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# A note on quadratic forms

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**Abstract**. We extend an interesting theorem of Yuan [12] for two quadratic forms to three matrices. Let $C_1, C_2, C_3$ be three symmetric matrices in $\Re^{n\times n}$, if $\max\{x^T C_1 x, x^T C_2 x, x^T C_3 x\} \geq 0$ for all $x \in \Re^n$, it is proved that there exist $t_i \geq 0$ $(i = 1, 2, 3)$ such that $\sum_{i=1}^3 t_i = 1$ and $\sum_{i=1}^3 t_i C_i$ has at most one negative eigenvalue.

**Key words.** quadratic forms – convex combination – matrix perturbation

## 1. Introduction

A very interesting result about two quadratic forms was given by Yuan [12]. It reads as follows:

**Theorem 1.** *Let $C_1, C_2 \in \Re^{n\times n}$ be two symmetric matrices and $A$ and $B$ be two closed sets in $\Re^n$ such that*

$$A \cup B = \Re^n \,. \tag{1}$$

*If we have*

$$x^T C_1 x \geq 0, x \in A, \quad x^T C_2 x \geq 0, x \in B\,, \tag{2}$$

*then there exists a $t \in [0, 1]$ such that the matrix*

$$tC_1 + (1-t)C_2 \tag{3}$$

*is positive semi-definite.*

The above theorem is very useful in the studying of optimal conditions for the two-ball trust region subproblem:

$$\min \quad g^T d + \frac{1}{2} d^T B d \tag{4}$$

$$\text{s.t.} \quad \| c + A^T d \| \leq \xi \tag{5}$$

$$\| d \|_2 \leq \Delta \,, \tag{6}$$

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which is an important subproblem of some trust region algorithms for nonlinear constrained optimization. Problem (4)–(6) was first given by Celis, Dennis and Tapia [2], and it is also called CDT problem. More details about the CDT problem and its applications can be found in [2], [4], [8], [12] and [13].

Crouzeix et al. [3] pointed out that Yuan's result is actually an alternate theorem. They also extended Theorem 1 to a locally convex topological linear space. For any symmetric matrices $C_1$ and $C_2$, Theorem 1 shows that

$$\max\{x^T C_1 x, x^T C_2 x\} \geq 0 \tag{7}$$

for all $x$, if and only if there exists a convex linear combination of $C_1$ and $C_2$ which is positiv semi-definite. The main object of this paper is to generalize this result from two matrices to three matrices. The two matrix case was also found very useful for analyzing global optimal conditions for the minimization of a quadratic function subject to one quadratic constraint (for example, see Moré [6]). As discussed by Peng and Yuan [7], the three matrix cas is closely related to the following problem

$$\min \quad x^T C_1 x \tag{8}$$

$$\text{s.t.} \quad x^T C_2 x \leq 0\,, \tag{9}$$

$$x^T C_3 x \leq 0\,. \tag{10}$$

An example was given by Crouzeix et al. [3] that Theorem 1 cannot be extended to more than two matrices and copositive matrices in a simple way. If $C_i$ $(i = 1, 2, 3)$ are symmetric matrices in $\Re^{n\times n}$, Peng and Yuan [7] showed that when 0 solves (8)–(10), there exists $(\alpha, \beta) \in \Re^2$, such that $C_1 + \alpha C_2 + \beta C_3$ has at most two negative eigenvalues. If $\alpha C_2 + \beta C_3$ is indefinite for all $(\alpha, \beta) \in \Re^2$ $((\alpha, \beta) \neq 0)$ and if the least eigenvalue of $C_1 + \alpha_0 C_2 + \beta_0 C_3$ is negative, where $(\alpha_0, \beta_0)$ maximizes the least eigenvalue of $C_1 + \alpha C_2 + \beta C_3$, it is shown by Peng and Yuan [7] that $C_1 + \alpha_0 C_2 + \beta_0 C_3$ has at most two negative eigenvalues and $\alpha_0, \beta_0$ must be greater than 0.

In this paper, we first give a different proof for Theorem 1. Then we present a similar result for quadratic forms with special structure. Finally we present a result about three quadratic forms which is stronger than the results in [7].

Throughout the paper, we use $\mathrm{co}_{i=1}^{r}(C_i)$ to represent the convex linear combination of matrices $C_i$, i.e. $\mathrm{co}_{i=1}^{r}(C_i) = \{\sum_{i=1}^{r} t_i C_i \,|\, \sum_{i=1}^{r} t_i = 1,\ t_i \geq 0\}$. $C_1$, $C_2$, $C_3$ and $C$ denote symmetric matrices in $\Re^{n\times n}$, and $\Re^n_+$ denotes the nonnegative orthant of $\Re^n$. We write $C \geq 0$ if $C$ is positive semi-definite, and $C > 0$ if $C$ is positive definite.

