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# Inventory Centralization Games with Price-Dependent Demand and Quantity Discount

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Consider a distribution system consisting of a set of retailers facing a single-period price-dependent demand of a single product. By taking advantage of the risk-pooling effect and the quantity/volume discount provided by suppliers or third-party carriers, the retailers may place joint orders and keep inventory at central warehouses before demand realization, and allocate inventory among themselves after demand realization to reduce their operating costs. Under rather general assumptions, we prove that there is a stable allocation of profits among the retailers in the sense that the resulting inventory centralization game has a nonempty core. We also show how to compute an allocation in the core.

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## 1. Introduction

In recent years, many companies have started exploring innovative collaboration strategies in efforts to improve their supply chain efficiency and, ultimately, the bottom line. There are numerous examples of firms employing collaborative strategies in the form of long-term alliances and collaborative logistics with an eye toward reducing their supply chain costs. For example, such collaborative strategies are used by the Good Neighbor Pharmacy and Affiliated Foods Midwest. The former is a cooperative network of 2,700 independently owned and operated pharmacies. The latter supplies more than 850 independent retails in the 12 Midwest states with a full line of grocery products.

To compete with big-box retailers, it is common for independent grocery stores, hardware stores, and pharmacies to form retailers' cooperative groups—business entities that employ economies of scale on behalf of retailer-members to get discounts from manufacturers and to pool marketing. To join a retailers' cooperative, a store would typically pay a membership fee and purchase stock in the cooperative in return for its voting share. In addition, a store is usually required to purchase a minimum amount of inventory from the cooperative. The operating profits of the cooperative are returned to the member stores in cash or stock rebate (see Stankevich 1996). Over the years, retail cooperative groups developed a variety of popular groupwide programs such as insurance, pension plans, inventory management, pricing assistance, logistics, warehousing, store design and layout, site selection, and employee training (see Ghosh 1994).

These innovative strategies raise a variety of important and challenging questions on managing supply chains. For example, for a group of companies in a supply chain, how should they cooperate? What possible outcomes can be achieved, and how do the players share the costs and benefits? Getting all players to agree on how to share costs and benefits was identified as one of the major barriers to collaborative commerce (see European Chemical Transportation Association (ECTA) 2006, NerveWire 2002).

Cost allocation also plays a critical role for a single firm in which several divisions or products share common resources. As firms create autonomous units with delegated responsibility toward personnel, adminstration, sales, and distribution, costs of common resources have to be fully allocated, otherwise the operating divisions may impute lower costs to the common factor inputs and consume the unallocated resources excessively (see Balachandran and Ramakrishnan 1996).

The goal of this paper is to analyze the cost or benefit allocation among several retailers in basic collaborative supply chain settings, referred to as *inventory centralization games*. In an inventory centralization game, we consider a distribution system with multiple retailers that may place joint orders and keep inventory at central warehouses. There are two main reasons for this type of cooperation. First, retailers can take advantage of the risk-pooling effect by delaying the allocation of inventory. Second, exploiting economies of scale allows retailers to reduce their costs or increase their profits. Additionally, in practice, suppliers usually provide quantity discounts to encourage large orders, whereas third-party carriers offer volume or quantity discounts, such as LTL (less-than-truckload) discounts, to their clients to encourage demand for larger, more profitable shipments.

Given retailers' interest in inventory centralization, it is critical that they allocate the cost or share the benefit in such a way that no retailer gains more by deviating from the cooperation. We call such cooperatively achieved allocations *core allocations* in this paper. This concept finds prominent use in cooperative game theory.

In this paper, we apply the concept of core to analyze inventory centralization games with stochastic, price-dependent demand and quantity discount. Specifically, we consider a distribution system consisting of a set of retailers that sell a single product. The demand of each retailer depends on its own selling price and a common random variable representing the market condition, referred to as a *market signal*. By taking advantage of economies of scale and risk-pooling effects, the retailers may form a coalition by placing joint orders through multiple warehouses and keeping inventory at warehouses. Due to long lead time, there is a single replenishment opportunity, and orders have to be placed long before the realization of the market signal. After the market signal is revealed, the inventory kept at the warehouses can then be allocated to the retailers. The objective of the retailers in a coalition is to maximize the expected total profits of all retailers in the coalition.

In the inventory centralization game literature, demand distribution is assumed to be given, which is of course consistent with traditional inventory models focusing on exogenously determined commodity prices. In recent years, however, with the development of information technology and e-commerce, a number of industries have used innovative pricing strategies as an effective tool to better match supply and demand and, therefore, significantly improve their operational efficiencies (see, for example, Kimes 1989, Gallego and van Ryzin 1994). Thus, in the inventory centralization games analyzed here, we assume that in addition to the inventory ordering and allocation decisions, retailers may set their own selling prices either before or after observing the market signal. Moreover, we assume that retailers are noncompeting, i.e., a retailer's price only affects its own demand. This assumption is valid if the retailers serve customers from nonoverlapping regions (for example, most Good Neighbor Pharmacy stores operate in separate nonintersecting market locations).

The main contribution of this paper is to show that inventory centralization games with price-dependent demand have nonempty cores under very general assumptions regarding ordering costs. Specifically, if the ordering costs are linear, we use convex programming duality theory to construct an allocation in the core of an inventory centralization game. More interestingly, under the assumption that the ordering cost follows a general quantity discount, we prove that an inventory centralization game in which all retailers share a single common warehouse has a nonempty core when (a) the retailers' pricing decisions are made after the revelation of the market signal, or (b) the retailers have identical cost parameters and their pricing decisions are made before the revelation of the market signal.

The main idea of our proof for inventory centralization games with general quantity discount is to construct a new inventory centralization game with a linear ordering cost, which is known to have a nonempty core, such that the maximum profit achieved by any subset of the retailers in the new game will not decrease, while ensuring the maximum profit achievable by all retailers remains unchanged. In addition to the proof of the nonemptiness of the core, our approach also suggests a mechanism to find an allocation in the core for the inventory centralization game with a general quantity discount.

Our assumption about the cost function is quite general. Indeed, the only assumption we make is that the larger the ordering/transporting quantity, the lower the average unit ordering/transportation cost. Our assumption includes several commonly used discounts: incremental discounts, all-units discounts under which suppliers offer price breaks for large orders (see Zipkin 2000), and the less-than-truckload volume discount (see Muriel and Simchi-Levi 2003). It is appropriate to point out that the all-units discount is neither continuous nor monotone.

The rest of this paper is organized as follows. In §2, we briefly review the related literature. In §3, we introduce the inventory centralization game model and some important solution concepts in the cooperative game theory. In §4, convex programming duality theory is employed to show that an inventory centralization game with linear ordering cost has a nonempty core, which is followed by an analysis on inventory centralization games with general quantity discounts in §5 and a procedure to compute a core allocation in §6. Finally, some concluding remarks and important extensions of our results are presented in §7.

## 2. Literature Review

In this section, we present a brief review of inventory centralization games and integrated inventory and pricing models. For a comprehensive review of applications of the cooperative game theory to supply chain management, see Nagarajan and Sošić (2006). We also refer to Demski and Kreps (1983) for a review and critique to models in managerial accounting, including the applications of cooperative games in cost allocation.

Unlike our paper, which allows for general quantity discount, inventory centralization games analyzed in the literature mainly focus on the risk-pooling effect. It is well known in the inventory literature that inventory centralization leads to cost reduction or profit increase (see, for example, Eppen 1979), which provides retailers with an incentive to form coalitions.

A special case of inventory centralization games is the newsvendor game, in which each retailer is a newsvendor with the same cost parameters and the transportation cost associated with reallocating inventory after observing the demand is negligible. Newsvendor games have been analyzed in several papers by assuming linear ordering cost structure. Under this assumption, Hartman et al. (2000) show that a newsvendor game has a nonempty core under special assumptions on demand distributions. This result has been generalized by Müller et al. (2002) and Slikker et al. (2001), who show that the core is always nonempty regardless of the demand distributions.

More general inventory games have been studied by Slikker et al. (2005) and by Ozen et al. (2008), who extend newsvendor games in two ways. First, retailers may place orders through multiple warehouses. Second, in their settings, the cost parameters of the retailers can be different and the transportation costs may be nonzero. Although the models are much more complicated than the one studied in Hartman et al. (2000), Slikker et al. (2005) and Ozen et al. (2008) still manage to show that the games have nonempty cores. In another paper, Ozen and Sošić (2006) extend the models by allowing the retailers to allocate inventory after observing the revelation of a market signal but before the realization of the demand.

