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Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Infinite Horizon Case

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We analyze an infinite horizon, single-product, periodic review model in which pricing and production/inventory decisions are made simultaneously. Demands in different periods are identically distributed random variables that are independent of each other, and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period, and all shortages are backlogged. Ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. The objective is to maximize expected discounted, or expected average, profit over the infinite planning horizon. We show that a stationary (s, S, p) policy is optimal for both the discounted and average profit models with general demand functions. In such a policy, the period inventory is managed based on the classical (s, S) policy, and price is determined based on the inventory position at the beginning of each period.

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1. Introduction. In recent years, scores of retail and manufacturing companies have started exploring innovative pricing strategies in an effort to improve their operations and, ultimately, the bottom line. Firms are employing methods such as dynamically adjusting price over time based on inventory levels or production schedules, as well as segmenting customers based on their sensitivity to price and lead time.

For instance, no company underscores the impact of the Internet on product pricing strategies more than Dell Computers. The exact same product is sold at different prices on Dell's Web site, depending on whether the purchase is made by a private consumer, a small, medium or large business, the federal government; or an education or health care provider. A more careful review of Dell's strategy (see Agrawal and Kambil 2000) suggests that even the price of the same product for the same industry is not fixed; it may change significantly over time.

Dell is not alone in its use of a sophisticated pricing strategy. Consider:

- Boise Cascade Office Products sells many products online. Boise Cascade states that prices for the 12,000 items ordered most frequently online might change as often as daily (Kay 1998).
- Ford Motor Co. uses pricing strategies to match supply and demand and target particular customer segments. Ford executives credit the effort with $3 billion in growth between 1995 and 1999 (Leibs 2000).

These developments call for models that integrate production decisions, inventory control, and pricing strategies. Such models and strategies have the potential to radically improve supply chain efficiencies in much the same way as revenue management has changed the airline industry; see Belobaba (1987) or McGill and van Ryzin (1999). Indeed, in the airline industry, revenue management provided growth and increased revenue by 5%, see Belobaba (1987). In fact, if it were not for the combined contributions of revenue management and airline schedule planning systems, American Airlines (Cook 2000) would have been profitable only one year in the decade beginning in 1990. In the retail industry, to name another example, dynamically pricing commodities can provide significant improvements in profitability, as shown by Gallego and van Ryzin (1994).

The coordination of replenishment strategies and pricing policies has been the focus of many papers, starting with the work of Whitin (1955), who analyzed the celebrated newsvendor problem with price-dependent demand. For a review, the reader is referred to Eliashberg and Steinberg (1991), Petruzzi and Dada (1999), Federgruen and Heching (1999), Yano and Gilbert (2002), Elmaghraby and Keskinocak (2003), or Chan et al. (2004).

Recently, Chen and Simchi-Levi (2004) considered a finite horizon, periodic review, single-product model with stochastic demand. Demands in different periods are independent of each other and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period, and all shortages are backlogged. The ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. Inventory holding and shortage costs are convex functions of the inventory level carried over from one period to the next. The objective is to find an inventory policy and pricing strategy maximizing expected profit over the finite horizon.

Chen and Simchi-Levi (2004) proved that when the demand process is additive, i.e., the demand process has two components, a deterministic part which is a function of the price and an additive random perturbation, an (s, S, p) policy is optimal. In such a policy the inventory strategy is an (s, S) policy: If the inventory level at the beginning of period t is below the reorder point, s_t, an order is placed to raise the inventory level to the order-up-to level, S_t. Otherwise, no order is placed. Price depends on the initial inventory level at the beginning of the period. Unfortunately, for general demand models, including multiplicative demand processes, Chen and Simchi-Levi showed that the (s, S, p) policy is not necessarily optimal. To characterize the optimal policy in this case, Chen and Simchi-Levi developed a new concept, the symmetric k-convexity, and employed it to prove that for general demand processes, an (s, S, A, p) policy is optimal. In such a policy, the optimal inventory strategy at period t is characterized by two parameters (s_t, S_t) and a set $A_t \in [s_t, (s_t+S_t)/2]$, possibly empty depending on the problem instance. When the inventory level x_t at the beginning of period t is less than s_t or $x_t \in A_t$, an order of size $S_t - x_t$ is made. Otherwise, no order is placed. Price depends on the initial inventory level at the beginning of the period.

In this paper we analyze the corresponding infinite horizon models under both the discounted and average profit criteria. We make assumptions similar to those in Chen and Simchi-Levi (2004), except that here all input parameters—i.e., demand processes, costs, and revenue functions—are assumed to be time independent. Surprisingly, by employing the symmetric k-convexity concept developed in Chen and Simchi-Levi (2004), we establish that a stationary (s, S, p) policy is optimal for both additive demand and general demand processes under the discounted and average profit criteria. Our approach is motivated by the classic papers by Iglehart (1963a, b), Veinott (1966), and Zheng (1991).

Unfortunately, the analysis of an infinite horizon dynamic program is quite difficult in general, since it usually involves the convergence of a sequence of finite horizon problems. Observe that in the case of the standard stochastic inventory problem, it is natural to expect that a stationary (s, S) policy is optimal for the infinite horizon model since an (s, S) policy is optimal for its finite horizon counterpart. Thus, an approach that involves the convergence of a sequence of finite horizon problems seems natural. This is exactly the approach applied by Iglehart (1963a) for the discounted cost case. However, this approach does not apply for the infinite horizon joint inventory and pricing model, since the optimal policy of the finite horizon would suggest the optimality of a stationary (s, S, A, p) policy.

A different approach for proving the optimality of a stationary (s, S) policy for the infinite horizon inventory control model was proposed by Zheng (1991) and is based on characterizing properties of the best (s, S) policy. Unfortunately, while his novel approach for the average case can be extended to analyze our model, his approach does not seem to be appropriate for the discounted case. We discuss this issue in §6 and §9.

Our proof of the optimality of (s, S, p) for the infinite horizon joint inventory and pricing model is essentially the same for both the discounted and average profit criteria.

It involves: (1) Characterizing the best stationary (s, S, p) policy. (2) Proving that the infinite horizon expected (discounted) profit function associated with the best stationary (s, S, p) policy solves the optimality equation and showing that the best (s, S, p) policy achieves the maximization in the optimality equation based on the concept of symmetric k-concavity. (3) Employing the optimality of an (s_t, S_t, A_t, p_t) policy for the finite horizon joint inventory and pricing model and arguing that optimal parameters s_t and S_t are bounded by employing the technique proposed by Veinott (1966). (4) Using these bounds and the optimality equation to prove that the infinite horizon profit function associated with the best stationary (s, S, p) policy is the maximum profit function and the best stationary (s, S, p) policy solves the joint inventory and pricing model.

Thus, in essence, our approach is similar to the one applied by Iglehart (1963b) for the average cost criterion. Of course, the pricing dimension in our model complicates the analysis considerably. Also, unlike the traditional infinite horizon stochastic inventory problem, the lack of a simple explicit expression for the profit function associated with a given stationary (s, S) policy and its corresponding optimal pricing strategy requires the application of recursive arguments in our proof. This significantly increases the complexity of the analysis. Finally, our proof employs the concept of symmetric k-concavity, while Iglehart (1963a, b) used the concept of k-convexity.

To put this research in perspective, we point out that our model is similar to the one proposed by Thomas (1974), who conjectures that an (s, S, p) policy is optimal for the finite horizon case under fairly general conditions. Polatoglu and Sahin (2000) also analyze a similar model in which unsatisfied demand is assumed to be lost. They show that an (s, S, p) policy is not optimal in general and they provide some conditions for the optimality of an (s, S, p) policy. These papers, as well as the paper by Chen and Simchi-Levi (2004), focus on periodic review models, while Feng and Chen (2002) consider an infinite horizon continuous review model under the average profit criterion in which the *interarrival time* is assumed to be *exponential* and a function of the selling price. Prices are restricted to a discrete set, and demand is assumed to be of unit size. For this model, the authors show that an (s, S, p) policy is optimal and provide the structure of the selling price. Their model and results are subsequently generalized by Chen and Simchi-Levi (2003). In fact, by employing an approach similar to the one used in this paper, Chen and Simchi-Levi (2003) show that an (s, S, p) policy is optimal for the infinite horizon continuous review model under both the discounted profit and the average profit criteria. In particular, the demand process may be price dependent and is quite general.

The paper is organized as follows. In §2 we review the main assumptions of our model and the concepts of k-convexity and symmetric k-convexity. In §3, we define some notation and show how one can find the best-pricing strategies for a given (s, S) inventory policy. We start §4 by identifying properties of the best (s, S) inventory policy for both the discounted and average profit cases. These properties, together with the concept of symmetric k-convexity, enable us to construct solutions for the optimality equations of the discounted and average profit problems in §5. In §6, we prove some useful bounds on the reorder level and order-up-to level for a corresponding finite horizon problem. In §7 and §8, we apply these bounds and the optimality equations to prove the optimality of a stationary (s, S, p) policy for the infinite horizon problems with the discounted and average profit criteria, respectively. Finally, in §9 we provide concluding remarks.

2. The model. Consider a firm facing stationary cost parameters and revenue functions as well as a stationary demand process over the infinite horizon. For each period t, let

$d_t =$ demand in period t

$p_t =$ selling price in period t

$\underline{p}, \bar{p}$ are the common lower and upper bounds on p_t, respectively.

Throughout this paper, we concentrate on demand functions similar to those considered in Chen and Simchi-Levi (2004). These demand functions are of the following form:

Assumption 1. *For any t, the demand function satisfies*

$$d_t = D_t(p_t, \epsilon_t) := \alpha_t D(p_t) + \beta_t, \tag{1}$$

where $\epsilon_t = (\alpha_t, \beta_t)$, and α_t, β_t are two random variables with $\alpha_t \geq 0$, $E\{\alpha_t\} = 1$, and $E\{\beta_t\} = 0$. The random perturbations, ϵ_t, are identically distributed with the same distribution as $\epsilon = (\alpha, \beta)$ and are independent across time. Furthermore, the function D is continuous and strictly decreasing.

As observed in Chen and Simchi-Levi (2004), by scaling and shifting, the assumptions $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$ can be made without loss of generality as long as α_t and β_t have finite means. A special case of this demand function is the *additive* demand function, where the demand function is of the form $d_t = D(p_t) + \beta_t$. This implies that only β_t is a random variable, while $\alpha_t = 1$. Another special case is a model with the *multiplicative* demand function. In this case, the demand function is of the form $d_t = \alpha_t D(p_t)$, where α_t is a random variable.

Let x_t be the inventory level at the beginning of period t just before placing an order. Similarly, y_t is the inventory level at the beginning of period t after placing an order. Lead time is assumed to be zero, and hence an order placed at the beginning of period t arrives immediately before demand for the period is realized. The ordering cost function includes both a fixed cost and a variable cost and is calculated for every t, $t = 1, 2, \ldots$, as

$$k\delta(y_t - x_t) + c(y_t - x_t),$$

where

$$\delta(u) := \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Unsatisfied demand is backlogged. Let x be the inventory level carried over from period t to the next period. Since we allow backlogging, the state space for the inventory levels is $(-\infty, \infty)$, and thus x may be positive or negative. A cost $h(x)$ is incurred at the end of period t, which represents inventory holding cost when $x > 0$ and shortage cost if $x < 0$.

Given a discount factor γ with $0 < \gamma \leq 1$, an initial inventory level, $x_1 = x$, and a pricing and replenishment policy, let

$$V_T^\gamma(x) = E\left\{\sum_{t=1}^{T} \gamma^{t-1}(-k\delta(y_t - x_t) - c(y_t - x_t) - h(x_{t+1}) + p_t D_t(p_t, \epsilon_t))\right\}, \tag{2}$$

be the T-period total expected discounted profit, where $x_{t+1} = y_t - D_t(p_t, \epsilon_t)$.

In the infinite horizon expected discounted profit model, the objective is to decide on ordering and pricing policies so as to maximize

$$\limsup_{T \to \infty} V_T^\gamma(x)$$

for $0 < \gamma < 1$ and any initial inventory level x. Similarly, in the infinite horizon expected average profit model, the objective is to maximize

$$\limsup_{T \to \infty} \frac{1}{T} V_T^\gamma(x),$$

for $\gamma = 1$ and any initial inventory level x.