## 2. Results and their proofs

First we introduce an interesting result of Brickman [1] which will be used for our new proof of Theorem 1 and for establishing Lemma 3 which is needed in the proof of Theorem 3.

**Lemma 1.** *Assume that $C_1, C_2$ are two symmetric matrices in $\Re^{n\times n}$. Define*

$$\tilde{R}(C_1, C_2) = \left\{ (x^T C_1 x, x^T C_2 x) | x \in \Re^n \right\}\,,$$

$$R(C_1, C_2) = \left\{ (x^T C_1 x, x^T C_2 x) | x \in \Re^n, \| \, x \, \| = 1 \right\}\,.$$

*Then* $\tilde{R}(C_1, C_2)$ *is a convex cone in* $\Re^2$. *If* $n \neq 2$, *then* $R(C_1, C_2)$ *is a convex set in* $\Re^2$.

It should be noted that the condition $n \neq 2$ is indispensable. A simple counter example can be found in [1].

Unlike the detailed analysis of the least eigenvalue of the convex linear combination of $C_1$ and $C_2$ used in [12], we give a simpler proof for Theorem 1 (for $n \neq 2$) by applying Lemma 1 and the separating theorem.

*Proof.* Denote $\Re^2_{--} = \{(x_1, x_2) | x_1 < 0, x_2 < 0\}$. Since (1) and (2) is equivalent to $\max\{x^T C_1 x, x^T C_2 x\} \geq 0$ for every $x \in \Re^n$, then

$$\Re^2_{--} \cap \tilde{R}(C_1, C_2) = \emptyset\,, \tag{11}$$

where $\tilde{R}(C_1, C_2)$ is defined in Lemma 1. By Lemma 1 and the separating theorem for convex cones, there exists $(\alpha, \beta) \in \Re^2$ $((\alpha, \beta) \neq 0)$ such that

$$\inf\{\alpha x^T C_1 x + \beta x^T C_2 x\} \geq 0 \geq \sup\{\alpha x_1 + \beta x_2\} \tag{12}$$

for every $x \in \Re^n$ and $(x_1, x_2) \in \Re^2_{--}$. It is obvious that $\alpha \geq 0, \beta \geq 0$. From (12) we have $\alpha C_1 + \beta C_2 \geq 0$.

□

It is worthwile to note that Theorem 1 is also true when $x$ is restricted in a subspace in $\Re^n$.

Crouzeix et al. [3] pointed out that Theorem 1 cannot be parallely extended to three matrices and they gave a nice counter example. In the following we give another example:

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -1 & -0.5 \\ -0.5 & 0 \end{pmatrix}. \tag{13}$$

It is easy to see that $\max\{x^T C_1 x, x^T C_2 x, x^T C_3 x\} \geq 0$. But for any convex combination $C = t_1 C_1 + t_2 C_2 + t_3 C_3$, we have $c_{22} < 0$, if $t_1 > 0$, and $c_{11} = -1$, if $t_1 = 0$, where $c_{ij}$ denotes the $i$-th row, $j$-th column element of matrix $C$. Therefore for every $C \in \text{co}(C_1, C_2, C_3)$, $C$ cannot be positiv semi-definite.

For the special case when all matrices are diagonal, we can easily generalize Theorem 1.

**Theorem 2.** *Let* $r$ *be any positive integer. If* $C_i$ $(i = 1, 2, \ldots, r)$ *are all diagonal matrices in* $\Re^{n \times n}$ *and* $\max_{1 \leq i \leq r}\{x^T C_i x\} \geq 0$ *for every* $x \in \Re^n$, *then there exists a* $C \in \text{co}_{i=1}^r(C_i)$ *such that* $C$ *is positive semi-definite.*

*Proof.* Let $C_i = \text{diag}(c_{11}^{(i)}, \ldots, c_{nn}^{(i)})$. It follows from our assumptions that:

$$\max_{1 \leq i \leq r} \left\{ \sum_{j=1}^{n} c_{jj}^{(i)} x_j^2 \right\} \geq 0\,, \tag{14}$$

for $x = (x_1, \ldots, x_n)^T \in \Re^n$. That is to say the linear inequality system:

$$\sum_{j=1}^{n} c_{jj}^{(i)} y_j < 0, \quad i = 1, \ldots, r \tag{15}$$

$$y \geq 0. \tag{16}$$

has no solution. By Theorem 21.1 in [9] we can show that there exists a nonzero $(\alpha_1, \ldots, \alpha_r) \in \Re_+^r$ satisfying

$$\sum_{i=1}^{r} \alpha_i c_{jj}^{(i)} \geq 0, \quad j = 1, \ldots, n, \tag{17}$$

i.e.

$$\sum_{i=1}^{r} \alpha_i C_i \geq 0. \tag{18}$$

Then it follows what we need.