The analysis in Hartman et al. (2000), Müller et al. (2002), Slikker et al. (2001, 2005), and Ozen et al. (2008) relies heavily on the structure of the underlying problems and its extension to more general settings may be difficult. To overcome this difficulty, Chen and Zhang (2009) present a unified way to analyze inventory centralization games using the duality theory of stochastic programming. Their duality approach is further extended in Chen and Zhang (2009) to analyze inventory centralization games with concave ordering cost.

Anupindi et al. (2001) study a two-stage model that is closely related to the one in Slikker et al. (2005) and Chen and Zhang (2009). However, in their model, the retailers do not fully cooperate. In the first stage, before demand realization, each retailer makes its own decision on how much to order. In the second stage, after observing the demands, the retailers can cooperate by reallocating their inventories. Granot and Sošić (2003) add an additional, noncooperative stage between the two stages, in which each retailer determines the amount of its residual inventories to share.

Our paper analyzes inventory centralization games with price-dependent demand and thus is related to the literature on the coordination of inventory control and pricing strategies. For a review of this literature, the reader is referred to Eliashberg and Steinberg (1993), Petruzzi and Dada (1999), Federgruen and Heching (1999), Yano and Gilbert (2002), Elmaghraby and Keskinocak (2003), or Chan et al. (2004). Our analysis and results for inventory centralization games with quantity discount depend on whether pricing decision is made before or after the revelation of a market signal, and thus is also related to the literature on price and production postponement. For more details on this literature, we refer to van Mieghem and Dada (1999) and Chod and Rudi (2005).

Our paper makes two important contributions to the literature. First, this paper is the first to incorporate pricing decisions in the inventory centralization game setting. So far, the literature on inventory centralization games exclusively assumes that price is exogenously determined. However, with the development of information technology and e-commerce, firms can adjust their prices at minimal costs to better match their supply and demand. Thus, it is critical to investigate the impact of pricing decisions in the inventory centralization game setting.

Second, this paper is the first to allow for general quantity discounts in the inventory centralization game setting. All the papers mentioned above do not take into consideration quantity discounts, i.e., they assume that the ordering cost is proportional to the order quantity. The only exception is Chen and Zhang (2009), which allows for concave ordering cost in the newsvendor game setting. The quantity discounts used here are much more general than the concave ordering cost analyzed in Chen and Zhang (2009) because they also include other commonly used discounts such as all-units discounts and LTL cost structure as special cases. Because we use rather general quantity discounts, the duality approach proposed in Chen and Zhang (2009) does not apply and consequently we develop a totally different approach in this paper.

## 3. The Inventory Centralization Game Model

Consider a distribution system consisting of a supplier, a set of $m$ warehouses denoted by $W = \{1, 2, \ldots, m\}$, and a set of $n$ retailers denoted by $N = \{1, 2, \ldots, n\}$. The retailers order from the supplier through the warehouses and sell a single type of goods in a single period.

The retailers are assumed to be noncompeting and allowed to make their selling price decision. Each retailer's demand depends on its own selling price and a common random variable—the market signal, $\omega$. To satisfy their demand, the retailers, taking advantage of risk-pooling effects, may form coalitions to place joint orders through the warehouses before observing the market signal, whereas the inventories are allocated to the retailers after the market signal is revealed. Let $Z_j \subseteq W$ be the set of warehouses that can be used to supply retailer $j$ if she does not cooperate with other retailers. If retailer $j$, together with some other retailers, decides to form a group $S$, referred to as a coalition, by placing joint orders and sharing inventory, her demand can be served by the inventory at any warehouse in $\bigcup_{j\in S} Z_j$.

Depending on when the pricing decision is made, we focus on two different models. The first model, referred to as the *nonanticipative pricing model*, assumes that the pricing decision is made before the market signal is revealed. On the other hand, the second model, referred to as the *postponed pricing model*, assumes that the pricing decision is made after the market signal is revealed.

The sequence of events is as follows. Before observing the realization of the market signal, each warehouse places an order by paying an ordering cost of $c_i(y_i)$ for an order quantity $y_i$ at warehouse $i$. The retailers then decide their selling prices $p_j(\omega)$. In the postponed pricing model, the pricing decision is made after the realization of the market signal $\omega$, whereas in the nonanticipative pricing model, the pricing decision is made before the realization of the market signal $\omega$. In this case, we impose the nonanticipative constraint on the prices: $\mathrm{E}[p_j(\omega)] = p_j(\omega)$ for all $\omega$ and $j \in N$, which implies that the pricing decision of each retailer is independent of the realization of the market signal.

After the market signal $\omega$ is revealed, all goods at the warehouses are allocated to the retailers, say $x_{ij}(\omega)$ units of goods are shipped from warehouse $i$ to retailer $j$. The transportation cost of sending one unit of goods from warehouse $i$ to retailer $j$ is $s_{ij}$. For each retailer $j$, if the total amount of goods received from the warehouses is more than the realized demand, a per-unit holding cost of $h_j$ for excess inventory is incurred. On the other hand, we make the following assumption regarding unsatisfied demand.

Assumption 1. *Unsatisfied demand at retailer $j$ is filled by an emergency order, which incurs a per-unit emergency ordering cost of $q_j$.*[1,2]

The demand of each retailer is random and depends on the realization of the market signal $\omega$ and its own selling price. Specifically, we concentrate on demand functions of the following forms:

Assumption 2. *For $j \in N$, the demand function of retailer $j$ satisfies*

$$\tilde{d}_j = D_j(p_j, \omega) := \beta_j(\omega) - \alpha_j(\omega) p_j, \tag{1}$$

*where $\alpha_j$ and $\beta_j$ are two nonnegative random variables, represented as functions of the market signal $\omega$.*[3]

To avoid technical complications, we assume that the sample space $\Omega$ of $\omega$ is finite. However, this assumption can be relaxed if necessary.

We further assume that $\underline{p}_j$ and $\bar{p}_j$ are the lower and upper bounds of $p_j(\omega)$, respectively. Thus, in the postponed pricing model, the feasible set of retailer $j$'s price decision $p_j(\cdot)$ is given by

$$P_j^{(p)} = \{p_j(\cdot)\colon\ \underline{p}_j \leqslant p_j(\omega) \leqslant \bar{p}_j,\ \forall\, \omega \in \Omega\},$$

whereas in the nonanticipative pricing model, the feasible set of retailer $j$'s price decision $p_j(\cdot)$ is given by imposing the nonanticipative constraint as follows:

$$P_j^{(n)} = \big\{p_j(\cdot)\colon \mathrm{E}[p_j(\omega)] = p_j(\omega),\\ \underline{p}_j \leqslant p_j(\omega) \leqslant \bar{p}_j,\ \forall\, \omega \in \Omega\big\}.$$

The inventory centralization problem for a coalition of retailers $S \subset N$ can be formulated as a two-stage stochastic programming model with recourse. In this model, $y_i$, $i = 1, 2, \ldots, m$, is the first-stage decision variable. After the market signal $\omega$ is revealed, a recourse decision should be made, which is the amount of goods sent from $i$ to $j$, i.e., $x_{ij}(\omega)$ for all $i \in \bigcup_{j\in S} Z_j$ and $j \in S$, and the selling price $p_j(\omega)$ (in the nonanticipative pricing model, we impose the nonanticipative constraint, which implies that pricing decisions are actually made at the first stage). Let $v_j(\omega)$ be the total amount of goods received by retailer $j$. For the coalition $S$, the objective is to maximize the expected total profit of all retailers in $S$.

Denote the maximum expected profit of the coalition $S$ by $V(S)$, which can be written as the optimal value of the following two-stage stochastic programming problem with recourse:

$$\begin{aligned}
V(S) = \max\ & \sum_{j\in S}\big\{\mathrm{E}[R_j(p_j(\omega),\omega) \\
&\qquad - f_j(v_j(\omega) - \beta_j(\omega) + \alpha_j(\omega)p_j(\omega))]\big\} \\
&\qquad - \sum_{i\in\cup_{j\in S}Z_j}\Big(c_i(y_i) + \sum_{j\in S} s_{ij}\mathrm{E}[x_{ij}(\omega)]\Big) \\
\text{s.t.}\ & y_i - \sum_{j\in S} x_{ij}(\omega) = 0, \quad i \in \bigcup_{j\in S} Z_j,\ \omega\in\Omega, \\
& v_j(\omega) - \sum_{i\in\cup_{j\in S}Z_j} x_{ij}(\omega) = 0, \quad j\in S,\ \omega\in\Omega, \\
& x_{ij}(\omega) \geqslant 0, \quad j\in S,\ i\in\bigcup_{j\in S} Z_j,\ \omega\in\Omega, \\
& p_j(\cdot)\in P_j, \quad j\in S,
\end{aligned} \tag{2}$$

where the maximization is taken over $(y_i, x_{ij}(\cdot), p_j(\cdot), v_j(\cdot))$, $R_j$ is the realized revenue function for a given selling price $r$ and realization of the marker signal $\omega$:

$$R_j(r, \omega) = r(\beta_j(\omega) - \alpha_j(\omega) r),$$

$f_j$ represents the inventory holding cost or emergency ordering cost

$$f_j(\chi) = h_j \chi^+ + q_j(-\chi)^+,$$

and $P_j = P_j^{(p)}$ for $j \in N$ in the postponed pricing model, whereas $P_j = P_j^{(n)}$ for $j \in N$ in the nonanticipative pricing model.