To find an optimal strategy that maximizes (2), let $v_t^\gamma(x)$ be the maximum total expected discounted profit over a t-period planning horizon when we start with an initial inventory level x. A natural dynamic program that can be applied to find the policy maximizing (2) is as follows. For $t = 1, 2, \ldots, T$,

$$v_t^\gamma(x) = cx + \max_{y\geq x,\, \bar{p}\geq p\geq \underline{p}} -k\delta(y-x) + f_t^\gamma(y, p) \tag{3}$$

with $v_0^\gamma(x) = 0$ for any x, where

$$f_t^\gamma(y, p) := -cy + E\{pD_t(p, \epsilon_t) - h(y - D_t(p, \epsilon_t)) + \gamma v_{t-1}^\gamma(y - D_t(p, \epsilon_t))\}.$$

Note that Assumption 1 implies that there is a one-to-one correspondence between the selling price $p_t \in [\underline{p}, \bar{p}]$ and the expected demand $D(p_t) \in [\underline{d}, \bar{d}]$, where

$$\underline{d} = D(\bar{p}) \qquad \text{and} \qquad \bar{d} = D(\underline{p}).$$

Therefore, we can present the formulation (3) only with respect to expected demand rather than with respect to price. For that purpose, denote the expected demand at period t by $d = D(p)$. Also, let

$$\phi_t^\gamma(x) = v_t^\gamma(x) - cx, \qquad h^\gamma(y) = h(y) + (1-\gamma)cy, \qquad \text{and} \qquad \hat{R}(d) = R(d) - cd,$$

where R is the expected revenue function with

$$R(d) = dD^{-1}(d),$$

which is a function of expected demand d. These functions, $\phi_t^\gamma(x)$, $h^\gamma(y)$, and $\hat{R}(d)$, allow us to transform the original problem to a problem with zero variable ordering cost.

Specifically, the dynamic program (3) can be written as

$$\phi_t^\gamma(x) = \max_{y\geq x} -k\delta(y-x) + g_t^\gamma(y, d_t^\gamma(y)), \tag{4}$$

with $\phi_0^\gamma(x) = -cx$ for any x, where

$$g_t^\gamma(y, d) = H^\gamma(y, d) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha d - \beta)\}, \tag{5}$$

$$H^\gamma(y, d) := -E\{h^\gamma(y - \alpha d - \beta)\} + \hat{R}(d),$$

and

$$d_t^\gamma(y) \in \arg\max_{\bar{d}\geq d\geq \underline{d}} g_t^\gamma(y, d). \tag{6}$$

Thus, most of our focus is on the transformed Problem (4), which has a similar structure to Problem (3). In this transformed problem one can think of h^γ as being the holding and shortage cost function, $\hat{R}$ as being the revenue function, the variable ordering cost is equal to zero, and $\phi_t^\gamma(x)$ is the maximum total expected discounted profit over a t-period planning horizon when starting with an initial inventory level x.

Let $Q^\gamma(x)$ be the single-period maximum expected profit when we start with an initial inventory level x; i.e.,

$$Q^\gamma(x) := \max_{\bar{d}\geq d\geq \underline{d}} H^\gamma(x, d). \tag{7}$$

For technical reasons, we need the following assumptions on the revenue function and the holding and shortage cost function.

Assumption 2. *The functions R and $-h$ are concave. Therefore, $H^\gamma(x, d)$ is jointly concave in (x, d). As a consequence, $Q^\gamma(x)$ is concave. Furthermore, we assume that,*

$$\lim_{|x|\to\infty} Q^\gamma(x) = \lim_{|x|\to\infty} Q^0(x) = -\infty.$$

In the above assumption, the concavity of function $Q^\gamma(x)$ follows from the joint concavity of function $H^\gamma(x, d)$, which is true because H^γ is essentially a composition of a concave function and an affine function. The convexity of the holding cost function is commonly assumed in the standard stochastic inventory control literature. Finally, the revenue function R is concave for $D_t(p) = b_t - a_t p$ $(a_t > 0, b_t > 0)$ or $D_t(p) = a_t p^{-b_t}$ $(a_t > 0, b_t > 1)$; both functions are popular in the economic literature (Petruzzi and Dada 1999).

The following concept of symmetric k-convexity, introduced in Chen and Simchi-Levi (2004), is important in the analysis of our model.

Definition 2.1. A real-valued function f is called sym-k-convex for $k \geq 0$, if for any x_0, x_1 and $\lambda \in [0, 1]$,

$$f((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda) f(x_0) + \lambda f(x_1) + \max\{\lambda, 1-\lambda\}k. \tag{8}$$

A function f is called sym-k-concave if $-f$ is sym-k-convex.

Observe that the classical concept of k-convexity introduced by Scarf (1960) is a special case of symmetric k-convexity. The following lemma describes properties of symmetric k-convex functions, which are introduced and proven in Chen and Simchi-Levi (2004).

Lemma 1. (a) *A real-valued convex function is also sym-0-convex, and hence sym-k-convex, for all $k \geq 0$. A sym-k_1-convex function is also a sym-k_2-convex function for $k_1 \leq k_2$.*

(b) *If $g_1(y)$ and $g_2(y)$ are sym-k_1-convex and sym-k_2-convex, respectively, then for $\alpha, \beta \geq 0$, $\alpha g_1(y) + \beta g_2(y)$ is sym-$(\alpha k_1 + \beta k_2)$-convex.*

(c) *If $g(y)$ is sym-k-convex and w is a random variable, then $E\{g(y-w)\}$ is also sym-k-convex, provided $E\{|g(y-w)|\} < \infty$ for all y.*

(d) *Assume that g is a continuous sym-k-convex function and $g(y) \to \infty$ as $|y| \to \infty$. Let S be a global minimizer of g, and s be any element from the set*

$$X := \big\{x \mid x \leq S, g(x) = g(S) + k \text{ and } g(x') \geq g(x) \text{ for any } x' \leq x\big\}.$$

Then we have the following results.

(i) *$g(s) = g(S) + k$ and $g(y) \geq g(s)$ for all $y \leq s$.*

(ii) *$g(y) \leq g(z) + k$ for all y, z with $(s+S)/2 \leq y \leq z$.*

3. Preliminaries. Our objective in this section is twofold. First, given a stationary (s, S, p) policy, we show how to determine the infinite horizon expected discounted or average profit. Second, given a stationary (s, S) inventory policy, we show how to construct the optimal pricing strategy.

As pointed out earlier, there is a one-to-one correspondence between price and expected demand through the mapping $d = D(p)$. Hence, from now on we use $(s, S, \mathbf{d})$ and (s, S, p) interchangeably, and we always assume $s \leq S$ when we refer to an (s, S) policy.

Given a stationary $(s, S, \mathbf{d})$ policy, let $I^\gamma(s, x, \mathbf{d})$ be the expected γ-discounted profit incurred during a horizon that starts with initial inventory level x and ends, at this period or a later period, with an inventory level less than s. Therefore, for $x < s$, $I^\gamma(s, x, \mathbf{d}) = 0$, and for $x \geq s$,

$$I^\gamma(s, x, \mathbf{d}) = H^\gamma(x, \mathbf{d}(x)) + \gamma E\{I^\gamma(s, x - \alpha\mathbf{d}(x) - \beta, \mathbf{d})\}. \tag{9}$$

Define $\tau(s, x, \mathbf{d})$ to be the number of periods it takes to drop the inventory level from x to a level below s. Thus, we have $\tau(s, x, \mathbf{d}) = 0$ for $x < s$, and

$$\tau(s, x, \mathbf{d}) = 1 + \tau(s, x - \alpha\mathbf{d}(x) - \beta, \mathbf{d}), \qquad \text{for } x \geq s.$$

Finally, let $M^\gamma(s, x, \mathbf{d})$ be the expected γ-discounted time to drop from initial inventory level x to a level below s. Observe that whenever $x < s$, we have $M^\gamma(s, x, \mathbf{d}) = 0$. On the other hand, when $x \geq s$ we have

$$M^\gamma(s, x, \mathbf{d}) = 1 + \gamma E\{M^\gamma(s, x - \alpha\mathbf{d}(x) - \beta, \mathbf{d})\}. \tag{10}$$

From the definition of τ and M^γ, we have that

$$M^\gamma(s, x, \mathbf{d}) = E\{1 + \gamma + \cdots + \gamma^{\tau(s, x, \mathbf{d})-1}\} = (1 - E\{\gamma^{\tau(s, x, \mathbf{d})}\})/(1-\gamma). \tag{11}$$

Let

$$c^\gamma(s, x, \mathbf{d}) = \frac{-k + I^\gamma(s, x, \mathbf{d})}{M^\gamma(s, x, \mathbf{d})} \tag{12}$$

for $x \geq s$ and $c^\gamma(s, x, \mathbf{d}) = 0$ for $x < s$.

The definitions of $I^\gamma(s, x, \mathbf{d})$, $M^\gamma(s, x, \mathbf{d})$, and $c^\gamma(s, S, \mathbf{d})$ imply the following properties.

Lemma 2. *Given a stationary $(s, S, \mathbf{d})$ policy, if $I^\gamma(s, S, \mathbf{d})$ and $M^\gamma(s, S, \mathbf{d})$ are bounded, then*

(i) *for $\gamma = 1$, $c^\gamma(s, S, \mathbf{d})$ is the long-run average profit*;

(ii) *for $0 < \gamma < 1$, the function*

$$c^\gamma(s, S, \mathbf{d})/(1-\gamma) + I^\gamma(s, x, \mathbf{d}) - c^\gamma(s, S, \mathbf{d})M^\gamma(s, x, \mathbf{d})$$

is the infinite horizon expected discounted profit starting with an initial inventory level x.

Proof. Notice that $c^\gamma(s, S, \mathbf{d})$ equals the ratio of the expected discounted profit in a cycle to the expected discounted length of a cycle, where a cycle is defined as the time between two periods in which the starting inventory level is less than s. Hence, part (i) follows directly from the elementary renewal reward theory (see Ross 1970), and so does the case $x < s$ for part (ii).

We now focus on part (ii) with $x \geq s$. The infinite horizon expected discounted profit consists of two parts: the expected discounted profit accrued up to the first time when we drop the inventory level from x to a level below s, which is $I^\gamma(s, x, \mathbf{d})$, and the expected discounted profit accumulated afterwards, which can be calculated as $E\{\gamma^{\tau(s, x, \mathbf{d})}\}c^\gamma(s, S, \mathbf{d})/(1-\gamma)$ since we start with an initial inventory level less than s. Therefore, the infinite horizon expected discounted profit is

$$I^\gamma(s, x, \mathbf{d}) + E\{\gamma^{\tau(s, x, \mathbf{d})}\}c^\gamma(s, S, \mathbf{d})/(1-\gamma).$$

Finally, the above observation, together with Equation (11), implies part (ii). □

To provide intuition about (ii), observe that $c^\gamma(s, S, \mathbf{d})$ is the expected discounted profit per period for the infinite horizon expected discounted profit problem starting with an initial inventory level less than s. Therefore, $c^\gamma(s, S, \mathbf{d})/(1-\gamma)$ is the infinite horizon expected discounted profit if we start with an initial inventory level, x, less than s, and this implies that (ii) holds since in this case both $I^\gamma(s, x, \mathbf{d})$ and $M^\gamma(s, x, \mathbf{d})$ are equal to zero. For $x \geq s$, observe that $c^\gamma(s, S, \mathbf{d})M^\gamma(s, x, \mathbf{d})$ is the expected discounted profit incurred during the expected discounted time $M^\gamma(s, x, \mathbf{d})$ if we start with an initial inventory level less than s. Thus, the difference between the infinite horizon expected discounted profit starting with an initial inventory level less s, and the infinite horizon expected discounted profit starting with the initial inventory level x, equals

$$I^\gamma(s, x, \mathbf{d}) - c^\gamma(s, S, \mathbf{d})M^\gamma(s, x, \mathbf{d}). \tag{13}$$

Hence, (ii) follows. Finally, it is appropriate to point out that Zheng (1991) uses a formulation similar to (12) and proves results similar to the one in Lemma 2.

We continue by assuming that the period demand is positive. Formally, this assumption says that for any realization of the random variables $\epsilon = (\alpha, \beta)$, $\alpha d + \beta \geq \alpha \underline{d} + \beta \geq \eta > 0$ for some η and any $d \in [\underline{d}, \bar{d}]$. This assumption will be relaxed later on.

In the following, we show how one can construct the best pricing strategy for a given (s, S) inventory policy. For any given (s, S), let $c^\gamma(s, S)$ be the optimal value of problem

$$\sup_{\mathbf{d}:\, \bar{d} \geq \mathbf{d}(\cdot) \geq \underline{d}} c^\gamma(s, S, \mathbf{d}). \tag{14}$$

Observe that the feasible set of Problem (14) consists of functions, $\mathbf{d}(\cdot)$.

Define

$$\phi^\gamma(x, s, S, s') = \begin{cases} 0, & \text{for } x < s', \\ \sup\limits_{\bar{d} \geq d \geq \underline{d}} g^\gamma(x, s, S, s', d), & \text{for } x \geq s', \end{cases} \tag{15}$$

where

$$g^\gamma(x, s, S, s', d) = H^\gamma(x, d) - c^\gamma(s, S) + \gamma E\{\phi^\gamma(x - \alpha d - \beta, s, S, s')\}.$$

Let $\phi^\gamma(x, s, S) = \phi^\gamma(x, s, S, s)$. Interestingly, as we explain at the end of this section, $\phi^\gamma(x, s, S)$ is (up to a constant) the total expected discounted profit associated with a stationary (s, S) inventory policy and its corresponding best pricing strategy.