□

Though Theorem 1 cannot be parallely extended to three matrices, we will show that there exists a $C \in \text{co}(C_1, C_2, C_3)$ such that $C$ has at most one negative eigenvalue if

$$\max \left\{ x^T C_1 x, x^T C_2 x, x^T C_3 x \right\} \geq 0, \ \forall x \in \Re^n. \tag{19}$$

If

$$\max \left\{ x^T C_2 x, x^T C_3 x \right\} \geq 0, \ \forall x \in \Re^n, \tag{20}$$

it follows from Theorem 1 that there exists $C \in \text{co}(C_2, C_3)$ such that $C$ is positive semi-definite. Because $\text{co}(C_2, C_3) \subset \text{co}(C_1, C_2, C_3)$, we only need to study under the assumption that (20) fails. This assumption and (19) imply that the following problem

$$\min \ x^T C_1 x \tag{21}$$

$$\text{s.t.} \ x^T C_2 x < 0, \tag{22}$$

$$x^T C_3 x < 0, \tag{23}$$

has optimal objective value 0. It should be noted that (19) does not guarantee a zero solution for problem (8)–(10). For example, if $C_i$ $(i = 1, 2, 3)$ are given by (13). We can see that (19) holds, but problem (8)–(10) is unbounded. First we establish a lemma which shows that we can focus our attentions to problem (8)–(10).

**Lemma 2.** *If (19) holds and $x^T C_i x \leq 0$ for $i = 1, 2, 3$, then either there is a convex linear combination of $C_1$, $C_2$ and $C_3$ which has at most one negative eigenvalue or $x^T C_i x = 0$ for all $i$.*

*Proof.* If the lemma is not true, there exists $x_0 \in \Re^n$ such that

$$x_0^T C_1 x_0 < 0\,, \tag{24}$$
$$x_0^T C_2 x_0 \le 0\,, \tag{25}$$
$$x_0^T C_3 x_0 \le 0\,. \tag{26}$$

It follows from (19) that either (25) or (26) holds as an equality. If $x_0^T C_3 x_0 < 0$ then $x_0^T C_2 x_0 = 0$. Therefore, $x_0$ is a local minimizer of $x^T C_2 x$, as (19) indicates $x^T C_2 x \ge 0$ for all $x$ close to $x_0$. Thus $C_2 x_0 = 0$ and $(x_0 + td)^T C_2 (x_0 + td) = t^2 d^T C_2 d$ for all $d \in \Re^n$, which implies that $C_2$ is positive semi-definite. This contradicts our assumptions. Similarly it is impossible to have $x_0^T C_2 x_0 < 0$. Thus it follows for (24)–(26) that

$$x_0^T C_1 x_0 < 0\,,\; x_0^T C_2 x_0 = x_0^T C_3 x_0 = 0\,. \tag{27}$$

If $C_2 x_0$ and $C_3 x_0$ are linearly independent, then there exists an $x$ sufficiently close to $x_0$ such that

$$x^T C_1 x < 0\,,\; x^T C_2 x < 0\,,\; x^T C_3 x < 0\,, \tag{28}$$

which contradicts (19). Thus $C_2 x_0$ and $C_3 x_0$ must be linearly dependent, which shows that there exists an $n-1$ dimensional subspace $S_{n-1}$ such that $C_2 x_0 \perp S_{n-1}$ and $C_3 x_0 \perp S_{n-1}$. Therefore for all $d \in S_{n-1}$ we have

$$(x_0 + td)^T C_2 (x_0 + td) = t^2 d^T C_2 d\,, \tag{29}$$
$$(x_0 + td)^T C_3 (x_0 + td) = t^2 d^T C_3 d\,. \tag{30}$$

(19), (27), (29) and (30) imply that

$$\max\{d^T C_2 d, d^T C_3 d\} \ge 0,\; \forall d \in S_{n-1}\,. \tag{31}$$

Applying Theorem 1 in subspace $S_{n-1}$ yields that there exists a matrix in $\mathrm{co}(C_2, C_3)$ having at most one negative eigenvalue, which contradicts the assumption. The contradictions prove the lemma.

□

The above lemma implies that if (19) holds and if every matrix in $\mathrm{co}(C_1, C_2, C_3)$ has more than one negative eigenvalue, then

$$\max\left\{x^T C_1 x, x^T C_2 x, x^T C_3 x\right\} \le 0 \tag{32}$$

is equivalent to

$$x^T C_1 x = x^T C_2 x = x^T C_3 x = 0\,. \tag{33}$$

Therefore, we only need to study the case when 0 solves problem (8)–(10).