In the above model, the term in the first summation in the objective function is the expected revenue minus the expected inventory holding cost and emergency ordering cost. The term in the second summation in the objective function is the regular ordering cost and the transportation cost. The first constraint implies that no warehouse holds inventory. As pointed out by Chen and Zhang (2009), this assumption is not critical and is made merely for the ease of presentation. The second constraint specifies that the total amount of goods received by a retailer equal the total amount sent to the retailer from the warehouses.

Now the pair $(N, V)$ with $V$ given by (2) for each coalition $S \subset N$ defines a cooperative inventory centralization game. Because some of our results depend on whether there is a single warehouse or multiple warehouses, we will use the notation $(N, V, m)$ to emphasize the cooperative inventory centralization game with $m$ warehouses when necessary.

At this point, it is appropriate to introduce some basic concepts of the cooperative game theory that will be used in this paper. The set of retailers $N$ is called the *grand coalition*. The function $V$, the maximum total value that a coalition $S$ can generate when the members of $S$ decide to secede from the grand coalition and cooperate only among themselves, is referred to as the *characteristic value function*.

A vector $l = (l_1, l_2, \ldots, l_N)$ is called an *efficient allocation* for the game $(N, V)$ if $\sum_{j\in N} l_j = V(N)$. The core of a cooperative game is a solution concept that requires that no subset of players has an incentive to secede.

Definition 1. An efficient allocation $l$ is in the *core* of the game $(N, V)$ if $\sum_{j\in N} l_j = V(N)$, and for any subset $S \subseteq N$, $\sum_{j\in S} l_j \geqslant V(S)$.

A game $(N, V)$ is called a *convex* (*value*) *game* if for every pair of subsets $S, T \subseteq N$, $V(S) + V(T) \leqslant V(S \cup T) + V(S \cap T)$, or equivalently, $V(S)$ is supermodular in $S$. It is well known that the core of a convex (value) game is always nonempty. Unfortunately, cooperative inventory centralization games may not be convex. Indeed, the newsvendor game (a cost game instead of a value game), which can be easily shown to be equivalent to a special case of the above cooperative inventory game, is not concave in general (see Ozen et al. 2005).

## 4. Inventory Games with Linear Ordering Cost

In this section, we assume that the ordering cost $c_i(y_i)$ is linear, and by slightly abusing the notation, we also use $c_i$ to denote the unit ordering cost. Because the realized revenue $R_j(p_j(\omega), \omega)$ is concave in $p_j(\omega)$ by Assumption 2 and $f_j(\chi)$ is convex, problem (2) is a concave maximization problem with linear constraints, which allows us to apply the elegant duality theory for convex minimization problems with linear constraints. For this purpose, define the Lagrangian function

$$
\begin{aligned}
&L_S(y, p, v, x, \lambda, \mu, \pi)\\
&= \sum_{j\in S} (\mathrm{E}[R_j(p_j(\omega), \omega) - f_j(v_j(\omega) - \beta_j(\omega) + \alpha_j(\omega) p_j(\omega))])\\
&\quad - \sum_{i\in\cup_{j\in S} Z_j} \left( c_i y_i + \sum_{j\in S} s_{ij} \mathrm{E}[x_{ij}(\omega)] \right)\\
&\quad + \sum_{i\in\cup_{j\in S} Z_j} \mathrm{E}\left[ \lambda_i(\omega) \left( y_i - \sum_{j\in S} x_{ij}(\omega) \right) \right]\\
&\quad + \sum_{j\in S} \mathrm{E}\left[ \mu_j(\omega) \left( \sum_{i\in\cup_{j\in S}} x_{ij}(\omega) - v_j(\omega) \right) \right]\\
&\quad + \sum_{j\in S,\, i\in\cup_{j\in S} Z_j} \mathrm{E}[\pi_{ij}(\omega) x_{ij}(\omega)]\\
&= \sum_{j\in S} \psi_j(p_j, v_j, \mu_j) + \sum_{i\in\cup_{j\in S} Z_j} y_i (\mathrm{E}[\lambda_i(\omega)] - c_i)\\
&\quad + \sum_{j\in S,\, i\in\cup_{j\in S} Z_j} \mathrm{E}[x_{ij}(\omega)(\pi_{ij}(\omega) - s_{ij} - \lambda_i(\omega) + \mu_j(\omega))],
\end{aligned}
$$

where $y = (y_i)_{i\in\cup_{j\in S} Z_j}$, $d = (d_j)_{j\in S}$, $v = (v_j)_{j\in S}$, $\lambda = (\lambda_i)_{i\in\cup_{j\in S} Z_j}$, $\mu = (\mu_j)_{j\in S}$, $\pi = (\pi_{ij})_{j\in S,\, i\in\cup_{j\in S} Z_j}$, and

$$
\begin{aligned}
\psi_j(p_j, v_j, \mu_j) = \mathrm{E}[R_j(p_j(\omega), \omega) - f_j(v_j(\omega) - \beta_j(\omega)&\\
+ \alpha_j(\omega) p_j(\omega)) - \mu_j(\omega) v_j(\omega)].&
\end{aligned} \tag{3}
$$

Consider the dual function $\gamma_S(\lambda, \mu, \pi)$ defined by

$$
\begin{aligned}
\gamma_S(\lambda, \mu, \pi) = \sup \quad & L_S(y, p, v, x, \lambda, \mu, \pi)\\
\text{s.t.} \quad & p_j(\cdot) \in P_j, \quad j \in S.
\end{aligned}
$$

The duality theorem for convex minimization problems with linear constraints implies that $V(S)$ is equal to the optimal objective value of the dual problem (see, for example, Bertsekas 1995, p. 299):

$$
\begin{aligned}
V(S) = \min \quad & \gamma_S(\lambda, \mu, \pi)\\
\text{s.t.} \quad & \pi_{ij}(\omega) \geqslant 0, \quad j \in S,\ i \in \bigcup_{j\in S} Z_j,\ \omega \in \Omega.
\end{aligned} \tag{4}
$$

Let $(\lambda^*, \mu^*, \pi^*)$ be optimal for the dual problem (4) with $S = N$. Then again, the duality theorem implies that

$$
\begin{aligned}
V(N) = \max \quad & L_N(y, p, v, x, \lambda^*, \mu^*, \pi^*)\\
\text{s.t.} \quad & p_j(\cdot) \in P_j, \quad j \in N.
\end{aligned} \tag{5}
$$

Define for $j \in N$,

$$
\begin{aligned}
l_j = \max \quad & \psi_j(p_j, v_j, \mu_j^*)\\
\text{s.t.} \quad & p_j(\cdot) \in P_j.
\end{aligned}
$$

We claim that $(l_1, l_2, \ldots, l_n)$ is in the core of the cooperative game $(N, V)$.