Notice that, in the recursive function, (15), we optimize a variable d for a given x; thus, our objective is to show that the recursive construction allows us to generate the function $\mathbf{d}(\cdot)$ that solves (14).

For this purpose, define

$$\psi^\gamma(x, s, S, s', \mathbf{d}) = I^\gamma(s', x, \mathbf{d}) - c^\gamma(s, S) M^\gamma(s', x, \mathbf{d}) \tag{16}$$

for a given feasible expected demand function $\mathbf{d}$ and let $\psi^\gamma(x, s, S, \mathbf{d}) = \psi^\gamma(x, s, S, s, \mathbf{d})$. Then from the recursions for I^γ (9) and M^γ (10), we have that

$$\psi^\gamma(x, s, S, s', \mathbf{d}) = \begin{cases} 0, & \text{for } x < s', \\ H^\gamma(x, \mathbf{d}(x)) - c^\gamma(s, S) & \\ \qquad + \gamma E\{\psi^\gamma(x - \alpha \mathbf{d}(x) - \beta, s, S, s', \mathbf{d})\}, & \text{for } x \geq s'. \end{cases} \tag{17}$$

On the other hand, from the definition of $c^\gamma(s, x, \mathbf{d})$, we have

$$\psi^\gamma(x, s, S, s', \mathbf{d}) = k + (c^\gamma(s', x, \mathbf{d}) - c^\gamma(s, S)) M^\gamma(s', x, \mathbf{d}). \tag{18}$$

The definition of $\phi^\gamma(x, s, S, s')$ and $\psi(x, s, S, s', \mathbf{d})$ will be useful in Proposition 1 as we compare the long-run average discounted profits $c^\gamma(s', S)$ and $c^\gamma(s, S)$. Notice the difference between (13) and ψ^γ.

Because we assume that the period demand is bounded below by a positive constant, it is clear that all the quantities I^γ, M^γ, ϕ^γ, and ψ^γ are well defined. In the following, we show how an optimal solution of Problem (14) can be constructed recursively given its optimal value $c^\gamma(s, S)$.

LEMMA 3. *For any x,*

$$\sup_{\mathbf{d}:\, \bar{d} \geq \mathbf{d}(\cdot) \geq \underline{d}} \psi^\gamma(x, s, S, s', \mathbf{d}) = \phi^\gamma(x, s, S, s').$$

In particular, $\phi^\gamma(S, s, S) = k$.

Proof. We argue by induction that $\psi^\gamma(x, s, S, s', \mathbf{d}) \leq \phi^\gamma(x, s, S, s')$ for any feasible function $\mathbf{d}$ and any x. It is clearly true for $x < s'$, since in this case both functions equal zero. Assume that it is true for any x with $x \leq y$ for some y. We prove that it is also true for $x \leq y + \eta$. In fact, for $x \geq s'$,

$$\begin{aligned}\psi^\gamma(x, s, S, s', \mathbf{d}) &= H^\gamma(x, \mathbf{d}(x)) - c^\gamma(s, S) + \gamma E\{\psi^\gamma(x - \alpha\mathbf{d}(x) - \beta, s, S, s', \mathbf{d})\}\\ &\leq H^\gamma(x, \mathbf{d}(x)) - c^\gamma(s, S) + \gamma E\{\phi^\gamma(x - \alpha\mathbf{d}(x) - \beta, s, S, s')\}\\ &\leq \sup_{\bar{d} \geq d \geq \underline{d}} H^\gamma(x, d) - c^\gamma(s, S) + \gamma E\{\phi^\gamma(x - \alpha d - \beta, s, S, s')\}\\ &= \phi^\gamma(x, s, S, s'),\end{aligned}$$

where the first inequality is justified by the induction assumption. On the other hand, for any given $\varepsilon > 0$, choose a function $\mathbf{d}_\varepsilon$ such that for any $x \geq s'$,

$$g^\gamma(x, s, S, s', \mathbf{d}_\varepsilon(x)) \geq \phi^\gamma(x, s, S, s') - \varepsilon.$$

We can prove by induction that $\psi^\gamma(x, s, S, s', \mathbf{d}_\varepsilon)$ converges to $\phi^\gamma(x, s, S, s')$ uniformly over any bounded set of x as $\varepsilon \downarrow 0$ since $\alpha d + \beta \geq \eta$ for any feasible d. Thus for any x,

$$\sup_{\mathbf{d}:\, \bar{d} \geq \mathbf{d}(\cdot) \geq \underline{d}} \psi^\gamma(x, s, S, s', \mathbf{d}) = \phi^\gamma(x, s, S, s').$$

Finally, (18) implies that

$$\psi^\gamma(S, s, S, \mathbf{d}) \leq k \qquad \text{and} \qquad \sup_{\mathbf{d}} \psi^\gamma(S, s, S, \mathbf{d}) = k.$$

Thus, $\phi^\gamma(S, s, S) = k$. □

In the above lemma, $\sup_{\mathbf{d}:\, \bar{d} \geq \mathbf{d}(\cdot) \geq \underline{d}} \psi^\gamma(x, s, S, s', \mathbf{d})$ is taken over a function, while (15) tells us how such a function can be constructed recursively. In fact, if for any $x \geq s'$ there exists $\mathbf{d}_0(x)$ such that $g^\gamma(x, s, S, s', \mathbf{d}_0(x)) = \phi^\gamma(x, s, S, s')$, then from the proof of Lemma 3 we have that $\psi^\gamma(x, s, S, s', \mathbf{d}_0) = \phi^\gamma(x, s, S, s')$. In particular, if $s' = s$, $\psi^\gamma(S, s, S, \mathbf{d}_0) = k$ implies that $c^\gamma(s, S, \mathbf{d}_0) = c^\gamma(s, S)$; i.e., $\mathbf{d}_0$ is an optimal solution for Problem (14). Furthermore, one can think of $\phi^\gamma(x, s, S)$ being equal to (up to a constant) the total expected discounted profit associated with a stationary (s, S) inventory policy and its corresponding best pricing strategy. Thus, from now on, we mainly focus on characterizing the optimal inventory policy.

4. Characterization of the best (s, S) inventory policy. In this section, we characterize the best stationary (s, S) inventory policy. This allows us to construct, in §5, a solution for the optimality equation for the infinite horizon joint inventory and pricing models. By the best, we mean the (s, S) inventory policy that gives the highest average discounted profit per period among all stationary (s, S) inventory policies associated with their best-pricing strategies.

Let c^γ be the optimal value of the problem

$$\sup_{(s, S)} c^\gamma(s, S). \tag{19}$$

Define

$$F^\gamma := \{(s, S) \mid c^\gamma(s, S) \geq \max Q^\gamma(x) - k,\, Q^\gamma(s) = c^\gamma(s, S) \text{ and } Q^\gamma(S) \geq c^\gamma(s, S)\}.$$

Observe that for $(s, S) \in F^\gamma$, we have $Q^\gamma(S) \geq Q^\gamma(s) \geq \max Q^\gamma(x) - k$. Hence, by Assumption 2, F^γ is a bounded set.

We prove in the following that the search for an optimal solution of (19) can be restricted to the bounded set F^γ.

Proposition 1. $c^\gamma = \sup_{(s,S)\in F^\gamma} c^\gamma(s,S)$.

Proof. To prove the proposition, we make the following observations.

(i) $c^\gamma \ge \max Q^\gamma(x) - k$. In fact, let x^γ be any maximum point of $Q^\gamma(x)$. Then $c^\gamma(x^\gamma - \eta, x^\gamma) = Q^\gamma(x^\gamma) - k$, since $I^\gamma(x^\gamma - \eta, x^\gamma, d) = H^\gamma(x, \mathbf{d}(x))$ and $M^\gamma(x^\gamma - \eta, x^\gamma, d) = 1$ for any expected demand function $\mathbf{d}$. Hence,

$$c^\gamma \ge c^\gamma(x^\gamma - \eta, x^\gamma) = \max Q^\gamma(x) - k.$$

(ii) $Q^\gamma(s) = c^\gamma(s,S)$. The proof is by contradiction, by showing that if this is not true that we can improve upon $c^\gamma(s,S)$.

(a) If $Q^\gamma(s) < c^\gamma(s,S)$, let s_1 be the smallest element in the set

$$\{x \mid x \ge s, Q^\gamma(x) = c^\gamma(s,S)\}.$$

Since $\phi^\gamma(S,s,S) = k \ge 0$, there exists $x \in [s,S]$ and $d \in [\underline{d}, \bar{d}]$ such that $H^\gamma(x,d) \ge c^\gamma(s,S)$. This, together with the continuity of Q^γ, implies that the above set is nonempty, s_1 is well defined, and $s < s_1 \le S$. From the recursive definition of $\phi^\gamma(x,s,S,s_1)$ we have that for any x,

$$\phi^\gamma(x,s,S,s_1) \ge \phi^\gamma(x,s,S),$$

since $\phi^\gamma(x,s,S) \le 0$ for $x \in [s,s_1]$. In particular, $\phi^\gamma(S,s,S,s_1) \ge k$. We claim $c^\gamma(s_1,S) \ge c^\gamma(s,S)$. In fact, Lemma 3, together with (18) and the fact that $\phi^\gamma(S,s,S,s_1) \ge k$, implies that

$$c^\gamma(s_1,S) = \sup_{\mathbf{d}:\, \bar{d} \ge \mathbf{d} \ge \underline{d}} c^\gamma(s_1,S,\mathbf{d}) \ge c^\gamma(s,S) = Q^\gamma(s_1).$$

If $c^\gamma(s_1,S) > Q^\gamma(s_1)$, we repeat this process and end up with a sequence $s_1 < s_2 < \cdots < S$ with $c^\gamma(s,S) = Q^\gamma(s_1) < c^\gamma(s_1,S) = Q^\gamma(s_2) < \cdots$. If the process stops in finite steps, say n steps, then $c^\gamma(s,S) \le c^\gamma(s_n,S) = Q^\gamma(s_n)$. Otherwise, let s^* be the limit of this sequence $\{s_n, n = 1,2,\ldots\}$ and $\tilde{c}^\gamma(s^*,S)$ be the limit of $c^\gamma(s_n,S)$. From the continuity of Q^γ as implied by its concavity, we have that $Q^\gamma(s^*) = \tilde{c}^\gamma(s^*,S)$. We argue that $\tilde{c}^\gamma(s^*,S) = c^\gamma(s^*,S)$. Define

$$\tilde{\phi}^\gamma(x,s^*,S) = \begin{cases} 0, & \text{for } x < s^*, \\ \sup_{\bar{d} \ge d \ge \underline{d}} H^\gamma(x,d) - \tilde{c}^\gamma(s^*,S) + \gamma E\{\tilde{\phi}^\gamma(x - \alpha d - \beta, s^*, S)\}, & \text{for } x \ge s^*. \end{cases}$$

One can prove by induction on x that $\phi^\gamma(x,s_n,S)$ converges to $\tilde{\phi}^\gamma(x,s^*,S)$ uniformly for x over any bounded set. Furthermore, we have that $\tilde{\phi}^\gamma(S,s^*,S) = k$ since $\phi^\gamma(S,s_n,S) = k$. Hence, from the definition (15) of $\phi^\gamma(x,s^*,S)$ and the fact that $\phi^\gamma(S,s^*,S) = k$, we have that $c^\gamma(s^*,S) = \tilde{c}^\gamma(s^*,S)$ and $\tilde{\phi}^\gamma(x,s^*,S)$ is identical to $\phi^\gamma(x,s^*,S)$. Therefore, $Q^\gamma(s^*) = c^\gamma(s^*,S) \ge c^\gamma(s,S)$.

(b) If $Q^\gamma(s) > c^\gamma(s,S)$, let s_1 be the largest element in the set

$$\{x \mid x \le s, Q^\gamma(x) = c^\gamma(s,S)\}.$$

The existence of s_1 is guaranteed by Assumption 2. Then from the recursions of I^γ (9) and M^γ (10), we have that for any x,

$$\phi^\gamma(x,s,S,s_1) \ge \phi^\gamma(x,s,S),$$

since $\phi^\gamma(x,s,S,s_1) \ge 0$ for $x \in [s_1,s]$. Following a similar argument to that for part (a), we can show that there exists a point s^* such that $Q^\gamma(s^*) = c^\gamma(s^*,S) \ge c^\gamma(s,S)$.