We will need the following Lemmas 3, 4, 6, 7 and 8 to obtain Theorem 3 which shows that there exists a matrix in $\mathrm{co}(C_1, C_2, C_3)$ which has at most one negative eigenvalue, if 0 solves problem (8)–(10).

**Lemma 3.** *Assume that $C_1, C_2 \in \Re^{n\times n}$ are symmetric and $n \geq 3$, then there exist $\alpha, \beta \in \Re$ satisfying $\alpha^2 + \beta^2 > 0$ such that $\alpha C_1 + \beta C_2$ is positive definite if and only if $N_{C_1} \cap N_{C_2} = \{0\}$, where $N_Q = \{x | x^T Qx = 0\}$.*

*Proof.* By the definition of $R(C_1, C_2)$ in Lemma 1, $N_{C_1} \cap N_{C_2} = \{0\}$ if and only if $0 \notin R(C_1, C_2)$. Because $n \geq 3$, $R(C_1, C_2)$ is a convex set in $\Re^2$ by Lemma 1. Moreover, it is obvious that $R(C_1, C_2)$ is closed. It follows from the separating theorem for closed convex sets that $0 \notin R(C_1, C_2)$ if and only if there exists $(\alpha, \beta) \in \Re^2$, such that $\alpha x^T C_1 x + \beta x^T C_2 x > 0$, for every $x \in \Re^n$, i.e. $\alpha C_1 + \beta C_2$ is positive definite in $\Re^n$. □

*Remark 1.* It is easy to see the above lemma is true for $n = 1$. But it fails for $n = 2$. Let

$$C_1 = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{34}$$

it is easy to verify that $N_{C_1} \cap N_{C_2} = \{0\}$. While for any $(\alpha, \beta) \in \Re^2$, $\alpha C_1 + \beta C_2$ cannot be positive definite. Moreover, Lemma 3 is also true if $x$ is restricted in a subspace in $\Re^n$ as long as the dimension of the subspace is not 2.

Our analysis also depends on the following theorem, which is given by Moré [6].

**Lemma 4.** *If $C_1$ and $C_2$ are two symmetric matrices in $\Re^{n\times n}$, then $C_1 + \alpha C_2$ is positive definite in $\Re^n$ for some $\alpha \in \Re$ if and only if*

$$x^T C_1 x > 0, \ \forall\, 0 \neq x \in \Re^n, \ x^T C_2 x = 0. \tag{35}$$

It should be noted that the lemma does not depend on the dimension of the linear space and it is true if $x$ is restricted in any subspace in $\Re^n$.

In the next three lemmas, we consider the problem of eigenvalue changes when a matrix is perturbed. Lemmas 6 and 7 will be used in the proof of Theorem 3. We give Lemma 5 and its proof because one can use the same techniques to prove Lemmas 6 and 7 whos proofs are omitted. Lemma 5 shows that if a given eigenvalue cannot be increased for all perturbation along a certain direction, then the peturbation matrix cannot be positive definite in the subspace spanned by eigenvectors related to the given eigenvalue.

**Lemma 5.** *Assume that $\lambda_k(\alpha)$ is the k-th largest eigenvalue of $C_1 + \alpha C_2$ and $\lambda_k(\alpha)$ has a local minimum at $\alpha_0$. Then $C_2$ is neither positive definite nor negative definite on any subspace $\mathcal{X} \supset \mathcal{X}_k$, where $C_1$, $C_2$ are two symmetric matrices in $\Re^{n\times n}$, $\mathcal{X}_k$ denotes the subspace spanned by the eigenvectors of $C_1 + \alpha_0 C_2$ related to $\lambda_k(\alpha_0)$.*

*Proof.* If the lemma is not true, without loss of generality, we assume:

$$C_1 + \alpha_0 C_2 = \mathrm{diag}(D_1, \lambda_k(\alpha_0) I_r, D_2), \tag{36}$$

$$D_1 = \mathrm{diag}(d_1, \ldots, d_{r_1}), \tag{37}$$

$$D_2 = \mathrm{diag}(d_{r_2+1}, \ldots, d_n), \tag{38}$$

$$d_n \leq \ldots \leq d_{r_2+1} < \lambda_k(\alpha_0) < d_{r_1} \leq \ldots \leq d_1, \tag{39}$$

$$r_1 + r = r_2, \ r_2 \geq k, \tag{40}$$

and $C_2$ is positive definite in $\mathcal{X}_k$. Thus there exists a $\varepsilon > 0$ such that