Theorem 1. *The vector $l = (l_1, l_2, \ldots, l_n)$ is in the core of the cooperative game $(N, V)$.*

Proof. Note that in the optimization problem (5), no constraint is imposed on the decision variables $y_i$ and $x_{ij}(\omega)$. Thus, we must have

$$
\mathrm{E}[\lambda_i^*(\omega)] - c_i = 0, \quad i \in \bigcup_{j\in N} Z_j,\ \omega \in \Omega, \tag{6}
$$

and

$$
\pi_{ij}^*(\omega) - s_{ij} - \lambda_i^*(\omega) + \mu_j^*(\omega) = 0, \quad j \in N,\ i \in \bigcup_{j\in N} Z_j,\ \omega \in \Omega. \tag{7}
$$

Therefore,

$$
L_S(y, p, v, x, \lambda^*, \mu^*, \pi^*) = \sum_{j\in S} \psi_j(p_j, v_j, \mu^*).
$$

This, together with (5), implies that

$$\sum_{j\in N} l_j = V(N).$$

In addition, because $(\lambda^*, \mu^*, \pi^*)$ is feasible for problem (4),

$$\sum_{j\in S} l_j = \gamma_S(\lambda^*, \mu^*, \pi^*) \geqslant V(S).$$

Thus, $l = (l_1, l_2, \ldots, l_n)$ is in the core of the cooperative game $(N, V)$. □

From the above proof, we know that the optimal dual variables $(\lambda^*, \mu^*, \pi^*)$ must satisfy constraints (6) and (7). We now provide some intuition of the dual variables and the constraints. In the dual, we attempt to allocate the ordering cost and the transportation cost to each unit goods received by the retailers. Specifically, let the dual variable $\mu_j^*(\omega)$ be the charge for each unit of goods received by retailer $j$ to compensate for its ordering cost and transportation cost and $\lambda_i^*(\omega)$ be a charge for each unit of goods sent out by warehouse $i$ to compensate its ordering cost if the market signal turns out to be $\omega$. Constraint (6) implies that the average unit charge by warehouse $i$ should be enough to cover its ordering cost $c_i$. Because $\pi_{ij}^*(\omega) \geqslant 0$, the dual constraint (7) implies that this unit charge $\mu_j^*(\omega)$ at retailer $j$ should be no more than the unit price, $\lambda_i^*(\omega)$, charged by warehouse $i$ plus the transportation cost $s_{ij}$. On the other hand, if there is a shipment from warehouse $i$ to retailer $j$, then $\pi_{ij}^*(\omega) = 0$ by the complementarity slackness condition, and the dual constraint (7) implies that this unit charge $\mu_j^*(\omega)$ is enough to compensate the unit price, $\lambda_i^*(\omega)$, charged by warehouse $i$ plus the transportation cost $s_{ij}$.

## 5. Inventory Games with Quantity Discounts

In this section, we assume that the supplier provides quantity discounts to encourage large orders, or a third-party carrier provides volume discounts to encourage larger shipments. Specifically, we make the following assumption.

ASSUMPTION 3. *We assume that $c_i(y)/y$ is nonincreasing. That is, the larger the ordering quantity, the lower the average unit ordering cost.*

For technical reasons, we assume that $c_i(y)$ is lower semicontinuous; that is, $\liminf_{y\to x} c_i(y) \geqslant c_i(x)$ for any $x$. Further, we assume that $c_i(y) \to \infty$ as $y \to \infty$. Under these assumptions, problem (2) has an optimal solution for any $S \subset N$.

Our assumption on the ordering cost is quite general. Indeed, we do not require $c_i(x)$ to be continuous, monotone, convex, or concave. Moreover, it includes several comsmonly used discounts: incremental discounts and all-unit discounts. The concave ordering cost analyzed in Chen and Zhang (2009) and the LTL volume discount function (see Muriel and Simchi-Levi 2003) are also important special cases.

Given this general ordering cost structure, unfortunately, the corresponding cooperative game may have an empty core. Indeed, in a special case of the inventory centralization games in which price is not a decision variable, Chen and Zhang (2009) show that for a distribution system with multiple warehouses, the core of the corresponding cooperative game may be empty even if the ordering costs involve only fixed costs and demand is deterministic.

Thus, in this section we focus on inventory centralization games with a single warehouse $(N, V, 1)$. Specifically, we analyze the postponed pricing model with a single warehouse and the nonanticipatory pricing model with a single warehouse and symmetric retailers in two separate subsections. Because we analyze inventory games with a single warehouse, in the following analysis we drop the index associated with the warehouses.

### 5.1. Single Warehouse, Postponed Pricing

In this subsection, we analyze an inventory centralization game $(N, V, 1)$ with postponed pricing. In this case, the value of a coalition $S$ can be defined as

$$\begin{aligned} V(S) = \max\ & -c(y) + g(y, S) \\ \text{s.t.}\ & y \geqslant 0, \end{aligned} \tag{8}$$

where

$$g(y, S) = \mathrm{E}[g_S(y, \omega)] \tag{9}$$

with

$$\begin{aligned} g_S(y, \omega) = \max\ & \sum_{j\in S} g_j(p_j, x_j, \omega) \\ \text{s.t.}\ & y - \sum_{j\in S} x_j = 0, \\ & x_j \geqslant 0, \quad j \in S, \\ & \underline{p}_j \leqslant p_j \leqslant \bar{p}_j, \quad j \in S, \end{aligned}$$

and

$$\begin{aligned} &g_j(p_j, x_j, \omega) \\ &\quad = R_j(p_j, \omega) - f_j(x_j - \beta_j(\omega) + \alpha_j(\omega)p_j) - s_j x_j. \end{aligned}$$

It is clear that, given the general quantity discount function $c(y)$, the objective function of the above optimization problem is neither convex nor concave. Thus, analyzing it directly appears to be quite challenging. To get around this challenge, we construct another inventory centralization game $(N, \widetilde{V})$ with a linear ordering cost, which is known to have a nonempty core, such that $\widetilde{V}(S) \geqslant V(S)$ for any $S \subset N$ and $\widetilde{V}(N) = V(N)$. If this could be done, then we could prove that any element in the core of the game $(N, \widetilde{V})$ is in the core of the game $(N, V)$.

To carry out the idea, we need the following result, Lemma 1, which implies that the bigger a coalition is, the larger the optimal ordering quantity should be. It is appropriate to point out that Lemma 1 involves the monotonicity of optimal solutions for a parameterized collection of optimization problems. A common approach to prove such monotonicity uses the concept of supermodularity. However, it turns out that the concept of supermodularity does not apply here. Indeed, because a newsvendor game (a cost game) is not concave in general (see Ozen et al. 2005), $V(S)$ may not be supermodular. Thus, the function $g(y, S)$ may not be supermodular in $(y, S)$; otherwise, the supermodularity of $g(y, S)$ would imply that $V(S)$ is supermodular in $S$ (see Topkis 1998, Theorem 2.7.6).

It is also important to point out that Lemma 1 is independent of how the retailers' demands are correlated. In addition, this result is true for any ordering cost as long as the relevant quantities are well defined.

Lemma 1. *For any given $S \subset N$, let $y^*(S)$ be the smallest optimal ordering quantity for the postponed pricing model* (8)–(9). *Then, we have $y^*(S_1) \leqslant y^*(S_2)$ for $S_1 \subseteq S_2$.*

Proof. We prove this result by contradiction.

Assume that there exist $S_1, S_2 \subseteq N$ with $S_1 \subset S_2$ such that $y^*(S_1) > y^*(S_2)$. Let $(x_j^1(\omega), p_j^1(\omega))_{j\in S_1}$ be the optimal inventory allocation and pricing associated with the optimal ordering quantity $y^*(S_1)$ for problem (8)–(9) with $S = S_1$. Similarly, let $(x_j^2(\omega), p_j^2(\omega))_{j\in S_2}$ be the optimal inventory allocation and pricing associated with the optimal ordering quantity $y^*(S_2)$ for problem (8)–(9) with $S = S_2$.

The definition of $y^*(S_1)$ and $y^*(S_2)$ implies that

$$\begin{aligned}-c(y^*(S_1)) &+ \sum_{j\in S_1} \mathrm{E}[g_j(p_j^1(\omega), x_j^1(\omega), \omega)]\\ &> -c(y^*(S_2)) + \sum_{j\in S_1} \mathrm{E}[g_j(p_j^3(\omega), x_j^3(\omega), \omega)] \qquad (10)\end{aligned}$$

for any $p_j^3(\cdot) \in P_j^{(p)}$ and $x_j^3(\cdot)$ with

$$y^*(S_2) = \sum_{j\in S_1} x_j^3(\omega) \quad \forall\, \omega \in \Omega. \qquad (11)$$

Similarly,

$$\begin{aligned}-c(y^*(S_2)) &+ \sum_{j\in S_2} \mathrm{E}[g_j(p_j^2(\omega), x_j^2(\omega), \omega)]\\ &\geqslant -c(y^*(S_1)) + \sum_{j\in S_2} \mathrm{E}[g_j(p_j^4(\omega), x_j^4(\omega), \omega)] \qquad (12)\end{aligned}$$

for any $p_j^4(\cdot) \in P_j^{(p)}$ and $x_j^4(\cdot)$ with

$$y^*(S_1) = \sum_{j\in S_2} x_j^4(\omega) \quad \forall\, \omega \in \Omega.$$

Specifically, let

$$p_j^4(\omega) = p_j^2(\omega), \quad x_j^4(\omega) = x_j^2(\omega) \quad \forall\, j \in S_2 \backslash S_1,\ \omega \in \Omega.$$