(iii) $Q^\gamma(S) \geq c^\gamma(s, S)$. We prove this by contradiction. If $Q^\gamma(S) < c^\gamma(s, S)$, then from the recursive definition of ϕ^γ (15) we have that

$$k = \phi^\gamma(S, s, S) < \sup_{\bar{d} \geq d \geq \underline{d}} \gamma E\{\phi^\gamma(S - \alpha d - \beta, s, S)\} \leq \sup_{x \leq S - \eta} \phi^\gamma(x, s, S).$$

Therefore, there exists S_1 with $S_1 \leq S - \eta$ such that $\phi^\gamma(S_1, s, S) > k$. From (16), we have $c^\gamma(s, S_1) \geq c^\gamma(s, S_1, d^\gamma_{(s,S)}) > c^\gamma(s, S)$. If $Q^\gamma(S_1) < c^\gamma(s, S_1)$, we can repeat the argument and find $S_{i+1} \leq S_i - \eta$, $i = 1, 2, \ldots,$ such that $c^\gamma(s, S_{i+1}) > c^\gamma(s, S_i)$ for $i = 1, 2, \ldots$. This process has to be finite since we have $s \leq S_{i+1} \leq S_i - \eta$. Assume we end up with S_n. Then, $Q^\gamma(S_n) \geq c^\gamma(s, S_n) \geq c^\gamma(s, S)$.

Observations (i)–(iii) imply that for the maximization Problem (19), it suffices to restrict the feasible set of (s, S) policies to the set F^γ. $\square$

For any $(s, S) \in F^\gamma$, since $Q^\gamma(s) = c^\gamma(s, S)$, one can show that $\phi^\gamma(x, s, S)$ is continuous in x and

$$\phi^\gamma(x, s, S) = \begin{cases} 0, & \text{for } x \leq s, \\ \max_{\bar{d} \geq d \geq \underline{d}} g^\gamma(x, s, S, s, d), & \text{for } x \geq s. \end{cases}$$

Furthermore, for $x \geq s$, the function

$$d^\gamma_{(s,S)}(x) \in \arg\max_{\bar{d} \geq d \geq \underline{d}} g^\gamma(x, s, S, s, d)$$

is well defined and by (16), (17), and Lemma 3, solves Problem (14).

Now we are ready to characterize the properties of the best (s, S) inventory policy. The characterization presented in the following lemma is key to our analysis of the discounted and average profit problems.

LEMMA 4. *There exists an optimal solution (s^γ, S^γ) to Problem (19) such that the functions $\phi^\gamma(x) := \phi^\gamma(x, s^\gamma, S^\gamma)$ and $Q^\gamma(x)$ (see (7) for the definition of this function), satisfy the following properties.*

(a) *$\phi^\gamma(x) \leq k$ for any x and $\phi^\gamma(S^\gamma) = k$.*
(b) *$Q^\gamma(s^\gamma) = c^\gamma$.*
(c) *$Q^\gamma(x) \geq c^\gamma$ for $x \in [s^\gamma, S^\gamma]$.*
(d) *$\phi^\gamma(x) \geq 0$ for any $x \leq S^\gamma$.*
(e) *$s^\gamma \leq x^\gamma$ for any maximum point x^γ of $Q^\gamma(x)$.*
(f) *$y^\gamma \leq S^\gamma$ for any minimum point y^γ of $h^\gamma(y)$.*

PROOF. Proposition 1 implies that for Problem (19), we can focus on (s, S) in the set F^γ. As we already pointed out, F^γ is a bounded set. We now prove that it is also closed, and hence compact. For this purpose, assume (s, S) is the limit of a sequence $(s_n, S_n) \in F^\gamma$. We claim that $(s, S) \in F^\gamma$. In fact, let $\tilde{c}^\gamma(s, S)$ be the limit of a subsequence $c^\gamma(s_{n_i}, S_{n_i})$. Then from the continuity of Q^γ, $Q^\gamma(S) \geq Q^\gamma(s) = \tilde{c}^\gamma(s, S)$. Define

$$\tilde{\phi}^\gamma(x, s, S) = \begin{cases} 0, & \text{for } x \leq s, \\ \max_{\bar{d} \geq d \geq \underline{d}} H^\gamma(x, d) - \tilde{c}^\gamma(s, S) + \gamma E\{\tilde{\phi}^\gamma(x - \alpha d - \beta, s, S)\}, & \text{for } x \geq s. \end{cases}$$

Since we assume the period demand is bounded below by a positive constant, one can prove by induction that $\phi^\gamma(x, s_{n_i}, S_{n_i})$ converges to $\tilde{\phi}^\gamma(x, s, S)$ uniformly for x over any bounded set. Furthermore, we have that $\tilde{\phi}^\gamma(S, s, S) = k$ since $\phi^\gamma(S_{n_i}, s_{n_i}, S_{n_i}) = k$. Hence, from the definition (15) of $\phi^\gamma(x, s, S)$ and the fact that $\phi^\gamma(S, s, S) = k$, we have that $c^\gamma(s, S) = \tilde{c}^\gamma(s, S)$ and $\tilde{\phi}^\gamma(x, s, S)$ is identical to $\phi^\gamma(x, s, S)$. Therefore, $c^\gamma(s_n, S_n)$ converges to $c^\gamma(s, S)$, which implies that $(s, S) \in F^\gamma$. As a consequence, F^γ is closed, and hence compact.

We are ready to prove the existence of the best (s, S) inventory policy. Assume that c^γ is the limit of $c^\gamma(\tilde{s}_n, \widetilde{S}_n)$ for a sequence $(\tilde{s}_n, \widetilde{S}_n) \in F^\gamma$. From the compactness of F^γ there is a subsequence $(\tilde{s}_{n_i}, \widetilde{S}_{n_i})$ of the sequence $(\tilde{s}_n, \widetilde{S}_n)$, such that

$$\lim_{i\to\infty}(\tilde{s}_{n_i}, \widetilde{S}_{n_i}) = (s^\gamma, S^\gamma)$$

for some $(s^\gamma, S^\gamma) \in F^\gamma$. As proven in the previous paragraph, we have

$$c^\gamma(s^\gamma, S^\gamma) = \lim_{i\to\infty} c^\gamma(\tilde{s}_{n_i}, \widetilde{S}_{n_i}) = c^\gamma,$$

and thus (s^γ, S^γ) is the best (s, S) inventory policy.

Hence,

- Part (a) follows from (16) and the fact that (s^γ, S^γ) solves Problem (19). In fact, for any $(s, x, \mathbf{d})$ policy, we have

$$\frac{-k + I^\gamma(s, x, \mathbf{d})}{M^\gamma(s, x, \mathbf{d})} = c^\gamma(s, x, \mathbf{d}) \le c^\gamma.$$

- Parts (b) and (c) hold since $(s^\gamma, S^\gamma) \in F^\gamma$ and Q^γ is concave.
- Part (d) follows from part (c) and the recursive definition of ϕ^γ in (15).
- From the argument of Observation (ii) in the proof of Proposition 1, it is easy to see that s^γ can be chosen as the smallest element in the set $\{x \mid Q^\gamma(x) = c^\gamma\}$. Therefore, part (c) implies that $s^\gamma \le x^\gamma$ for any maximum point x^γ of $Q^\gamma(x)$, and hence part (e) holds.

We now prove part (f). For any minimum point y^γ of $h^\gamma(x)$, we prove by induction that $\phi^\gamma(x)$ is nondecreasing for $x \le y^\gamma$, and consequently we can choose S^γ such that $y^\gamma \le S^\gamma$. Without loss of generality, assume that $s^\gamma \le y^\gamma$. First, $\phi^\gamma(x)$ is nondecreasing for $x \le s^\gamma$. Now assume it is true for any x with $x \le y$ for some $y \le y^\gamma$. Then for x and x' such that $s^\gamma \le x \le x' \le \min\{y + \eta, y^\gamma\}$, we have

$$\begin{aligned}
\phi^\gamma(x) &= \max_{\bar{d} \ge d \ge \underline{d}} H^\gamma(x, d) - c^\gamma + \gamma E\{\phi^\gamma(x - \alpha d - \beta)\} \\
&\le \max_{\bar{d} \ge d \ge \underline{d}} H^\gamma(x', d) - c^\gamma + \gamma E\{\phi^\gamma(x' - \alpha d - \beta)\} \\
&= \phi^\gamma(x'),
\end{aligned}$$

where the inequality holds since $x \le x' \le y^\gamma$, $h^\gamma(x)$ is convex, and $\phi^\gamma(x)$ is nondecreasing for $x \le y$ by induction assumption. Therefore, $\phi^\gamma(x)$ is nondecreasing for $x \le y^\gamma$. Thus, part (f) follows. $\square$

To provide some intuition, recall that $Q^\gamma(x)$ is the single-period maximum expected profit when we start with an inventory level x; $c^\gamma(s, S)$ can be viewed as the average discounted profit per period for a given (s, S) policy and its associated best price strategy. Thus, if (b) does not hold, one can change the reorder point, s^γ, and improve the average discounted profit per period. If (c) does not hold, one can decrease S^γ and increase the average discounted profit per period.

Lemma 4 is essentially parallel to the characterization of the best (s, S) policy, given by Lemma 1 in Zheng (1991) for the standard infinite-horizon inventory control models. Notice that the single-period maximum expected profit $Q^\gamma(x)$ plays the same role as the single-period expected inventory holding and shortage cost $G_\alpha(x)$ in Zheng. Also, it is easy to see from the proof of Lemma 4 that the result of the lemma can be generalized to cases when $Q^\gamma(x)$ is quasi-concave; this is similar to the assumption that $-G_\alpha(x)$ is quasi-concave in Zheng.

Of course, the proof for Lemma 4 is significantly more involved than the proof of Lemma 1 in Zheng (1991). In fact, the pricing decision variable presented in our model makes it necessary to refer to the recursive techniques like the one used in (15) throughout this paper, while in Zheng's case, a closed form expression for the long-run average discounted cost associated with a given (s, S) policy is readily available. Furthermore, dealing with continuous state spaces significantly increases the complexity of the analysis here. Finally, we point out other differences between Lemma 1 in Zheng and our Lemma 4. On the one hand, Lemma 4 part (b) is stronger than Lemma 1 part (iii) in Zheng because we deal with continuous inventory levels, while in Zheng the inventory level is discrete. On the other hand, Lemma 4 part (e) and part (f) are weaker than Lemma 1 part (ii) in Zheng due to the introduction of the pricing variable in our model.

5. Optimality equations. The characterization of the best (s, S) inventory policy identified in Lemma 4 allows us to construct a solution for the optimality equations for the infinite horizon models under both the discounted profit and average profit criteria. This in turn allows us to prove the optimality of a stationary (s, S) inventory policy for the discounted (§7) and the average (§8) profit criteria. For this purpose, we start by showing that the function ϕ^γ defined in Lemma 4 is symmetric k-concave.

LEMMA 5. *ϕ^γ is symmetric k-concave for the general demand model.*

PROOF. We prove, by induction, that ϕ^γ satisfies

$$\phi^\gamma(x_\lambda) \geq (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \max\{\lambda, 1-\lambda\}k \tag{20}$$

for any $x_0 < x_1$ and $\lambda \in [0, 1]$, where $x_\lambda = (1-\lambda)x_0 + \lambda x_1$.

Since $\phi^\gamma(x) = 0$ for $x \leq s^\gamma$, it is obvious that (20) holds for $x_1 \leq s^\gamma$. Now assume that (20) holds for any x_0 and x_1 with $x_0 < x_1 \leq y$ for some y. We show that (20) also holds for any x_0 and x_1 with $x_0 < x_1 \leq y + \eta$. We distinguish between three cases.

Case 1. $x_0 > s^\gamma$. Letting $d_\lambda = (1-\lambda)d^\gamma_{(s^\gamma, S^\gamma)}(x_0) + \lambda d^\gamma_{(s^\gamma, S^\gamma)}(x_1)$, we have that

$$\begin{aligned}
\phi^\gamma(x_\lambda) &\geq H^\gamma(x_\lambda, d_\lambda) - c^\gamma + \gamma E\{\phi^\gamma(x_\lambda - \alpha d_\lambda - \beta)\} \\
&\geq (1-\lambda)(H^\gamma(x_0, d^\gamma_{(s^\gamma, S^\gamma)}(x_0)) - c^\gamma + \gamma E\{\phi^\gamma(x_0 - \alpha d^\gamma_{(s^\gamma, S^\gamma)}(x_0) - \beta)\}) \\
&\quad + \lambda(H^\gamma(x_1, d^\gamma_{(s^\gamma, S^\gamma)}(x_1)) - c^\gamma + \gamma E\{\phi^\gamma(x_1 - \alpha d^\gamma_{(s^\gamma, S^\gamma)}(x_1) - \beta)\}) \\
&\quad - \gamma \max\{\lambda, 1-\lambda\}k \\
&\geq (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where the second inequality follows from the concavity of H^γ, the fact that for any feasible d, $x_0 - \alpha d - \beta \leq x_1 - \alpha d - \beta \leq y$, and the induction assumption.

Case 2. $x_0 \leq s^\gamma$ and $x_\lambda \leq S^\gamma$. (20) holds because by Lemma 4 parts (a) and (d), $\phi^\gamma(x_1) \leq k$ and $\phi^\gamma(x_\lambda) \geq 0$.