$$x^T C_2 x > \varepsilon\,, \ \forall x \in \mathcal{X}_{k,1}\,, \tag{41}$$

where $\mathcal{X}_{k,1} = \{x | x \in \mathcal{X}_k,\ \| x \| = 1\}$. Hence there exists an open neighborhood of $\mathcal{X}_{k,1}$, say $U$, such that

$$x^T C_2 x > \varepsilon\,, \ \forall x \in U\,. \tag{42}$$

It is easy to show that there exists a $\varepsilon_0 > 0$ such that

$$x^T (C_1 + \alpha_0 C_2) x \geq \lambda_k(\alpha_0) + \varepsilon_0\,, \tag{43}$$

for all $x \in \text{span}\{e_{r_2+1}, \ldots, e_n\}^{\perp} \setminus U$ satisfying $\| x \| = 1$. Because

$$n - k + 1 > \dim(\text{span}\{e_{r_2+1}, \ldots, e_n\})\,, \tag{44}$$

then for any subspace $\mathcal{X}$ with dimension $n - k + 1$, there exists

$$v \in \mathcal{X}\,,\ \| v \| = 1\,,\ v \in \text{span}\{e_{r_2+1}, \ldots, e_n\}^{\perp}\,. \tag{45}$$

Therefore by the famous Courant–Fischer theorem (see [10]), there exists a $v \in \Re^n$ satisfying (45) such that

$$\begin{aligned}\lambda_k(\alpha_0 + \alpha) &= \min_{\dim(\mathcal{X})=n-k+1}\ \max_{x\in\mathcal{X}, \|x\|_2=1} x^T (C_1 + \alpha_0 C_2) x + \alpha x^T C_2 x \\ &\geq v^T (C_1 + \alpha_0 C_2) v + \alpha v^T C_2 v\,. \end{aligned} \tag{46}$$

If $v \in U$, (45) and (42) give that

$$v^T (C_1 + \alpha_0 C_2) v \geq \lambda_k(\alpha_0)\,,\ v^T C_2 v > \varepsilon\,. \tag{47}$$

If $v \notin U$, it follows from (43) that

$$v^T (C_1 + (\alpha_0 + \alpha) C_2) v \geq \lambda_k(\alpha_0) + \varepsilon_0/2\,. \tag{48}$$

for all sufficiently small $\alpha > 0$. Now (46)–(48) imply that $\lambda_k(\alpha_0 + \alpha) > \lambda_k(\alpha_0)$ for sufficiently small $\alpha > 0$, which contradicts our assumption. Thus we complete our proof.

□

*Remark 2.* It is easy to see from our proof that if $\lambda_k(\alpha_0)$ is a one sided local maximum, $\lambda_k(\alpha_0) \geq \lambda_k(\alpha)$, for $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$ $(\varepsilon > 0)$, then $C_2$ cannot be positive definite in $\mathcal{X}$. Similarly one can prove that if $\lambda_k(\alpha_0) \geq \lambda_k(\alpha)$, for $\alpha \in (\alpha_0 - \varepsilon, \alpha_0)$ $(\varepsilon > 0)$, then $C_2$ cannot be negative definite in $\mathcal{X}$.

The following two lemmas are generalizations of the above lemma from one parameter to two parameters. Their proofs are more complicated and are omitted because basically they follow the same approach of the proof of Lemma 5.

**Lemma 6.** *Assume $\lambda_k(\alpha, \beta)$ is the k-th largest eigenvalue of $C_1 + \alpha C_2 + \beta C_3$ and $\lambda_k(\alpha_0, \beta_0)$ is a local maximum. Then $\alpha C_2 + \beta C_3$ cannot be positive definite in subspace $\mathcal{X} \supset \mathcal{X}_k$ for $(\alpha, \beta) \in \Re^2$, where $C_1, C_2, C_3$ are symmetric matrices in $\Re^{n\times n}$, $\mathcal{X}_k$ denotes the subspace spanned by the eigenvectors of $C_1 + \alpha_0 C_2 + \beta_0 C_3$ related to $\lambda_k(\alpha_0, \beta_0)$.*

*Remark 3.* Similarly to Remark 2, for $C_1 + \alpha C_2 + \beta C_3$, if $\lambda_k(\alpha_0, \beta_0) \geq \lambda_k(\alpha, \beta)$ for all $(\alpha, \beta) \in N_\varepsilon(\alpha_0, \beta_0)$, and $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, then we can show that for $(\alpha, \beta) \in \Re^2_+$, $\alpha C_2 + \beta C_3$ cannot be positive definite in $\mathcal{X}$. Similarly if $\lambda_k(\alpha_0, \beta_0) \geq \lambda_k(\alpha, \beta)$ for $(\alpha, \beta) \in N_\varepsilon(\alpha_0, \beta_0)$, and $\beta \geq \beta_0$, then $\alpha C_2 + \beta C_3$ cannot be positive definite in $\mathcal{X}$. Where $N_\varepsilon(\alpha_0, \beta_0)$ denotes a ball in $\Re^2$ centered at $(\alpha_0, \beta_0)$ with a radius $\varepsilon$.