This, together with inequality (12), implies that

$$\begin{aligned}-c(y^*(S_2)) &+ \sum_{j\in S_1} \mathrm{E}[g_j(p_j^2(\omega), x_j^2(\omega), \omega)]\\ &> -c(y^*(S_1)) + \sum_{j\in S_1} \mathrm{E}[g_j(p_j^4(\omega), x_j^4(\omega), \omega)] \qquad (13)\end{aligned}$$

for any $p_j^4(\cdot) \in P_j^{(p)}$ and $x_j^4(\cdot)$ with

$$y^*(S_1) - y^*(S_2) = \sum_{j\in S_1} (x_j^4(\omega) - x_j^2(\omega)) \quad \forall\, \omega \in \Omega. \qquad (14)$$

Adding the two inequalities (10) and (13) together gives us that

$$\begin{aligned}&\sum_{j\in S_1} \mathrm{E}[g_j(p_j^1(\omega), x_j^1(\omega), \omega) + g_j(p_j^2(\omega), x_j^2(\omega), \omega)]\\ &\quad > \sum_{j\in S_1} \mathrm{E}[g_j(p_j^3(\omega), x_j^3(\omega), \omega) + g_j(p_j^4(\omega), x_j^4(\omega), \omega)] \qquad (15)\end{aligned}$$

for any $p_j^3(\cdot), p_j^4(\cdot) \in P_j^{(p)}$, $(x_j^3(\cdot))_{j\in S_1}$ satisfying (11) and $(x_j^4(\cdot))_{j\in S_1}$ satisfying (14).

Define

$$\lambda(\omega) = \frac{y^*(S_1) - y^*(S_2)}{y^*(S_1) - \sum_{j\in S_1} x_j^2(\omega)}.$$

Because

$$\sum_{j\in S_1} x_j^2(\omega) \leqslant \sum_{j\in S_2} x_j^2(\omega) = y^*(S_2) < y^*(S_1),$$

we have that $\lambda(\omega) \in [0, 1]$. For $j \in S_1$, let

$$x_j^3(\omega) = (1 - \lambda(\omega)) x_j^1(\omega) + \lambda(\omega) x_j^2(\omega),$$

$$p_j^3(\omega) = (1 - \lambda(\omega)) p_j^1(\omega) + \lambda(\omega) p_j^2(\omega),$$

and

$$x_j^4(\omega) = \lambda(\omega) x_j^1(\omega) + (1 - \lambda(\omega)) x_j^2(\omega),$$

$$p_j^4(\omega) = \lambda(\omega) p_j^1(\omega) + (1 - \lambda(\omega)) p_j^2(\omega).$$

It is clear that $(x_j^3(\cdot))_{j\in S_1}$ satisfies (11) and $(x_j^4(\cdot))_{j\in S_1}$ satisfies (14). In addition,

$$x_j^3(\omega) + x_j^4(\omega) = x_j^1(\omega) + x_j^2(\omega)$$

and

$$p_j^3(\omega) + p_j^4(\omega) = p_j^1(\omega) + p_j^2(\omega).$$

Thus, the concavity of the realized revenue function $R_j$ implies that

$$\begin{aligned}R_j(p_j^3(\omega), \omega) &+ R_j(p_j^4(\omega), \omega)\\ &\geqslant R_j(p_j^1(\omega), \omega) + R_j(p_j^2(\omega), \omega),\end{aligned}$$

and the convexity of $f_j$ implies that

$$\begin{aligned}&-f_j(x_j^3(\omega)-\beta_j(\omega)+\alpha_j(\omega)p_j^3(\omega))\\&\quad -f_j(x_j^4(\omega)-\beta_j(\omega)+\alpha_j(\omega)p_j^4(\omega))\\&\qquad \geqslant -f_j(x_j^1(\omega)-\beta_j(\omega)+\alpha_j(\omega)p_j^1(\omega))\\&\qquad\quad -f_j(x_j^2(\omega)-\beta_j(\omega)+\alpha_j(\omega)p_j^2(\omega)).\end{aligned}$$

Adding the two inequalities together and taking expectation with respect to $\omega$ gives us an inequality that contradicts inequality (15). Thus, $y^*(S_1)\leqslant y^*(S_2)$. □

We now prove that we can construct a new inventory centralization game $(N,\widetilde{V},1)$ with linear ordering cost such that for any $S\subset N$, $\widetilde{V}(S)\geqslant V(S)$, whereas $\widetilde{V}(N)=V(N)$ for the postponed pricing model (8)–(9). For this purpose, we define for any given scalar $\hat{c}\geqslant 0$ an inventory centralization game $(N,V_{\hat{c}},1)$ with the ordering cost being $\hat{c}x$. In this game, for any $S\subseteq N$,

$$\begin{aligned}V_{\hat{c}}(S)=\max\ & -\hat{c}y+g(y,S)\\ \text{s.t.}\ & y\geqslant 0.\end{aligned}$$

Lemma 2. *There exists a scalar $\hat{c}^*$ such that for any $S\subset N$, $V_{\hat{c}^*}(S)\geqslant V(S)$ and $V_{\hat{c}^*}(N)=V(N)$.*

Proof. We consider two cases. First, assume that zero is an optimal solution for problem (8) (together with (9)) for the grand coalition, i.e., $S=N$ in (8). Lemma 1 implies that zero is an optimal solution for problem (8) for any $S\subseteq N$. On the other hand, if we choose a sufficiently large $\hat{c}^*$, say $\hat{c}^*\geqslant \max_{j\in N}\max\{\bar{p}_j,q_j\}$, it is easy to see that zero is also an optimal solution for problem $\max_{y\geqslant 0}-\hat{c}^*y+g(y,S)$. In this case, $V_{\hat{c}^*}(S)=V(S)$ for any $S\subseteq N$.

We now assume that zero is not an optimal solution for problem (8) for the grand coalition. Let $y^*$ be an optimal solution for problem (8) for the grand coalition, i.e., $S=N$ in (8). Upon denoting $c^*=c(y^*)/y^*$, we have that

$$\begin{aligned}V(N)&=-c(y^*)+g(y^*,N)\\&=-c^*y^*+g(y^*,N)\\&\leqslant \max_{y\geqslant 0}-c^*y+g(y,N)\\&=V_{c^*}(N).\end{aligned}$$

On the other hand, because we assume that zero is not an optimal solution for problem (8) for the grand coalition, we have that

$$\begin{aligned}V(N)&=\max_{y\geqslant 0}-c(y)+g(y,N)\\&>g(0,N)\\&=\lim_{\hat{c}\to\infty}V_{\hat{c}}(N).\end{aligned}$$

The continuity of $V_{\hat{c}}(N)$ as a function of $\hat{c}$, together with the above two inequalities, implies that there exists a $\hat{c}^*$ such that $V(N)=V_{\hat{c}^*}(N)$.

Define $\hat{x}=\sup\{x\geqslant 0\colon c(x)/x\geqslant \hat{c}^*\}$. Let $\hat{y}^*$ be the smallest optimal solution for the problem $\min_{y\geqslant 0}-\hat{c}^*y+g(y,N)$. We claim that $\hat{y}^*\leqslant\hat{x}$.

Assume to the contrary that $\hat{y}^*>\hat{x}$. The definition of $\hat{x}$ together with the monotonicity of $c(x)/x$ implies that $c(\hat{y}^*)/\hat{y}^*<\hat{c}^*$. Thus,

$$\begin{aligned}V_{\hat{c}^*}(N)&=-\hat{c}^*\hat{y}^*+g(\hat{y}^*,N)\\&<-c(\hat{y}^*)+g(\hat{y}^*,N)\\&\leqslant -c(y^*)+g(y^*,N)\\&=V(N),\end{aligned}$$

which contradicts the fact that $V(N)=V_{\hat{c}^*}(N)$. Thus, $\hat{y}^*\leqslant\hat{x}$.

