \leqslant \mathbf{k}$ and an additional resource capacity constraint $z_1 + z_2 \leqslant k_0$. Notice that for the model of Ceryan et al., we can prove Proposition 1(b) by using the same argument (however in the complementary product case, Proposition 1(a) may fail if we impose the constraint $z_1 + z_2 \leqslant k_0$).

Compared with these three papers, we deal with both the complementary product case and the substitutable product case in a unified framework. We are not aware of any paper in the literature that analyzes integrated inventory and pricing models with complementary products. Moreover, we develop theoretical results on $v_t(A\mathbf{x})$ in the substitutable case that are not available in the literature.

Even though these three papers present results on v_t that are almost identical to part of Proposition 1, our approach is significantly simpler. In fact, all these papers establish the submodularity of $v_t(\mathbf{x})$ recursively by analyzing the first-order optimality condition (the Karush-Kuhn-Tucker (KKT) conditions) of problem (6). Their approaches are lengthy and require some additional technical conditions on the objective functions (e.g., smoothness almost everywhere). Moreover, all three papers ignore the bound constraints $\mathbf{l} \leqslant \mathbf{p} \leqslant \mathbf{u}$ on prices in their analysis, though such constraints are imposed and are particularly important for linear demand models. For example, Zhu and Thonemann (2009) discuss the range of optimal prices *only after* deriving their structural results. Song and Xue (2007) mention that the price vector $\mathbf{p}$ belongs to some compact set in their introduction section but does not explicitly deal with it in their analysis.

To end this section, we provide some comments on when our results should be used in applications. The methodology and concepts summarized in Topkis (1998) provide the key mathematical structures that permit monotone comparative statics. Among those concepts, supermodularity is most convenient and in most cases essential to derive comparative statics properties in dynamic settings. Facing a parametric optimization problem, it is probably a good idea to start with KKT conditions by ignoring some complicating constraints and the differentiability requirement to build some basic understanding of possible monotone comparative statics properties. If one believes that such properties exist, it is advisable to first check whether Theorem 2.7.6 in Topkis (1998) is applicable to establish a rigorous proof. If it does not work, one can try our results, possibly carrying out simple variable transformations. If neither Theorem 2.7.6 nor our results work, one may then work with KKT conditions, which often involve tedious analysis (in particular when inequality constraints are present) and require restrictive conditions such as smoothness of related functions. In this case, it is very likely that the problem possesses special properties that have to be exploited.

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