Case 3. $x_0 \leq s^\gamma \leq S^\gamma \leq x_\lambda$.

$$\begin{aligned}
\phi^\gamma(x_\lambda) &\geq (1-\mu)\phi^\gamma(S^\gamma) + \mu\phi^\gamma(x_1) - \max\{\mu, 1-\mu\}k \\
&\geq \mu(\phi^\gamma(x_1) - k) \\
&\geq \lambda(\phi^\gamma(x_1) - k) \\
&\geq (1-\lambda)\phi^\gamma(x_0) + \lambda\phi^\gamma(x_1) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where μ is chosen such that $x_\lambda = (1-\mu)S^\gamma + \mu x_1$ with $0 \leq \mu \leq \lambda$. The first inequality follows from Case 1, the second inequality holds because $\phi^\gamma(S^\gamma) = k$ by Lemma 4 part (a), the third inequality holds because $0 \leq \mu \leq \lambda$ and, by Lemma 4 part (a), $\phi^\gamma(x_1) \leq k$, and the last inequality follows from the fact that $\phi^\gamma(x_0) = 0$ since $x_0 \leq s^\gamma$.

Therefore, by induction, ϕ^γ is symmetric k-concave. □

REMARK. In the special case of additive demand functions, we can show that ϕ^γ is k-concave. For a proof, the reader is referred to Chen (2003).

We are ready to prove that (ϕ^γ, c^γ) satisfies the equation

$$\phi^\gamma(x) + c^\gamma = \max_{y\geq x}\left\{ \max_{\bar{d}\geq d\geq \underline{d}} -k\delta(y-x) + H^\gamma(y,d) + \gamma E\{\phi^\gamma(y-\alpha d-\beta)\}\right\} \tag{21}$$

and that (s^γ, S^γ) is the policy that attains the first maximization in Equation (21).

Notice that when $\gamma = 1$, (21) is the optimality equation for the average profit problem. On the other hand, when $0 < \gamma < 1$, define

$$\hat{\phi}^\gamma(x) = c^\gamma/(1-\gamma) + \phi^\gamma(x).$$

Then, (21) implies that

$$\hat{\phi}^\gamma(x) = \max_{y\geq x,\, \bar{d}\geq d\geq \underline{d}} -k\delta(y-x) + H^\gamma(y,d) + \gamma E\{\hat{\phi}^\gamma(y-\alpha d-\beta)\},$$

which is the optimality equation for the γ-discounted profit problem for $0 < \gamma < 1$, i.e., Problem (4).

THEOREM 5.1. (ϕ^γ, c^γ) *satisfies Equation* (21) *and* (s^γ, S^γ) *attains the first maximization in Equation* (21).

PROOF. For any x, define

$$O^\gamma(x) := \max_{\bar{d}\geq d\geq \underline{d}} H^\gamma(x,d) - c^\gamma + \gamma E\{\phi^\gamma(x-\alpha d-\beta)\}.$$

From (15) and Lemma 4 part (b), one can see that $O^\gamma(x) = Q^\gamma(x) - c^\gamma$ for $x \leq s^\gamma$ and $O^\gamma(x) = \phi^\gamma(x)$ for $x \geq s^\gamma$. We have the following observations.

(a) $O^\gamma(x) \leq O^\gamma(s^\gamma) = 0$ for $x \leq s^\gamma$. This follows from Lemma 4 parts (b) and (e), the concavity of Q^γ, and the fact that $O^\gamma(x) = Q^\gamma(x) - c^\gamma$ for $x \leq s^\gamma$.

(b) $O^\gamma(x) \leq O^\gamma(S^\gamma) = k$ for any x. This result follows from part (a) and Lemma 4 part (a) because $O^\gamma(x) = \phi^\gamma(x)$ for $x \geq s^\gamma$.

(c) $O^\gamma(y) \geq O^\gamma(z) - k$, for any y, z with $s^\gamma \leq y \leq z$. Since $O^\gamma(x) = \phi^\gamma(x)$ for $x \geq s^\gamma$, we only need to show that $\phi^\gamma(y) \geq \phi^\gamma(z) - k$. For $y \leq S^\gamma$, we have

$$\phi^\gamma(y) \geq 0 \geq \phi^\gamma(z) - k$$

by Lemma 4 parts (a) and (d). For $y \geq S^\gamma$, $\phi^\gamma(y) \geq \phi^\gamma(z) - k$ follows from Lemma 4 part (a), Lemma 5, and Lemma 1 part (d).

Observations (a), (b), and (c) imply that the optimal y in Equation (21) follows the (s^γ, S^γ) policy: If $x \leq s^\gamma$, then $y = S^\gamma$, otherwise $y = x$. Thus, (ϕ^γ, c^γ) satisfies (21). □

The above results are proven under the assumption that $\alpha\underline{d} + \beta \geq \eta > 0$. This assumption can be relaxed for the discounted case. Furthermore, all the results also hold for the average case if $\Pr\{\alpha\underline{d} + \beta = 0\} < 1$. See Appendix A for the proof in each case.

6. Bounds. In §5, we constructed a solution for the optimality Equation (21) and proved that the (s^γ, S^γ) policy attains the first maximization in (21). Unfortunately, this does not necessarily imply the optimality of a stationary (s, S) inventory policy. Indeed, for the average profit criterion, the function $\phi^1(x)$ is unbounded, and no conclusion can be drawn easily by using standard dynamic programming arguments. Similarly, in the case of the discounted profit criterion, when $\gamma \in (0, 1)$, the infinite horizon problem is the so-called undiscounted dynamic program under Assumption **P** in Bertsekas (1976). In this case, one might try to apply Proposition 10, p. 260 in Bertsekas (1976), as was done by Zheng (1991) for his problem. However, it is not clear that this proposition can be applied because

Proposition 10 in Bertsekas requires that the optimality equation holds for the optimal cost function, while here $\hat{\phi}^\gamma$ is the expected discounted profit function associated with the stationary (s^γ, S^γ) inventory policy. We are not aware of any result that characterizes conditions under which we can claim the optimality of a stationary (s, S) inventory policy directly from the optimality Equation (21).

Thus, we need a different argument. In particular, we establish some connection between the finite horizon model and the solution for the optimality equation for the discounted (§7) and average (§8) profit criteria to prove the optimality of the stationary (s^γ, S^γ) inventory policy. For this purpose, we need some bounds on some of the parameters of the optimal policy for the finite horizon model. Our approach in this section is motivated by the classical work of Veinott (1966). In fact, Veinott suggested a different way to prove the optimality of an (s, S) policy for the standard inventory control problem. Even though we are not able to prove the optimality of an (s, S, p) policy for the joint inventory and pricing models based on Veinott's approach, it provides us with a very useful technique to derive lower bounds and upper bounds for the optimal parameters.

Consider the dynamic program (4). A straightforward extension of the analysis in Chen and Simchi-Levi (2004) shows that an (s, S, A, p) policy is optimal for this problem.

For every t, $t = 1, \ldots$, let $(s_t^\gamma, S_t^\gamma, A_t^\gamma, p_t^\gamma)$ be the parameters of the optimal policy. We show that s_t^γ and S_t^γ are uniformly bounded. Specifically, define

$$\underline{S}^\gamma = \min_{\bar{d}\geq d\geq \underline{d}} \left\{\arg\max_x H^\gamma(x, d), \arg\max_x H^0(x, d)\right\}, \quad \text{and} \quad \bar{s}^\gamma = \max_{\bar{d}\geq d\geq \underline{d}} \left\{\arg\max_x H^\gamma(x, d)\right\},$$

$$\underline{s}^\gamma = \max\{x \mid x \leq \underline{S}^\gamma, H^\mu(\underline{S}^\gamma, d) \geq H^\mu(x, d) + k, \text{ for } \mu = 0, \gamma \text{ and all feasible } d\},$$

and

$$\bar{S}^\gamma = \min\{x \mid x \geq \bar{s}^\gamma, H^\gamma(\bar{s}^\gamma, d) \geq H^\gamma(x, d) + k, \text{ for all feasible } d\}.$$

The existence of $\underline{s}^\gamma$ and $\bar{S}^\gamma$ follows from Assumption 2.

Lemma 6. *For $t \geq 1$,*

$$\phi_t^\gamma(x) \geq \phi_t^\gamma(x') - k, \quad \text{for } x \leq x', \tag{22}$$

$$g_t^\gamma(y', d) - g_t^\gamma(y, d) \leq H^\gamma(y', d) - H^\gamma(y, d) + k, \quad \text{for } y \leq y', \tag{23}$$

and

$$g_t^\gamma(y', d_t^\gamma(y')) \leq g_t^\gamma(y, d_t^\gamma(y)) + k, \quad \text{for } y' \geq y \geq \bar{s}^\gamma. \tag{24}$$

Proof. By induction. For $t = 0$, $\phi_t^\gamma(x) = -cx$ is nonincreasing since the variable ordering cost $c \geq 0$. Hence, (22) holds for $t = 0$. For $t \geq 1$, (22) follows directly from (4). (23) follows from (5) and (22) for period $t - 1$. (24) follows from (23), the definition of $\bar{s}^\gamma$, and the concavity of H^γ. □

Lemma 7.

$$g_1^\gamma(y', d) - g_1^\gamma(y, d) = H^0(y', d) - H^0(y, d) \geq 0, \quad \text{for } y \leq y' \leq \underline{S}^\gamma, \tag{25}$$

$$g_t^\gamma(y', d) - g_t^\gamma(y, d) \geq H^\gamma(y', d) - H^\gamma(y, d) \geq 0, \quad \text{for } y \leq y' \leq \underline{S}^\gamma \text{ and } t > 1, \tag{26}$$

$$g_t^\gamma(y', d_t^\gamma(y')) \geq g_t^\gamma(y, d_t^\gamma(y)), \quad \text{for } y \leq y' \leq \underline{S}^\gamma \text{ and } t \geq 1, \tag{27}$$

and

$$\phi_t^\gamma(x') \geq \phi_t^\gamma(x), \quad \text{for } x \leq x' \leq \underline{S}^\gamma \text{ and } t \geq 1. \tag{28}$$

Proof. Equation (25) follows from the definition of $\underline{S}^\gamma$ and the fact that $g_1^\gamma(y, d) = H^0(y, d)$. We prove the remaining three inequalities by induction. Assume that (26) holds for some $t > 1$. Equation (27) follows directly from (25) (if $t = 1$) or (26) (if $t > 1$). Furthermore, for any $x \le x' \le \underline{S}^\gamma$,

$$\begin{aligned}\phi_t^\gamma(x') &= \max\Big\{g_t^\gamma(x', d_t^\gamma(x')), -k + \max_{y>x'} g_t^\gamma(y, d_t^\gamma(y))\Big\}\\ &\ge \max\Big\{g_t^\gamma(x, d_t^\gamma(x)), -k + \max_{y>x} g_t^\gamma(y, d_t^\gamma(y))\Big\}\\ &= \phi_t^\gamma(x),\end{aligned}$$

where the inequality follows from (27). This proves inequality (28). Finally, (5), (28), and the definition of $\underline{S}^\gamma$ imply that (26) holds for $t+1$ and any feasible d, since H^γ is concave. □

We are ready to present our bounds on s_t^γ and S_t^γ.

Lemma 8. *For every t, $s_t^\gamma \in [\underline{s}^\gamma, \bar{s}^\gamma]$ and $S_t^\gamma \in [\underline{S}^\gamma, \bar{S}^\gamma]$.*

Proof. We first show that for every t and $y \le \underline{s}^\gamma$,

$$g_t^\gamma(y, d_t^\gamma(y)) \le -k + g_t^\gamma(\underline{S}^\gamma, d_t^\gamma(\underline{S}^\gamma)),$$

which implies that an order is placed for this level of inventory, y, and hence $s_t^\gamma \ge \underline{s}^\gamma$.

For $t > 1$, we have that for $y \le \underline{s}^\gamma$,

$$\begin{aligned}g_t^\gamma(y, d_t^\gamma(y)) &= H^\gamma(y, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha_t d_t^\gamma(y) - \beta_t)\}\\ &\le -k + H^\gamma(\underline{S}^\gamma, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(\underline{S}^\gamma - \alpha_t d_t^\gamma(y) - \beta_t)\}\\ &\le -k + H^\gamma(\underline{S}^\gamma, d_t^\gamma(\underline{S}^\gamma)) + \gamma E\{\phi_{t-1}^\gamma(\underline{S}^\gamma - \alpha_t d_t^\gamma(\underline{S}^\gamma) - \beta_t)\}\\ &= -k + g_t^\gamma(\underline{S}^\gamma, d_t^\gamma(\underline{S}^\gamma)),\end{aligned}$$

where the first inequality follows from the definition of $\underline{s}^\gamma$ and (28) and the second inequality from the definition of d_t^γ.

Consider now $t = 1$. Using the fact that $g_t^\gamma(y, d) = H^0(y, d)$ and the definition of $d_t^\gamma(x)$, $\underline{s}^\gamma$, and $\underline{S}^\gamma$, we have $g_t^\gamma(y, d_t^\gamma(y)) \le -k + g_t^\gamma(\underline{S}_t^\gamma, d_t(\underline{S}_t^\gamma))$ for $y \le \underline{s}^\gamma$.