Define a new function $\tilde{c}(x)$ as follows:

$$\tilde{c}(x)=\begin{cases}\hat{c}^*x & \text{for } 0\leqslant x<\hat{x},\\ c(x) & \text{otherwise.}\end{cases}$$

The following properties of $\tilde{c}(x)$ will be useful for our analysis. First, $\tilde{c}(x)\leqslant c(x)$ for any $x$. This follows directly from the monotonicity of $c(x)/x$. Second, $\tilde{c}(x)$ preserves the lower semicontinuity of $c(x)$. To show this, it suffices to prove that $\tilde{c}(x)$ is lower semicontinuous at $x=\hat{x}$. Note that

$$\liminf_{y\to\hat{x}+}\tilde{c}(y)=\liminf_{y\to\hat{x}+}c(y)\geqslant c(\hat{x})=\tilde{c}(\hat{x}),$$

whereas

$$\liminf_{y\to\hat{x}-}\tilde{c}(y)=\hat{c}^*\hat{x}\geqslant\liminf_{y\to\hat{x}+}\hat{x}c(y)/y\geqslant c(\hat{x})=\tilde{c}(\hat{x}),\qquad(16)$$

where the first inequality and the second inequality follow from the definition of $\hat{x}$ and the lower semicontinuity of $c(x)$, respectively. Note that (16) implies that

$$\tilde{c}(y)\leqslant\hat{c}^*y\quad\text{for any } 0\leqslant y\leqslant\hat{x}.\qquad(17)$$

Third, $\hat{y}^*$ is also optimal for the problem $\max_{y\geqslant 0}-\tilde{c}(y)+g(y,N)$. Indeed, the definition of $\hat{y}^*$, together with the fact $\hat{y}^*\leqslant\hat{x}$, implies that for any $0\leqslant y<\hat{x}$,

$$\begin{aligned}-\tilde{c}(\hat{y}^*)+g(\hat{y}^*,N)&\geqslant -\hat{c}^*\hat{y}^*+g(\hat{y}^*,N)\\&\geqslant -\hat{c}^*y+g(y,N)\\&=-\tilde{c}(y)+g(y,N),\end{aligned}$$

where the first inequality follows from (17). For $y\geqslant\hat{x}$,

$$\begin{aligned}&-\tilde{c}(\hat{y}^*)+g(\hat{y}^*,N)\geqslant -\hat{c}^*\hat{y}^*+g(\hat{y}^*,N)\\&\quad=-c(y^*)+g(y^*,N)\geqslant -c(y)+g(y,N)\\&\quad=-\tilde{c}(y)+g(y,N),\end{aligned}$$

where the first equality follows from the definition of $\hat{y}^*$ and $\hat{c}^*$. Thus, $\hat{y}^*$ is also optimal for the problem $\max_{y\geqslant 0}-\tilde{c}(y)+g(y,N)$.

We are now ready to prove that for any $S \subset N$, $V_{\hat{c}^*}(S) \geqslant V(S)$. Let $\tilde{y}^*(S)$ be the smallest optimal solution for the problem $\max_{y\geqslant 0} -\tilde{c}(y) + g(y, S)$. Note that $\tilde{c}(x)$ is lower semicontinuous. Hence, $\tilde{y}^*(S)$ is well defined. Lemma 1 implies that for any $S \subset N$, $\tilde{y}^*(S) \leqslant \hat{y}^* \leqslant \hat{x}$.

We claim that

$$\hat{c}^* \tilde{y}^*(S) = \tilde{c}(\tilde{y}^*(S)). \tag{18}$$

Indeed, if $\tilde{y}^*(S) < \hat{x}$, we have from the definition of $\tilde{c}(\cdot)$ that $\hat{c}^* \tilde{y}^*(S) = \tilde{c}(\tilde{y}^*(S))$. On the other hand, if $\tilde{y}^*(S) = \hat{x}$ and $\hat{c}^* \tilde{y}^*(S) > \tilde{c}(\tilde{y}^*(S)) = c(\tilde{y}^*(S))$, we have that $\hat{y}^* = \tilde{y}^*(S)$ and

$$V_{c^*}(N) = -\hat{c}^* \hat{y}^* + g(\hat{y}^*, N) < -c(\hat{y}^*) + g(\hat{y}^*, N) \leqslant V(N),$$

which is a contradiction. Thus, in this case, (18) follows from (17).

Finally, we have that

$$\begin{aligned}
V_{\hat{c}^*}(S) &= \max_{y\geqslant 0} -\hat{c}^* y + g(y, S) \\
&\geqslant -\hat{c}^* \tilde{y}^*(S) + g(\tilde{y}^*(S), S) \\
&= -\tilde{c}(\tilde{y}^*(S)) + g(\tilde{y}^*(S), S) \\
&= \max_{y\geqslant 0} -\tilde{c}(y) + g(y, S) \\
&\geqslant \max_{y\geqslant 0} -c(y) + g(y, S) \\
&= V(S),
\end{aligned}$$

where the second equality follows from (18) and the last inequality from the fact that $\tilde{c}(y) \leqslant c(y)$ for any $y \geqslant 0$.

The proof is now complete. $\square$

We are now ready to present the main result of this paper.

Theorem 2. *Under Assumption* 3, *the inventory centralization game* $(N, V, 1)$ *with the characteristic value function defined by* (8)–(9) *has a nonempty core. Let* $(N, V_{\hat{c}^*}, 1)$ *be the inventory centralization game with marginal ordering cost* $\hat{c}^*$, *where* $\hat{c}^*$ *is defined in Lemma* 2. *Then, any element in the core of* $(N, V_{\hat{c}^*}, 1)$ *is also in the core of* $(N, V, 1)$.

Proof. The proof is straightforward. Let $l = (l_j)_{j\in N}$ be an element in the core of $(N, V_{\hat{c}^*}, 1)$. We have that for any $S \subseteq N$,

$$\sum_{j\in S} l_j \geqslant V_{\hat{c}^*}(S) \geqslant V(S).$$

In addition,

$$\sum_{j\in N} l_j = V_{\hat{c}^*}(N) = V(N).$$

Hence, $l = (l_j)_{j\in N}$ is also in the core of $(N, V, 1)$. Because $(N, V_{\hat{c}^*}, 1)$ is an inventory centralization game with a liner ordering cost, Theorem 1 implies that it has a nonempty core. Thus, $(N, V, 1)$ has a nonempty core as well. $\square$

### 5.2. Single Warehouse, Nonanticipatory Pricing, Symmetric Retailers

In this subsection, we analyze the inventory centralization game $(N, V, 1)$ with nonanticipative pricing. Unfortunately, it is not clear whether Lemma 1, which plays a key role in the analysis for the game $(N, V, 1)$ with postponed pricing, can be extended to the game $(N, V, 1)$ with nonanticipative pricing. Note that in the proof of Lemma 1, we construct two price vectors defined by $p_j^3(\omega) = (1-\lambda(\omega))p_j^1(\omega) + \lambda(\omega)p_j^2(\omega)$ and $p_j^4(\omega) = \lambda(\omega)p_j^1(\omega) + (1-\lambda(\omega))p_j^2(\omega)$ for $j \in S_1$ to derive a contradiction. For the nonanticipative pricing model, the feasible prices need to be independent of the realization of $\omega$. However, even if $p_j^1(\omega)$ and $p_j^2(\omega)$ are independent of $\omega$, the way that $p_j^3(\omega)$ and $p_j^4(\omega)$ are constructed in the proof of Lemma 1 cannot guarantee that they would be independent of $\omega$ because $\lambda(\omega)$ may depend on $\omega$. Thus, the proof of Lemma 1 may not be easily extended to the nonanticipative pricing model, which prohibits us from extending Lemma 1 and thus Lemma 2, as well as Theorem 2, to the general nonanticipative pricing model.

Interestingly, Lemma 1 can be extended to the special case of the nonanticipative pricing model with symmetric retailers. In this case, the retailers are assumed to have identical cost parameters, i.e., $s_j = s$, $h_j = h$, $q_j = q$ for $j \in N$. Under this assumption, the nonanticipative pricing model is equivalent to the case that the aggregate demand of the retailers will be directly satisfied by the inventory from the central warehouse without explicitly allocating the inventory to the retailers. Thus, in this case, the value of a coalition $S$ can be defined by (8) with

$$\begin{aligned}
g(y, S) = \max\quad & w_S(y, p) \\
\text{s.t.}\quad & \underline{p}_j \leqslant p_j \leqslant \bar{p}_j, \quad j \in S,
\end{aligned} \tag{19}$$

where

$$w_S(y, p) = \sum_{j\in S} \mathrm{E}[R_j(p_j, \omega)] - \mathrm{E}\left[f\left(y - \sum_{j\in S}\beta_j(\omega) + \sum_{j\in S}\alpha_j(\omega)p_j\right)\right] - sy$$

and $f(\chi) = h\chi^+ + q(-\chi)^+$.