**Lemma 7.** *Assume that $\lambda_k$ is the k-th largest eigenvalue of the symmetric matrix $C_1$ and $\lambda_i = \lambda_k$ $(i = r_1 + 1, \ldots, r_2)$, where $r_2 > r_1$ are two nonnegative integers. Let $\mathcal{X}_k$ be the subspace spanned by the eigenvectors of $C_1$ related to $\lambda_k$. Denote $\lambda_k(\alpha)$ the k-th eigenvalue of matrix $C_1 + \alpha C_2$. Then we have that*

*(1) If $C_2$ is indefinite in $\mathcal{X}_k$, then $\lambda_{r_1+1}(\alpha) > \lambda_{r_1+1}$, $\lambda_{r_2}(\alpha) < \lambda_{r_2}$ when $\alpha$ is small enough and $\alpha > 0$;*
*(2) If $C_2$ is positive semi-definite in $\mathcal{X}_k$ and $C_2 \neq 0$ in $\mathcal{X}_k$, then $\lambda_i(\alpha) \geq \lambda_i$ $(i = r_1 + 1, \ldots, r_2)$ when $\alpha$ is small enough and $\alpha > 0$. Moreover, $\lambda_{r_1+1}(\alpha) > \lambda_{r_1+1}$.*

To establish our main result, we also need the following lemma.

**Lemma 8.** *If for every $C \in \mathrm{co}(C_1, C_2, C_3)$, C has at least two negative eigenvalues. Then $\lambda_{n-1}(\alpha, \beta)$ attains its maximum in $\Re^2_+$, where $\lambda_{n-1}(\alpha, \beta)$ is defined as in Lemma 6.*

*Proof.* By the assumption, $\bar{\lambda}_{n-1}(\alpha, \beta) < 0$, for all $\alpha^2 + \beta^2 = 1$ satisfying $\alpha \geq 0,\ \beta \geq 0$, where $\bar{\lambda}_{n-1}(\alpha, \beta)$ denotes the $(n-1)$-largest eigenvalue of $\alpha C_2 + \beta C_3$.

Due to the continuity of $\bar{\lambda}_{n-1}(\alpha, \beta)$,

$$\bar{\lambda}_{n-1} = \max_{\alpha^2+\beta^2=1, \alpha\geq 0, \beta\geq 0} \bar{\lambda}_{n-1}(\alpha, \beta) < 0\,. \tag{49}$$

It is easy to see that

$$\begin{aligned} \lambda_{n-1}(\alpha, \beta) &\leq \bar{\lambda}_{n-1}(\alpha, \beta) + \| C_1 \|_2 \\ &\leq \bar{\lambda}_{n-1}\sqrt{\alpha^2 + \beta^2} + \| C_1 \|_2\,, \end{aligned} \tag{50}$$

which, together with (49), shows that $\lambda_{n-1}(\alpha, \beta) \to -\infty$ as $\alpha^2 + \beta^2 \to \infty$ $(\alpha \geq 0,\ \beta \geq 0)$. Therefore there exists $(\alpha_0, \beta_0) \in \Re^2_+$ such that $\lambda_{n-1}(\alpha_0, \beta_0)$ maximizes $\lambda_{n-1}(\alpha, \beta)$.

□

With the above results, we can show an important theorem for our main result in the following.

**Theorem 3.** *If 0 solves the problem (8)–(10), then there exists a matrix $C \in \mathrm{co}(C_1, C_2, C_3)$, such that C has at most one negative eigenvalue.*

*Proof.* If the theorem is not true, there exist $C_1, C_2, C_3$ such that 0 solves (8)–(10), and every matrix in $\mathrm{co}(C_1, C_2, C_3)$ has at least two negative eigenvalues. By Lemma 8, assume $(\alpha_0, \beta_0) \in \Re^2_+$ such that $\lambda_{n-1}(\alpha_0, \beta_0)$ maximizes $\lambda_{n-1}(\alpha, \beta)$ in $\Re^2_+$, where $\lambda_{n-1}(\alpha, \beta)$ is defined in Lemma 6. Define

$$\mathcal{X} = \mathrm{span}\{x|(C_1 + \alpha_0 C_2 + \beta_0 C_3)x = \lambda_i x, \lambda_i < 0\}\,. \tag{51}$$

We will discuss three cases according to the values of $\alpha_0, \beta_0$.