To show that $s_t^\gamma \le \bar{s}^\gamma$, we apply inequality (24), which implies that no order is placed when $y \ge \bar{s}^\gamma$. Hence, $s_t^\gamma \in [\underline{s}^\gamma, \bar{s}^\gamma]$.

To show that $S_t^\gamma \le \bar{S}^\gamma$, it suffices to show that for $y \ge \bar{S}^\gamma$ we have

$$g_t^\gamma(\bar{s}^\gamma, d_t^\gamma(\bar{s}^\gamma)) \ge g_t^\gamma(y, d_t^\gamma(y)).$$

In fact, for $y \ge \bar{S}^\gamma$,

$$\begin{aligned}g_t^\gamma(\bar{s}^\gamma, d_t^\gamma(\bar{s}^\gamma)) &= H^\gamma(\bar{s}^\gamma, d_t^\gamma(\bar{s}^\gamma)) + \gamma E\{\phi_{t-1}^\gamma(\bar{s}^\gamma - \alpha_t d_t^\gamma(\bar{s}^\gamma) - \beta_t)\}\\ &\ge H^\gamma(\bar{s}^\gamma, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(\bar{s}^\gamma - \alpha_t d_t^\gamma(y) - \beta_t)\}\\ &\ge k + H^\gamma(y, d_t^\gamma(y)) + \gamma E\{\phi_{t-1}^\gamma(y - \alpha_t d_t^\gamma(y) - \beta_t)\} - \gamma k\\ &\ge g_t^\gamma(y, d_t^\gamma(y)),\end{aligned}$$

where the first inequality follows from the definition of d_t^γ, the second inequality from the definition of $\bar{S}^\gamma$ and (22), and the last inequality from definition (5).

Finally, inequality (27) implies that the function $g_t^\gamma(y, d_t^\gamma(y))$ is nondecreasing for $y \le \underline{S}^\gamma$. Hence, $S_t^\gamma \ge \underline{S}^\gamma$, and as a result, $S_t^\gamma \in [\underline{S}^\gamma, \bar{S}^\gamma]$. □

7. Discounted profit case. Consider the discounted profit case with a discount factor $0 < \gamma < 1$ and recall the definition of $\hat{\phi}^\gamma(x)$. Lemma 2 tells us that $\hat{\phi}^\gamma(x)$ is the infinite horizon expected discounted profit for the stationary $(s^\gamma, S^\gamma, d^\gamma_{(s^\gamma, S^\gamma)})$ policy when starting with an initial inventory level x.

The following convergence result relates the t-period maximum total expected discounted profit function, $\phi^\gamma_t(x)$, and $\hat{\phi}^\gamma(x)$.

THEOREM 7.1. *For any $M \geq \max\{\bar{S}^\gamma, S^\gamma\}$ and any $t \geq 1$, we have that*

$$\max_{x \leq M} |\phi^\gamma_t(x) - \hat{\phi}^\gamma(x)| \leq \gamma^{t-1} \max_{x \leq M} |\phi^\gamma_1(x) - \hat{\phi}^\gamma(x)|. \tag{29}$$

PROOF. By induction. For $t = 1$, inequality (29) holds as equality. Consider $t > 1$. From (4) and (21), we have that for any $x \leq M$,

$$\begin{aligned}
\phi^\gamma_t(x) - \hat{\phi}^\gamma(x) &= \max_{M \geq y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y-x) + H^\gamma(y, d) + \gamma E\{\phi^\gamma_{t-1}(y - \alpha d - \beta)\} \\
&\quad - \max_{M \geq y \geq x, \bar{d} \geq d \geq \underline{d}} -k\delta(y-x) + H^\gamma(y, d) + \gamma E\{\hat{\phi}^\gamma(y - \alpha d - \beta)\} \\
&\leq \max_{M \geq y \geq x} -k\delta(y-x) + H^\gamma(y, d^\gamma_t(x)) + \gamma E\{\phi^\gamma_{t-1}(y - \alpha d^\gamma_t(x) - \beta)\} \\
&\quad - (-k\delta(y-x) + H^\gamma(y, d^\gamma_t(x)) + \gamma E\{\hat{\phi}^\gamma(y - \alpha d^\gamma_t(x) - \beta)\}) \\
&= \gamma \max_{M \geq y \geq x} E\{\phi^\gamma_{t-1}(y - \alpha d^\gamma_t(x) - \beta) - \hat{\phi}^\gamma(y - \alpha d^\gamma_t(x) - \beta)\} \\
&\leq \gamma^{t-1} \max_{x \leq M} |\phi^\gamma_1(x) - \hat{\phi}^\gamma(x)|,
\end{aligned}$$

where the first equation follows from Theorem 5.1, Lemma 8, and the assumption that $M \geq \max\{\bar{S}^\gamma, S^\gamma\}$; the first inequality from the definition of d^γ_t (see (6)); and the last inequality from the induction assumption.

By employing a similar approach, we can prove that for $x \leq M$,

$$\hat{\phi}^\gamma(x) - \phi^\gamma_t(x) \leq \gamma^{t-1} \max_{x \leq M} |\phi^\gamma_1(x) - \hat{\phi}^\gamma(x)|.$$

Hence, (29) holds for all t. □

The theorem thus implies that the t-period maximum total expected discounted profit function, $\phi^\gamma_t(x)$, converges to the infinite horizon expected discounted profit function, $\hat{\phi}^\gamma(x)$, associated with the stationary $(s^\gamma, S^\gamma, d^\gamma_{(s^\gamma, S^\gamma)})$ policy, and as a consequence, this policy is optimal for the infinite horizon expected discounted profit problem.

8. Average profit case. In this section we analyze the average profit case, and hence assume that $\gamma = 1$. To prove that a stationary $(s, S, \mathbf{d})$ policy is optimal for the average profit case, we apply a similar approach to the one used by Iglehart (1963b) for the traditional stochastic inventory model. Specifically, we show that the long-run average profit of the best $(s, S, \mathbf{d})$ policy, c^1, is the limit of the maximum average profit per period over a t-period planning horizon.

THEOREM 8.1. *For any x,*

$$\phi^1_t(x)/t - c^1 \to 0, \qquad \text{as } t \to \infty.$$

PROOF. We prove by induction that for any given $M \geq \max\{\bar{S}^1, S^1\}$, there exist r and R such that

$$tc^1 + \phi^1(x) + r \leq \phi^1_t(x) \leq tc^1 + \phi^1(x) + R, \qquad \text{for } x \leq M \text{ and any } t. \tag{30}$$

First, for $x \le \min\{\underline{s}^1, s^1\}$, $\phi^1(x)$ and $\phi^1_t(x)$ are constants. Hence, for $t = 1$, there exist two parameters r and R such that (30) holds for $x \le M$.

Second, assume (30) is true for $t-1$. Since $S^1_t \le \bar{S}^1 \le M$, for $x \le M$ we have

$$\phi^1_t(x) = \max_{M \ge y \ge x, \bar{d} \ge d \ge \underline{d}} -k\delta(y-x) + H^1(y,d) + E\{\phi^1_{t-1}(y - \alpha d - \beta)\},$$

and hence

$$\begin{aligned} \phi^1_t(x) &\le \max_{M \ge y \ge x, \bar{d} \ge d \ge \underline{d}} -k\delta(y-x) + H^1(y,d) + E\{\phi^1(y - \alpha d - \beta)\} + (t-1)c^1 + R \\ &\le \max_{y \ge x, \bar{d} \ge d \ge \underline{d}} -k\delta(y-x) + H^1(y,d) - c^1 + E\{\phi^1(y - \alpha d - \beta)\} + tc^1 + R \\ &= \phi^1(x) + tc^1 + R, \end{aligned}$$

where the first inequality follows from the induction assumption (30), the second inequality holds because we removed the constraint $M \ge y$, and the equality follows from the optimality equation, (21).

The left-hand-side inequality (i.e., the lower bound) of (30) can be established in a similar fashion.

By choosing M arbitrarily large, (30) implies that

$$\phi^1_t(x)/t - c^1 \to 0, \qquad \text{as } t \to \infty$$

for any x. $\square$

The theorem thus suggests that starting with any initial inventory level, the maximum average profit per period over a t-period planning horizon converges to a constant c^1, the long-run average profit of the best $(s, S, \mathbf{d})$ policy. Therefore, the best $(s, S, \mathbf{d})$ policy, the stationary $(s^\gamma, S^\gamma, d^\gamma_{(s^\gamma, S^\gamma)})$ policy, is optimal for the infinite horizon average profit problem.

9. Concluding remarks. In this section we summarize our main results. Recall that for the finite horizon case, Chen and Simchi-Levi (2004) proved that an (s, S, p) policy is not necessarily optimal for general demand processes. Indeed, by developing and employing the concept of symmetric k-convex functions, Chen and Simchi-Levi showed that in this case an (s, S, A, p) policy is optimal.

Surprisingly, in the current paper we show, using the concept of symmetric k-convexity, that a stationary (s, S, p) policy is optimal in the infinite horizon case for both the discounted and average profit criteria. This result holds for the general demand process defined by Assumption 1, which includes additive and multiplicative demand functions; both are common in the economics literature.

One limitation of the models analyzed in this paper as well as in the paper by Chen and Simchi-Levi (2004) is the zero lead-time assumption. This is not the case for standard stochastic inventory control problems. For these problems, the structural results of the optimal policy can generally be extended to models with deterministic lead time. The idea is to transfer a model with positive lead time to one with a similar structure, but zero lead time (Scarf 1960). However, this technique is not valid in our case because for our models with positive lead time, the two decisions, the ordering decision and the pricing decision, will take effect at different times.

As pointed out in §6, the optimality equation characterized in Equation (21) is not sufficient for proving the optimality of a stationary (s, S, p) policy. Indeed, the technique applied by Zheng (1991) for the discounted cost criterion does not work in our case. Thus, we developed lower and upper bounds on the reorder points and order-up-to levels in §6 to prove that a stationary (s, S, p) policy is optimal for the discounted profit case. However, the technique developed by Zheng for the average cost case does work for our models. In fact, this technique allows us to prove without invoking §§5, 6, and 8 (see Appendix B) that a stationary (s, S, p) policy is optimal for the average profit criterion even if Q^γ is only assumed to be quasi-concave.

An interesting question is whether we can extend the results of the continuous pricing model to a discrete price environment. This question is addressed in Appendix C, where we demonstrate that a stationary (s, S) inventory policy is not necessarily optimal for our demand model with discrete prices. This is consistent with the counterexample provided by Thomas (1974), which shows that with discrete prices, an (s, S) inventory policy is not necessarily optimal even for a single-period problem. This has important computational consequences. Indeed, to solve those models numerically, we first have to discretize the inventory variables and the pricing variables; unfortunately, our counterexample in Appendix C suggests that this may destroy the structure of the optimal policy. However, for the average profit case, one can apply the following: First, discretize the inventory variables, and then for each discretized inventory level x, choose the price $\mathbf{d}(x) \in \arg\max_{\bar{d}\geq d\geq \underline{d}} H^\gamma(x, d)$. Notice that by doing that, we can maintain the concavity or quasiconcavity of Q^γ for the discretized model, therefore the optimality of a stationary (s, S, p) policy still holds. Finally, we point out that the counterexample provided in Appendix C illustrates that, in the case of discrete prices, even if a stationary (s, S, p) policy is optimal for the average profit case, a stationary (s, S, p) policy might not be optimal for the discounted profit case.

Upon finishing the first revision of this paper, we were notified of a working paper by Feng and Chen (2003), which analyzes a similar model to ours. Assuming that the single-period profit functions are quasiconcave and focusing on *discrete prices*, they propose an algorithm to find the best (s, S, p) policy and prove that a stationary (s, S, p) policy is optimal under *the average profit criterion*. One problem with their approach is that with discrete prices, a stationary (s, S, p) policy may fail to be optimal, as illustrated by Appendix C. Indeed, in order to guarantee the quasi concavity of their single-period profit functions, Feng and Chen (2003) require the price decision variable to be continuous. Unfortunately, in their discrete price model, there is no guarantee that the single-period profit function is still quasi concave.

Appendix A. In this appendix we show that all the results in §3, §4, and §5 hold for the discounted case and for the average case if $\Pr\{\alpha \underline{d} + \beta = 0\} < 1$.

To do that, we construct a sequence of random variables α_η such that

(R_a) $E\{\alpha_\eta\} = 1$;

(R_b) α_η is bounded below by a positive constant;

(R_c) α_η converges to α in distribution as $\eta \downarrow 0$.