We can show that Lemma 1, the existence of monotone optimal solutions, can be extended to the game $(N, V, 1)$ with nonanticipative pricing and symmetric retailers with $V$ defined by (8) and (19) (see the appendix for the proof). Note that Lemma 2 built upon the monotonicity of the optimal solutions is independent of how $g(y, S)$ is defined. Hence, Lemma 2, and thus Theorem 2, can be directly extended with no modification in their proofs, which immediately implies that the core of the game $(N, V, 1)$ with nonanticipative pricing and symmetric retailers is nonempty.

## 6. Calculating an Allocation in the Core for the Game $(N, V, 1)$

Our approach to prove the nonemptiness of the core of a cooperative game $(N, V, 1)$ with quantity discount suggests a way to find an allocation in the core in three steps. First, solve

$$V(N) = \max_{y\geqslant 0} -c(y) + g(y, N).$$

Second, given $V(N)$, find a $\hat{c}^*$ such that

$$V(N)=\max_{y\geqslant 0} -\hat{c}^* y+g(y,N).$$

Third, find an allocation in the core of the inventory centralization game $(N, V_{\hat{c}^*}, 1)$ with linear ordering cost by employing the duality approach in §4. Theorem 2 implies that this allocation is in the core of $(N, V, 1)$.

We now give a detailed description of the algorithm.

Algorithm 1. *Step* 0. Set the tolerance $\epsilon > 0$.

*Step* 1. Solve

$$V(N)=\max_{y\geqslant 0} -c(y)+g(y,N),$$

where $g(y,N)$ is defined by (9) or (19) depending on whether it is a postponed pricing model or a nonanticipative pricing model. Let $y^*$ be an optimal solution.

If $y^*=0$ is optimal for the above problem, let $\hat{c}^* = \max_{j\in N}\max\{\bar{p}_j, q_j\}$ and go to Step 3. Otherwise, let $\underline{c} = c(y^*)/y^*$ and $\bar{c}=\max_{j\in N}\max\{\bar{p}_j, q_j\}$.

*Step* 2. Let $\hat{c}=(\underline{c}+\bar{c})/2$. Solve

$$V_{\hat{c}}(N)=\max_{y\geqslant 0} -\hat{c}y+g(y,N).$$

If $V_{\hat{c}}(N)=V(N)$ or $\bar{c}-\underline{c}<\epsilon$, let $\hat{c}=\hat{c}$ and go to Step 3. If $V_{\hat{c}}(N)<V(N)$, $\bar{c}=\hat{c}$; if $V_{\hat{c}}(N)>V(N)$, $\underline{c}=\hat{c}$. Go to Step 2.

*Step* 3. Solve the dual problem to get the optimal solution $\mu^*$:

$$\begin{aligned}\min\ & \sum_{j\in N}\gamma_j(\mu_j)\\ \text{s.t. } & \mathrm{E}[\lambda(\omega)]=\hat{c},\\ & \mu_j(\omega)\leqslant \lambda(\omega)+s_j, \quad j\in N,\ \omega\in\Omega,\end{aligned}$$

where

$$\begin{aligned}\gamma_j(\mu_j)=\sup\ & \psi_j(p_j, v_j, \mu_j)\\ \text{s.t. } & p_j\in P_j,\end{aligned}$$

and $\psi_j$ is defined in (3).

Define

$$l_j=\gamma_j(\mu_j^*).$$

The vector $l=(l_1, l_2, \ldots, l_n)$ gives an approximate core allocation of $(N, V, 1)$.

We now make some comments about the algorithm. In Step 1, for general quantity discounts, the objective function does not have a nice concavity property, and thus there may not exist efficient algorithms. However, the commonly used quantity discounts such as all-units discount, incremental discount, and LTL cost structure are all piecewise linear. Thus, we may solve the above optimization problem for each linear piece and then find the highest optimal objective values derived from all pieces. To be more precise, let $0=\tau_1<\tau_2<\cdots<\tau_k<\tau_{k+1}=\infty$ for some integer $k$ and assume that

$$c(y)=b_i+c_i y \quad \forall\, y\in[\tau_i, \tau_{i+1}),\ i=1,2,\ldots,k,$$

for scalars $b_i$ and $c_i$ with $b_{i+1}+c_{i+1}\tau_{i+1}\leqslant b_i+c_i\tau_{i+1}$ to guarantee the lower semicontinuity of $c(y)$. Then,

$$V(N)=\max_{i=1,\ldots,k}\ \max_{\tau_i\leqslant y\leqslant\tau_{i+1}} -(b_i+c_i y)+g(y,N).$$

Note that the inner optimization is a concave maximization problem, which can be solved efficiently.

In Step 2, for each $\hat{c}$ derived from the binary search, we solve a concave maximization problem, which again can be solved efficiently. In Step 3, to derive the dual, we use equalities (6) and (7) to simplify the formulation derived in §4. The dual can be solved by a variety of dual methods for convex minimization problems (see Bertsakas 1995). Finally, the allocation $l$ is an approximate core allocation in the sense that $|\sum_{j\in N} l_j - V(N)|\leqslant\delta$ and $\sum_{j\in S} l_j\leqslant V(S)+\delta$ for $S\subset N$ for some small positive scalar $\delta$ that depends on $\epsilon$, and the accuracy of solving the dual in Step 3.

We now illustrate the above algorithm through a simple numerical example. Specifically, we assume that $\beta_j(\omega)$ and $\alpha_j(\omega)$, $j\in N$, are deterministic. In addition, $\bar{p}_j=\beta_j/\alpha_j$ and $\underline{p}_j=0$. The ordering cost is given by an all-units discount with unit ordering cost $c_1$ for order quantity below $\tau$ and unit ordering cost $c_2$ for order quantity at or above $\tau$. We assume that $c_1>c_2+\epsilon/2>c_2>0$, $c_2\leqslant\beta_j/\alpha_j-\epsilon/2$ for $j\in N$, and $\tau\leqslant\sum_{j\in N}(\beta_j-\alpha_j p_j^0)$ with $p_j^0=(\beta_j+\alpha_j c_2)/(2\alpha_j)$. Choose $h_j$ and $q_j$ significantly larger than $c_1$, $c_2$, and $\beta_j$, which implies that at optimality the order quantity equals the total demand.

We are now ready to carry out the above algorithm. In Step 1, it is easy to show that

$$V(N)=\sum_{j\in N}\frac{(\beta_j-\alpha_j c_2)^2}{4\alpha_j}.$$

In this case, the total ordering quantity is exactly $\tau$ and the optimal price of retailer $j$ is given by $p_j^0$. In Step 2, even though our choice of $\tau$ immediately implies that $\hat{c}^*=c_2$, our algorithm ends up with a $\hat{c}$ such that $\hat{c}\in(c_2, c_2+\epsilon/2)$. In Step 3,

$$\begin{aligned}\gamma_j(\mu_j)=\max_{0\leqslant p_j\leqslant\beta_j/\alpha_j,\, v_j}\ & p_j(\beta_j-\alpha_j p_j)\\ & -f_j(v_j-(\beta_j-\alpha_j p_j))-\mu_j v_j.\end{aligned}$$

Because $h_j$ and $q_j$ are sufficiently large, we have that at optimality, $v_j=\beta_j-\alpha_j p_j$. Through careful computation, we have that

$$\gamma_j(\mu_j)=\begin{cases}\dfrac{(\beta_j-\alpha_j\mu_j)^2}{4\alpha_j} & \text{if } |\alpha_j\mu_j|\leqslant\beta_j,\\ -\beta_j\mu_j & \text{if } \alpha_j\mu_j\leqslant-\beta_j,\\ 0 & \text{otherwise.}\end{cases}$$

Finally, solving $\min_{\mu_j \leqslant \hat{c}} \gamma_j(\mu_j)$ gives $\mu_j^* = \hat{c}$. Thus, the vector $l = (l_1, l_2, \ldots, l_n)$ with $l_j = (\beta_j - \alpha_j \hat{c})^2/(4\alpha_j)$ gives an approximate core allocation of $(N, V, 1)$.