*Case 1.* $\alpha_0 = \beta_0 = 0$.

It follows from Remark 3 that $\alpha C_2 + \beta C_3$ cannot be positive definite in $\mathcal{X}$ for every $\alpha \geq 0, \beta \geq 0$. Therefore there exists $0 \neq x_0 \in \mathcal{X}$ satisfying $x_0^T C_2 x_0 \leq 0, x_0^T C_3 x_0 \leq 0$. Otherwise $x^T C_3 x > 0$ for $0 \neq x \in \mathcal{X}, x^T C_2 x \leq 0$, then there exists $\alpha \in \Re$ such that $\alpha C_2 + C_3 > 0$ in $\mathcal{X}$ by Lemma 4. The first sentence of this paragraph implies that $\alpha < 0$, thus $x^T C_2 x \leq 0$ for every $x \in \mathcal{X}, x^T C_3 x \leq 0$, which gives a contradiction. The definition of $\mathcal{X}$ and $\alpha_0 = \beta_0 = 0$ give that

$$x_0^T C_1 x_0 = x_0^T (C_1 + \alpha_0 C_2 + \beta_0 C_3) x_0 < 0\,, \tag{52}$$

which contradicts the assumption that 0 solves (8)–(10).

*Case 2.* $\alpha_0 + \beta_0 > 0, \alpha_0 \beta_0 = 0$.

Without loss of generality, assume $\alpha_0 > 0, \beta_0 = 0$. It follows from Remark 3 that $\alpha C_2 + \beta C_3$ cannot be positive definite in $\mathcal{X}$ for all $(\alpha, \beta) \in \Re^2$ with $\beta \geq 0$. If $x^T C_3 x > 0$ for $0 \neq x \in \mathcal{X}, x^T C_2 x = 0$. Then there exists $\alpha \in \Re$ such that $\alpha C_2 + C_3 > 0$ in $\mathcal{X}$ by Lemma 4, which gives a contradiction. Therefore there exists $0 \neq x_0 \in \mathcal{X}$, such that $x_0^T C_2 x_0 = 0, x_0^T C_3 x_0 \leq 0$. Thus (52) holds, which contradicts our assumption.

*Case 3.* $\alpha_0 > 0, \beta_0 > 0$.

It follows from Lemma 6 that $\alpha C_2 + \beta C_3$ cannot be positive definite in $\mathcal{X}$ for every $(\alpha, \beta) \in \Re^2$. If $\dim(\mathcal{X}) \geq 3$, it follows from Lemma 3 that there exists $x_0 \in \mathcal{X}$ such that $x_0^T C_2 x_0 = x_0^T C_3 x_0 = 0$, which yields (52). This is a contradiction. Therefore we can assume that $\dim(\mathcal{X}) = 2$.

If $\dim(\mathcal{X}_{n-1}) = 1$ (where $\mathcal{X}_k$ is defined in Lemma 6). Since $\alpha C_2 + \beta C_3$ cannot be positive definite in $\mathcal{X}_{n-1}$ for every $(\alpha, \beta) \in \Re^2$ by Lemma 6, $C_2, C_3$ must be equal to 0 in $\mathcal{X}_{n-1}$, which yields (52) for every $x_0 \in \mathcal{X}_{n-1}$. This is a contradiction.

If $\dim(\mathcal{X}_{n-1}) = 2$, i.e. $\mathcal{X}_{n-1} = \mathcal{X}$, then $\lambda_n(\alpha_0, \beta_0) = \lambda_{n-1}(\alpha_0, \beta_0)$ and $(\alpha_0, \beta_0)$ maximizes $\lambda_{n-1}(\alpha, \beta), \lambda_n(\alpha, \beta)$ in $\Re^2_+$ simultaneously. If there exists $(\alpha, \beta) \in \Re^2$ such that $\alpha C_2 + \beta C_3$ is indefinite in $\mathcal{X}$, $(\alpha_0, \beta_0)$ cannot maximize $\lambda_{n-1}(\alpha, \beta), \lambda_n(\alpha, \beta)$ in $\Re^2_+$ simultaneously by (1) of Lemma 7. Since $\alpha C_2 + \beta C_3$ cannot be either positive definite or indefinite in $\mathcal{X}$ for every $(\alpha, \beta) \in \Re^2$, $\alpha C_2 + \beta C_3$ must be either positive semi-definite or negative semi-definite in $\mathcal{X}$ for $(\alpha, \beta) \in \Re^2$, then $\alpha C_2 + \beta C_3$ must be equal to 0 in $\mathcal{X}$ for every $(\alpha, \beta) \in \Re^2$ by (2) of Lemma 7. Therefore $C_2$ and $C_3$ are equal to 0 in $\mathcal{X}$. It then follows that (52) holds for every $x_0 \in \Re^n$, which gives a contradiction.