Let $F(x)$ be the cumulative probability distribution of α. Without loss of generality, assume that $\int_1^\infty x\,dF(x) > 0$, and for any $\eta < 1$, let

$$q_\eta = \frac{1-F(\eta)}{\int_\eta^\infty (x-\eta)\,dF(x)} \frac{\int_0^\eta (\eta - x)\,dF(x)}{F(\eta)},$$

and

$$p_\eta = \frac{q_\eta F(\eta)}{1-F(\eta)}.$$

Since for $0 < \eta < 1$, $\int_\eta^\infty (x-\eta)\,dF(x) \geq \int_1^\infty x\,dF(x) - \eta(1-F(\eta))$ and $\int_0^\eta (\eta - x)\,dF(x) \leq \eta F(\eta)$, we have $q_\eta = O(\eta)$ and $p_\eta = O(\eta)$. (Note that here we use $O(\cdot)$ to denote the big-O notation, which should be distinguished from the function O^γ used in Theorem 5.1.) Furthermore,

$$F(\eta)(1+q_\eta) + (1-p_\eta)(1-F(\eta)) = 1, \tag{31}$$

and

$$\eta F(\eta)(1+q_\eta)+(1-p_\eta)\int_\eta^\infty x\,dF(x)=\int_0^\infty x\,dF(x)=1. \tag{32}$$

Define a function F_η such that

$$F_\eta(x)=\begin{cases}0, & \text{for } x<\eta,\\ (1+q_\eta)F(\eta)+(1-p_\eta)(F(x)-F(\eta)), & \text{for } x\ge\eta.\end{cases}$$

Equation (31) implies that F_η is a distribution function. Let α_η be a random variable with distribution F_η. Then by (32), $E\{\alpha_\eta\}=1$ and the requirements (R_a), (R_b), and (R_c) are satisfied.

We are ready to relax the assumption that $\alpha\underline{d}+\beta$ is bounded below by a positive constant. For this purpose, consider a similar model with α replaced by α_η for $\eta>0$. We refer to this model as the *modified problem*.

In the modified problem, $\alpha_\eta\underline{d}+\beta$ is bounded below by a positive constant. Let $c_\eta^\gamma(s,S)$ be the average discounted profit per period for the stationary (s,S) policy associated with the best price under this modified model. Define

$$F_\eta^\gamma := \big\{(s,S)\mid c_\eta^\gamma(s,S)\ge -k+\max Q_\eta^\gamma(x),\, Q_\eta^\gamma(s)=c_\eta^\gamma(s,S) \text{ and } Q_\eta^\gamma(S)\ge c_\eta^\gamma(s,S)\big\},$$

where $Q_\eta^\gamma(x)=\max_{\bar d\ge d\ge \underline{d}} H_\eta^\gamma(x,d)$ and $H_\eta^\gamma(x,d)=\hat R(d)-E\{h^\gamma(x-\alpha_\eta d-\beta)\}$. From the construction of α_η, one can directly verify that $H_\eta^\gamma(x,d)$ converges to $H^\gamma(x,d)$ uniformly over any bounded set of x as $\eta\to 0$, which in turn implies that Q_η^γ converges to Q^γ uniformly over any bounded set of x as $\eta\to0$. Therefore, by Assumption 2, F_η^γ is uniformly bounded for $0<\eta\le\bar\eta$.

Let $(s_\eta^\gamma,S_\eta^\gamma)$ be the best (s,S) policy under the modified model with parameter η, and let $c_\eta^\gamma=c_\eta^\gamma(s_\eta^\gamma,S_\eta^\gamma)$. Define

$$\phi_\eta^\gamma(x)=\begin{cases}0, & \text{for } x\le s_\eta^\gamma,\\ \max\limits_{\bar d\ge d\ge \underline{d}} H_\eta^\gamma(x,d)-c_\eta^\gamma+\gamma E\{\phi_\eta^\gamma(x-\alpha_\eta d-\beta)\}, & \text{for } x\ge s_\eta^\gamma.\end{cases} \tag{33}$$

Since F_η^γ is uniformly bounded for $0<\eta\le\bar\eta$ for some $\bar\eta>0$, there exists a limit point, say (s_0^γ,S_0^γ), for some subsequence $(s_{\eta_i}^\gamma,S_{\eta_i}^\gamma)$, where $\eta_i\to0$ as $i\to\infty$. As we prove in Lemma 9 (which is presented at the end of this appendix), there exists a subsequence of $\{\phi_{\eta_i}^\gamma\}$ (without loss of generality, assume the sequence itself) that converges to a continuous function ϕ_0^γ uniformly over any bounded set of x.

Lemma 4 part (b), together with the fact that Q_η^γ converges to Q^γ uniformly over a bounded set as $\eta\to0$, implies that $c_{\eta_i}^\gamma$ converges to a point c_0^γ. This property, together with the fact that $\phi_{\eta_i}^\gamma$ converges to a function ϕ_0^γ uniformly over any bounded set of x as $i\to\infty$, implies that ϕ_0^γ satisfies the recursion (33) with $\eta=0$, where $H_0^\gamma(x,d)=H^\gamma(x,d)$ and $\alpha_0=\alpha$. Furthermore, since $\phi_\eta^\gamma(S_\eta^\gamma)=k$, $\phi_0^\gamma(S_0^\gamma)=k$, and hence $c_0^\gamma=c^\gamma(s_0^\gamma,S_0^\gamma)$. Therefore, Lemma 4 and Lemma 5 hold and $(\phi_0^\gamma,c_0^\gamma)$ satisfies (21). Moreover, the technique used in Lemma 9 allows us to prove that $I^\gamma(s,x,\mathbf{d})$ and $M^\gamma(s,x,\mathbf{d})$ are well defined. Thus, all the results in §3, §4, and §5 hold for the discounted case and for the average case if $\Pr\{\alpha\underline{d}+\beta=0\}<1$.

LEMMA 9. *There exists a subsequence of $\{\phi_{\eta_i}^\gamma\}$ that converges to a continuous function uniformly over any compact set of x as $\eta_i\to0$, if $0<\gamma<1$ or $\gamma=1$ and* $\Pr\{\alpha\underline{d}+\beta=0\}<1$.

PROOF. We employ the Arzela-Ascoli Theorem (that can be found in most standard textbooks on functional analysis), which states that if a sequence of continuous functions defined in a compact metric space is uniformly bounded and equicontinuous, then it has a uniformly convergent subsequence.

Let $\underline{x} \leq \inf_{0<\eta\leq\bar{\eta}} s_\eta^\gamma$, $\bar{x}$ be any point with $\bar{x} \geq \underline{x}$, and $I = [\underline{x}, \bar{x}]$. We will prove

(1) Uniform boundedness: There exists a constant M such that $|\phi_\eta^\gamma(x)| \leq M$ for any $0 < \eta \leq \bar{\eta}$, and $x \in I$.

(2) Equicontinuity: For any given $\epsilon > 0$, there exists a constant $\delta > 0$ such that for any $x \in I$ and $y \in I$ with $|x - y| < \delta$, $|\phi_\eta^\gamma(x) - \phi_\eta^\gamma(y)| < \epsilon$ for any $0 < \eta \leq \bar{\eta}$.

We will distinguish between the discounted case and the average case. First we focus on the discounted case. We show the following:

(a) Uniform boundedness. From Lemma 4, we have $\phi_\eta^\gamma(x) \leq k$ for any x and $\eta > 0$. Now assume that $\underline{M} = \min_{x\in I,\, \bar{d}\geq d\geq \underline{d},\, 0<\eta\leq\bar{\eta}} H_\eta^\gamma(x, d) - c_\eta^\gamma$. Then we have that for any $x \geq s_\eta^\gamma$,

$$\begin{aligned}\phi_\eta^\gamma(x) &= \max_{\bar{d}\geq d\geq \underline{d}} H_\eta^\gamma(x, d) - c_\eta^\gamma + \gamma E\{\phi_\eta^\gamma(x - \alpha_\eta d - \beta)\}\\ &\geq \underline{M} + \gamma \min_{y\in I} \phi_\eta^\gamma(y),\end{aligned}$$

which implies that $\min_{y\in I} \phi_\eta^\gamma(y) \geq \underline{M}/(1-\gamma)$. Therefore, $|\phi_\eta^\gamma(x)|$ is uniformly bounded for any $x \in I$ and $0 < \eta \leq \bar{\eta}$.

(b) Equicontinuity. For any constant $\delta > 0$, if $x, y \geq s_\eta^\gamma$ with $|x - y| < \delta$, we have that

$$\phi_\eta^\gamma(x) - \phi_\eta^\gamma(y) \leq \max_{\bar{d}\geq d\geq \underline{d}} H_\eta^\gamma(x, d) - H_\eta^\gamma(y, d) + \gamma \max_{|x'-y'|<\delta} |\phi_\eta^\gamma(x') - \phi_\eta^\gamma(y')|, \tag{34}$$

and if $x \geq s_\eta^\gamma \geq y$ with $|x - y| < \delta$, we have

$$\phi_\eta^\gamma(x) - \phi_\eta^\gamma(y) \leq \max_{\bar{d}\geq d\geq \underline{d}} H_\eta^\gamma(x, d) - c_\eta^\gamma + \gamma \max_{|x'-y'|<\delta} |\phi_\eta^\gamma(x') - \phi_\eta^\gamma(y')|. \tag{35}$$

Notice that $H_\eta^\gamma(x, d)$ is continuous for any (η, x, d) with $\eta \geq 0$. Hence, for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that for any $d \in [\underline{d}, \bar{d}]$, $|x - y| < \delta$, and $0 \leq \eta \leq \bar{\eta}$,

$$|H_\eta^\gamma(x, d) - H_\eta^\gamma(y, d)| < \epsilon,$$

and for any $|x - s_\eta^\gamma| < \delta$ and $0 \leq \eta \leq \bar{\eta}$,

$$|Q_\eta^\gamma(x) - c_\eta^\gamma| < \epsilon.$$

This, together with inequalities (34) and (35), imply that for any $\eta > 0$,

$$\max_{|x'-y'|<\delta} |\phi_\eta^\gamma(x') - \phi_\eta^\gamma(y')| \leq \epsilon/(1-\gamma).$$

Therefore, $\{\phi_\eta^\gamma\}_{0<\eta\leq\bar{\eta}}$ is equicontinuous over set I.

Now we concentrate on the average case. We have the following:

(a) Uniform boundedness. From Lemma 4, $\phi_\eta^1(x) \leq k$ for any $\eta > 0$ and x. Assume that $\underline{M} = \min_{x\in I,\, \bar{d}\geq d\geq \underline{d},\, 0<\eta\leq\bar{\eta}} H_\eta^1(x, d) - c_\eta^1$. If $\underline{M} \geq 0$, $\phi_\eta^1(x) \geq 0$ for any $x \in I$. If $\underline{M} < 0$, we have that for any $x \geq s_\eta^1$,

$$\phi_\eta^1(x) \geq \max_{\bar{d}\geq d\geq \underline{d}} \underline{M} + E\{\phi_\eta^1(x - \alpha d - \beta)\}.$$

Construct a function ψ such that

$$\psi(x) = \begin{cases} 0, & \text{for } x \leq s_\eta^1,\\ \underline{M} + E\{\psi(x - \alpha\bar{d} - \beta)\}, & \text{for } x \geq s_\eta^1.\end{cases}$$

Since the above recursion is the renewal-type equation, we have that $\psi(x) = \underline{M}(1 + m(x - s_\eta^1))$, where $m(x) = \sum_{n=1}^{\infty} F_n(x)$ and $F_n(x)$ is the distribution of the summation of n independently and identically distributed random variables, each of which has the same distribution as $\alpha\bar{d} + \beta$. Since $\Pr\{\alpha\underline{d} + \beta = 0\} < 1$, we have $m(x) < \infty$. (See Ross 1970 for related results about the renewal theory and the renewal-type equation.) Furthermore, it is easy to see that $\phi_\eta^1(x) \geq \psi(x)$ for any x. Therefore, for any $x \in I$, we have

$$\phi_\eta^1(x) \geq \psi(x) = \underline{M}(1 + m(x - s_\eta^1)) \geq \underline{M}(1 + m(\bar{x} - \underline{x})).$$

Hence, $\{\phi_\eta^1\}_{0<\eta\leq\bar{\eta}}$ is uniformly bounded over set I.

(b) Equicontinuity. $H_\eta^1(x, d)$ is continuous in (η, x, d) and hence, $H_\eta^1(x, d)$ is uniformly continuous over any bounded set of (η, x, d). Thus, for any given $\epsilon > 0$, we can choose $\delta > 0$, satisfying the following two requirements:

— $|H_\eta^1(x, d) - H_\eta^1(y, d)| < \epsilon$ for any $0 < \eta < \bar{\eta}$, any $x, y \in I$ with $|x - y| < \delta$ and $d \in [\underline{d}, \bar{d}]$;

— $|Q_\eta^1(x) - c_\eta^1| < \epsilon$ for any $|x - s_\eta^1| < \delta$.