Now let $N = \{1, 2, 3\}$, $c_1 = 2$, $c_2 = 3$, $\tau = 5$, $\beta_1 = 5$, $\beta_2 = 7$, $\beta_3 = 9$, and $\alpha_j = 1$, $j = 1, 2, 3$. Let $\epsilon$ be a very small positive number. Then, $\hat{c}$ would be very close to $c_2 = 1$. In this case, our algorithm computes an allocation that approximates the core allocation $l = (l_1, l_2, l_3) = (4, 9, 16)$. We can also compute the characteristic value function $V$: $V(\{1\}) = 1$, $V(\{2\}) = 5$, $V(\{3\}) = 15$, $V(\{1, 2\}) = 13$, $V(\{1, 3\}) = 20$, $V(\{2, 3\}) = 25$, and $V(\{1, 2, 3\}) = 29$. Observe that by placing orders together, all retailers receive profits higher than what can be achieved by acting alone ($l_j > V(\{j\})$). However, a retailer with a smaller market size (smaller $\beta_j$) benefits more (higher $l_j/V(\{j\})$ from the cooperation. In this specific example, one can also see that $l$ is the unique element of the core because $V(\{1, 2\}) + V(\{1, 3\}) + V(\{2, 3\}) = 2V(\{1, 2, 3\})$.

## 7. Concluding Remarks

In this paper, we analyze inventory centralization games with price-dependent demand. We employ convex programming duality theory to show that the core of an inventory centralization game with price-dependent demand is nonempty when the ordering cost is linear. This observation is true if the pricing decisions are made either before or after observing the market signal. The duality approach presented in this paper presents a mechanism to find an allocation in the core.

Under the assumption that the replenishment is made through a single warehouse, we show that the core is nonempty for an inventory centralization game with quantity discount based on either the nonanticipative pricing model with identical cost parameters for all retailers or the postponed pricing model. In addition, our proof also suggests a procedure to find an allocation in the core. This is quite significant given the generality of the ordering cost function.

To prove the nonemptiness of an inventory centralization game with a general quantity discount, we construct another inventory game with linear ordering cost. In the construction, we keep the maximal profit of the grand coalition unchanged. Meanwhile, the maximal profit for each coalition does not decrease. This construction allows one to claim that any allocation in the core of the newly constructed inventory game is in the core of the inventory game with quantity discount. We expect that this technique may be applicable to other settings beyond inventory centralization games.

It would be interesting to extend our model, results, and analysis to more general settings. First, our model can be easily extended to handle the three-stage inventory centralization game with demand update proposed in Ozen and Sošić (2006). Specifically, all our results and analysis hold with minor modification for the following three-stage inventory centralization game. In this game, the demand function is given by $\beta(\xi) - \alpha(\xi)p$, where $\xi$ is a random variable different from $\omega$. At the first stage, ordering decisions are made. At the second stage, the retailers can collect market information and receive a market signal $\omega$. The retailers then use the market signal to refine their estimation of the random variable $\xi$. At the last stage, after the revelation of the market signal but before the realization of the random variable $\xi$, inventory is allocated to the retailers. Pricing decision can be made either at the second stage before the revelation of the market signal or at the third stage after the revelation of the market signal but before the realization of $\xi$, which gives the nonanticipative pricing model and the postponed pricing model correspondingly. It is appropriate to point out that the three-stage inventory centralization game analyzed in Ozen and Sošić (2006) does not incorporate pricing decision and quantity discount.

Second, extending our model to incorporate competing retailers is appealing and significantly more complicated. Indeed, in this case, even defining the value of a coalition is tricky because the demand of each retailer depends on the pricing decisions of all retailers, no matter whether they are in the same coalition or not. To overcome this difficulty, it might be reasonable to analyze the formation of coalition structures. For related work, see Nagarajan and Sošić (2007).

Third, as retailers band together to place joint orders and share inventory at central warehouses, it is likely that the handling cost at the central warehouses would increase. How the cost increase impacts the collaboration among the retailers depends on the cost structure and is an interesting question for future exploration.

Finally, we would like to extend our model to multiple-period settings. If the ordering costs are linear, we can similarly employ the convex programming duality approach used in §4 to show the nonemptiness of the core. Even though we conjecture that multiple-period inventory centralization games with general quantity discounts may have empty cores, it is possible that the core of a multiple-period inventory centralization game with concave ordering cost is nonempty. For some initial attempt along this direction, we refer to Chen and Zhang (2006), who analyze the economic lot-sizing game with concave ordering cost and deterministic demand.

## Appendix

### Proof of the Extension of Lemma 1 to the Case with Nonanticipatory Pricing and Symmetric Retailers

Proof. By contradiction. Assume to the contrary that there exist $S_1, S_2 \subseteq N$ with $S_1 \subset S_2$ such that $y^*(S_1) > y^*(S_2)$. Let $(p_j^1)_{j \in S_1}$ be the optimal pricing associated with the optimal ordering quantity $y^*(S_1)$ for problem (8) and (19) with $S = S_1$. Similarly, let $(p_j^2)_{j \in S_2}$ be the optimal pricing associated with the optimal ordering quantity $y^*(S_2)$ for problem (8) and (19) with $S = S_2$.

It is clear that

$$-c(y^*(S_1))+w_{S_1}(y^*(S_1),p^1) > -c(y^*(S_2))+w_{S_1}(y^*(S_2),p^1)$$

and

$$-c(y^*(S_2))+w_{S_2}(y^*(S_2),p^2) \geqslant -c(y^*(S_1))+w_{S_2}(y^*(S_1),p^2).$$

Adding the two inequalities together, we have that

$$\begin{aligned}
&\mathrm{E}\Bigg[f\Bigg(y^*(S_1)-\sum_{j\in S_1}\beta_j(\omega)+\sum_{j\in S_1}\alpha_j(\omega)p_j^1\Bigg)\Bigg] \\
&\quad+\mathrm{E}\Bigg[f\Bigg(y^*(S_2)-\sum_{j\in S_2}\beta_j(\omega)+\sum_{j\in S_2}\alpha_j(\omega)p_j^2\Bigg)\Bigg] \\
&<\mathrm{E}\Bigg[f\Bigg(y^*(S_2)-\sum_{j\in S_1}\beta_j(\omega)+\sum_{j\in S_1}\alpha_j(\omega)p_j^1\Bigg)\Bigg] \\
&\quad+\mathrm{E}\Bigg[f\Bigg(y^*(S_1)-\sum_{j\in S_2}\beta_j(\omega)+\sum_{j\in S_2}\alpha_j(\omega)p_j^2\Bigg)\Bigg].
\end{aligned}$$

Because $f$ is convex, the above inequality contradicts the fact that

$$\begin{aligned}
&f\Bigg(y^*(S_1)-\sum_{j\in S_1}\beta_j(\omega)+\sum_{j\in S_1}\alpha_j(\omega)p_j^1\Bigg) \\
&\quad-f\Bigg(y^*(S_2)-\sum_{j\in S_1}\beta_j(\omega)+\sum_{j\in S_1}\alpha_j(\omega)p_j^1\Bigg) \\
&\geqslant f\Bigg(y^*(S_1)-\sum_{j\in S_2}\beta_j(\omega)+\sum_{j\in S_2}\alpha_j(\omega)p_j^2\Bigg) \\
&\quad-f\Bigg(y^*(S_2)-\sum_{j\in S_2}\beta_j(\omega)+\sum_{j\in S_2}\alpha_j(\omega)p_j^2\Bigg).
\end{aligned}$$

Thus, for the nonanticipative pricing model under the simplified conditions (8) and (19), we have $y^*(S_1)\leqslant y^*(S_2)$. □

## Endnotes

1. Similar assumptions are made in Simchi-Levi et al. (2004). It is appropriate to point out that relaxing this assumption by allowing lost sales imposes a significant challenge because it destroys the concavity property of the expected revenue function needed for our analysis in this paper.
2. In the case where pricing is not a decision, it is straightforward to verify that a single-period lost-sales model can be equivalently reformulated as a model with unsatisfied demand being filled by an emergency order. Thus, the inventory centralization games analyzed in the literature, which assume that pricing is not a decision and unsatisfied demand is lost, are special cases of the model analyzed in this paper.
3. In our assumption, demand is linear in price, which is widely adopted in the literature. However, it is appropriate to point out that this assumption can be relaxed. Indeed, following Chen and Simchi-Levi (2004a, b), we may analyze demand functions of the following form:

$$\tilde{d}_j = D_j(p_j,\omega) := \beta_j(\omega)-\alpha_j(\omega)\phi_j(p_j),$$

where $\phi_j(\cdot)$ is a strictly increasing function. Of course, to have a tractable model, certain technical conditions are needed to guarantee the concavity of the revenue function (see Simchi-Levi et al. 2004 for details). For simplicity, we focus on the linear demand model in Assumption 2.

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