Thus the proof is completed.

□

Now we have obtained our main theorem, which is stronger than the results in [7].

**Theorem 4.** *Let $C_1, C_2, C_3$ be three symmetric matrices in $\Re^{n\times n}$. If*

$$\max\left\{x^T C_1 x, x^T C_2 x, x^T C_3 x\right\} \geq 0\,, \quad \text{for every } x \in \Re^n\,, \tag{53}$$

*then there exists a matrix $C = \sum_{i=1}^3 t_i C_i$ $(\sum_{i=1}^3 t_i = 1, t_i \geq 0, i = 1, 2, 3)$, such that $C$ has at most one negative eigenvalue.*

*Proof.* It follows obviously from Lemma 2 and Theorem 3. □

**Theorem 5.** *Let $C_i$ $(i = 1, 2, 3)$ be three symmetric matrices in $\Re^{n\times n}$. Then*

$$\max_{1\leq r\leq 3}\left\{v^T(C_i \oplus C_i)v\right\} \geq 0, \forall v \in \Re^{2n}\,, \tag{54}$$

*if and only if there exists $C \in \mathrm{co}_{i=1}^3(C_i)$ such that $C \geq 0$ in $\Re^n$, where $X \oplus Y$ denotes the direct sum:*

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

*Proof.* We only need to verify the "only if " part. It follows from Theorem 4 that there exists a $D \in \mathrm{co}_{i=1}^3(C_i \oplus C_i)$ such that $D$ has at most one negative eigenvalue. Due to the special structure of $D$, the multiplicity of every eigenvalue must be even. Therefore $D$ must be positive semi-definite. It follows the conclusion stated in theorem. □

In fact, [1] pointed out that $\tilde{W}(C_1, C_2, C_3) = \{y = (y_1, y_2, y_3) \in \Re^3 | y_i = v^T(C_i \oplus C_i)v,\ i = 1, 2, 3,\ \text{for } v \in \Re^{2n}\}$ is a convex cone in $\Re^3$, so one can also use separating theorem for convex sets to prove the above theorem.

A natural conjecture for multi-quadratic forms is as follows: if

$$\max_{1\leq i\leq r}\left\{x^T C_i x\right\} \geq 0,\ \forall x \in \Re^{n\times n}\,, \tag{55}$$

then there exists a matrix $C \in \mathrm{co}_{i=1}^r(C_i)$ such that $C$ has at most $r-2$ negative eigenvalues. Theorem 1 and Theorem 4 are the cases when $r = 2$ and $r = 3$, respectively. Whether the conjecture is true for $r > 3$ is still unknown.

## References


1. Brickman, L. (1961): On the field of values of a matrix. Proc. Am. Math. Soc. **12**, 61–66
2. Celis, M.R., Dennis, J.E., Tapia, R.A. (1985): A trust region algorithm for nonlinear equality constrained optimization. In: Boggs, P.T., Byrd, R.H., Schnabel, R.B., eds., Numerical Optimization, pp. 71–82. SIAM, Philadelphia
3. Crouzeix, J.P., Martínez-Legaz, J.E., Seeger, A. (1995): An alternative theorem for quadratic forms and extensions. Linear Algebra Appl. **215**, 121–134
4. Heinkenschloss, M. (1994): On the solution of a two ball trust region subproblem. Math. Program. **64**, 249–276

5. Martínez-Legaz, J.E., Seeger, A. (1994): Yuan's Alternative theorem and the maximization of the minimum eigenvalue. J. Optim. Theory Appl. **82**, 159–167
6. Moré, J.J. (1993): Generalization of the trust region problem. Optim. Methods Software **2**, 189–209
7. Peng, J., Yuan, Y. (1997): Optimality conditions for the minimization of a quadratic with two quadratic constraints. SIAM J. Optim. **7**, 574–594
8. Powell, M.J.D., Yuan, Y. (1991): A trust region algorithm for equality constrained optimization. Math. Program. **49**, 189–211
9. Rockafellar, T. (1970): Convex Analysis. Princeton University Press
10. Sun, J. (1987): Matrix Perturbation Analysis (in Chinese). Science Press, Beijing
11. Uhlig, F. (1979): A recurring theorem about pairs of quadratic forms and extensions: A survey. Linear Algebra Appl. **25**, 219–237
12. Yuan, Y. (1990): On a subproblem of trust region algorithms for constrained optimization. Math. Program. **47**, 53–63
13. Yuan, Y. (1991): A dual algorithm for minimizing a quadratic function with two quadratic constraints. J. Comput. Math. **9**, 348–359