For any $x, y \geq s_\eta^1$, we have that

$$\phi_\eta^1(x) - \phi_\eta^1(y) \leq \max_{\bar{d}\geq d\geq \underline{d}} H_\eta^1(x, d) - H_\eta^1(y, d) + E\{\phi_\eta^1(x - \alpha d - \beta) - \phi_\eta^1(y - \alpha d - \beta)\}, \tag{36}$$

and for $x \geq s_\eta^1 \geq y$, we have that

$$\phi_\eta^1(x) - \phi_\eta^1(y) \leq \max_{\bar{d}\geq d\geq \underline{d}} H_\eta^1(x, d) - c_\eta^1 + E\{\phi_\eta^1(x - \alpha d - \beta) - \phi_\eta^1(y - \alpha d - \beta)\}. \tag{37}$$

Construct a function ψ such that

$$\psi(x) = \begin{cases} 0, & \text{for } x \leq s_\eta^1 - \delta, \\ \epsilon + E\{\psi(x - \alpha\underline{d} - \beta)\}, & \text{for } x \geq s_\eta^1 - \delta. \end{cases}$$

As before, this recursion is a renewal-type equation. Hence, we have that $\psi(x) = \epsilon(1 + m(x - s_\eta^1 + \delta))$, where $m(x) = \sum_{n=1}^{\infty} F_n(x)$ and $F_n(x)$ is the distribution of the summation of n independently and identically distributed random variables, each of which has the same distribution as $\alpha\underline{d} + \beta$. Since $\Pr\{\alpha\underline{d} + \beta = 0\} < 1$, we have $m(x) < \infty$ and $\psi(x)$ is a nondecreasing function of x. (Again, see Ross 1970 for related results about the renewal theory and the renewal-type equation.) Since $\psi(x)$ is nondecreasing in x, we have

$$\psi(x) = \begin{cases} 0, & \text{for } x \leq s_\eta^1 - \delta, \\ \max\limits_{\bar{d}\geq d\geq \underline{d}} \epsilon + E\{\psi(x - \alpha d - \beta)\}, & \text{for } x \geq s_\eta^1 - \delta. \end{cases}$$

Furthermore, one can show that for any $x, y \in I$ with $x \geq y$ and $|x - y| < \delta$, $|\phi_\eta^1(x) - \phi_\eta^1(y)| \leq \psi(x)$. Therefore, inequalities (36) and (37) imply that for $x, y \in I$ with $x \geq y$ and $|x - y| < \delta$,

$$|\phi_\eta^1(x) - \phi_\eta^1(y)| \leq \psi(x) = \epsilon(1 + m(x - s_\eta^1 - \delta)) \leq \epsilon(1 + m(\bar{x} - \underline{x} + \delta)).$$

Hence, $\{\phi_\eta^1\}_{0<\eta\leq\bar{\eta}}$ is equicontinuous over set I.

We now construct a subsequence of $\{\phi_{\eta_i}^\gamma\}_{i=1}^\infty$ that converges to a continuous function uniformly over any bounded set by using the famous diagonization procedure as follows. First, we have that $\{\phi_\eta^\gamma\}_{0<\eta\leq\bar{\eta}}$ is uniformly bounded and equicontinuous over set I. Therefore, there exists a subsequence of $\{\phi_{\eta_i}^\gamma\}_{i=1}^\infty$, say $\{\phi_{0,i}^\gamma\}_{i=0}^\infty$, which converges to a continuous function uniformly over set I (in the sense of $\|\cdot\|_\infty$). Similarly, there exists a subsequence of $\{\phi_{0,i}^\gamma\}_{i=0}^\infty$, say $\{\phi_{1,i}^\gamma\}_{i=0}^\infty$, which converges to a continuous function uniformly over set $[\underline{x}, \bar{x} + 1]$. Continuing with this process, we can construct a subsequence of $\{\phi_{n-1,i}^\gamma\}_{i=0}^\infty$, say $\{\phi_{n,i}^\gamma\}_{i=0}^\infty$, which converges to a continuous function uniformly over set $[\underline{x}, \bar{x} + n]$. Then the following subsequence $\{\phi_{n,n}^\gamma\}_{n=0}^\infty$ converges to a continuous function uniformly over any compact set. □

Appendix B. In this appendix, we show that the quasi concavity of Q^γ suffices for the optimality of a stationary (s, S, p) policy under the average profit criterion.

The idea, which is proposed by Zheng (1991), is to focus on a relaxed model, i.e., a model where negative orders are allowed and a fixed cost k is charged whenever a negative order is placed. Here is the optimality equation for the relaxed model.

$$\hat{\phi}^\gamma(x) + c^\gamma = \max_y \Big\{ \max_{\bar{d} \ge d \ge \underline{d}} -k\delta(|y-x|) + H^\gamma(y, d) + \gamma E\{\hat{\phi}^\gamma(y - \alpha d - \beta)\} \Big\}. \tag{38}$$

Lemma 4 allows us to construct a solution for (38) as follows:

$$\begin{aligned} \hat{\phi}^\gamma(x) &= 0, \qquad \text{for } x < s^\gamma, \\ \hat{\phi}^\gamma(x) &= \hat{O}^\gamma(x), \qquad \text{for } x \in [s^\gamma, S^\gamma], \\ \hat{\phi}^\gamma(x) &= \max\{0, \hat{O}^\gamma(x)\}, \qquad \text{for } x > S^\gamma, \end{aligned} \tag{39}$$

where $\hat{O}^\gamma(x) = \max_{\bar{d} \ge d \ge \underline{d}} H^\gamma(x, d) - c^\gamma + \gamma E\{\hat{\phi}^\gamma(x - \alpha d - \beta)\}$, which is similar to the function O^γ defined in the proof of Theorem 5.1. Again, we can start by assuming that $\alpha\underline{d} + \beta \ge \eta$ for some $\eta > 0$ and relax this assumption by employing the same technique used in Appendix A. Since the argument is essentially the same, the analysis is omitted.

Now we show that $(\hat{\phi}^\gamma, c^\gamma)$ is a solution for (38). First, we claim that $\hat{O}^\gamma(x) \le k$ for any x and $\hat{O}^\gamma(S^\gamma) = k$. Since for $x \le S^\gamma$, $\hat{O}^\gamma(x) = O^\gamma(x) \le k$, we only need to prove $\hat{O}^\gamma(x) \le k$ for $x > S^\gamma$.

Assume to the contrary that $\hat{O}^\gamma(x^*) > k$ for some $x^* > S^\gamma$. Then there exists some y with $y \le x^*$, such that $\hat{O}^\gamma(y) < 0$; otherwise, $\hat{\phi}^\gamma(x) = \phi^\gamma(x)$ for $s^\gamma \le x \le x^*$, and Lemma 4 part (a) implies that $k < \hat{O}^\gamma(x^*) = \hat{\phi}^\gamma(x^*) \le k$. Let $y^* = \inf\{y\colon \hat{O}^\gamma(y) < 0\}$. Then we have $y^* \ge S^\gamma$ and $Q^\gamma(y^*) \le c^\gamma$. However, by induction one can show $\hat{\phi}^\gamma(x) \le k$ for any $x \ge y^*$, which contradicts the fact that $\hat{\phi}^\gamma(x^*) = \hat{O}^\gamma(x^*) > k$. Hence, $\hat{O}^\gamma(x) \le k$ for $x > S^\gamma$.

Now we are ready to show that $(\hat{\phi}^\gamma, c^\gamma)$ is a solution for (38). In fact, the definitions of $\hat{\phi}^\gamma$ and $\hat{O}^\gamma$, together with Lemma 4, imply that $\hat{O}^\gamma(x) \le 0$ for $x \le s^\gamma$, $\hat{O}^\gamma(S^\gamma) = k$, and $\hat{O}^\gamma(x) \le k$ for any x. Therefore, it is easy to see that the function $\hat{\phi}^\gamma$ defined in (39), together with c^γ, is a solution for the optimality Equation (38) for the relaxed model. The following modified (s^γ, S^γ) policy attains the first maximization of the optimality Equation (38): Place an order to increase the inventory level to S^γ when the initial inventory level is less than s^γ; make a negative order to reduce the inventory level to S^γ or do nothing (depending on which one is more profitable) when the initial inventory level is above S^γ; and do nothing when the initial inventory level is between s^γ and S^γ. Thus, by employing Theorem 2.1 from Ross (1983, p. 93), this stationary modified (s^γ, S^γ) inventory policy solves the relaxed model since $\hat{\phi}^\gamma$ is bounded. (Notice that Theorem 2.1 in Ross 1983 is proven for discrete state spaces. However, one can extend this result by essentially following the proof given in Ross 1983 to problems involving even continuous state spaces.)

Finally, we prove that the stationary (s^1, S^1) inventory policy is optimal for the original model under the average profit criterion. The argument goes as follows: For the relaxed model, the stationary modified (s^1, S^1) inventory policy suggested in the above paragraph differs from the stationary (s^1, S^1) inventory policy in at most one period when a negative order is placed to reduce the inventory level to S^1. Once the inventory level is below S^1, it will never exceed S^1 again. Hence, the two inventory policies, the stationary modified (s^1, S^1) inventory policy and the stationary (s^1, S^1) inventory policy, give the same long-run average profit per period, which implies that the stationary (s^1, S^1) inventory policy is also optimal for the relaxed model. Thus, it is optimal for the original model. Finally, it is appropriate to point out that our result holds when Q^γ is only assumed to be quasi-concave because it suffices for Lemma 4, as we already observed.

Appendix C. In this appendix we demonstrate that a stationary (s, S) inventory policy is not necessarily optimal for the infinite horizon joint inventory control and pricing problem with discrete prices.

Consider the following example with deterministic demand:

$$c = 0, \qquad d \in \{3, 4\}, \qquad R(d) = d(7 + \varepsilon - d), \qquad h(x) = \rho|x|, \tag{40}$$

where $\rho > 0$, and ε is sufficiently small. We will investigate two examples.

First, let $k = 0$ and $\varepsilon < 0$. Assume by contradiction that a stationary base stock policy is optimal. Then,

$$Q^\gamma(x) = \begin{cases} 12 + 3\varepsilon - \rho|x-3|, & \text{for } x \le 3.5 + 0.5\varepsilon/\rho, \\ 12 + 4\varepsilon - \rho|x-4|, & \text{otherwise.} \end{cases}$$

Since 3 is the global maximizer of $Q^\gamma(x)$, one can see that in this case the optimal base stock level is $s = 3$ and $c^\gamma = 12 + 3\varepsilon$. However, the function

$$\phi(x) = \begin{cases} 0, & \text{if } x \le 3, \\ \max\{-\rho|x-3| + \gamma\phi(x-3), \varepsilon - \rho|x-4| + \gamma\phi(x-4)\}, & \text{if } x \ge 3, \end{cases}$$

does not satisfy the optimality Equation (21). In particular, when $x = 3.5$, it is optimal to place an order because making an order to raise the inventory level to $y = 4$ yields a higher expected profit. Therefore, a stationary base stock policy may not be optimal for the discounted case. However, the stationary base stock policy with base stock level $s = 3$ is still optimal for the average case because in a finite number of periods the inventory level will drop to below the base stock level, and from then on the inventory level will never exceed the base stock level. We notice that even though the stationary base stock policy is optimal for the average case, the base stock policy does not achieve the maximization in the optimality Equation (21).

We now show, by looking at a modified example, that a stationary (s, S) may not be optimal for the average profit case ($\gamma = 1$). Assume that $k > \varepsilon > 0$ and $\rho \gg 1$ in (40). In addition, we introduce a small random perturbation to the demand. Specifically, in this case, the realized demand is αd, where $d \in \{3, 4\}$ and $\Pr\{\alpha = 1\} = 1 - 2(\eta + \eta^2)$, $\Pr\{\alpha = 0.25\} = \eta$, $\Pr\{\alpha = 1.75\} = \eta$, $\Pr\{\alpha = 0.2\} = \eta^2$, $\Pr\{\alpha = 1.8\} = \eta^2$. Hence, $E\{\alpha d\} = 1$. Let $Q^1_\eta(x)$ be the single-period maximum expected profit for a given inventory level x for this modified model. In the following, we assume that $\rho\eta$ is sufficiently small.

Notice that $S_0 = 4$ is the global maximizer of function Q^1. For any feasible expected demand function, the inventory level will drop from $S_0 = 4$ to a level no more than $S_0 - 0.6$ in just one period. Thus, the average profit per period associated with the stationary policy (s^*, S_0), $s^* = S_0 - 0.5$ and its corresponding best pricing strategy is $\hat{c} = -k + Q^1_\eta(S_0) = -k + Q^1(S_0) + O(\rho\eta)$. Let $s_0 = 3 - (k-\varepsilon)/\rho$, $a_0 = 3 + (k-\varepsilon)/\rho$, $b_0 = 4 - k/\rho$, and $d_0 = 4 + k/\rho$. It is easy to see that s_0, a_0, b_0, and d_0 are the solutions for the equation $Q^1(x) = -k + Q^1(S_0)$.

In the following, we argue by contradiction that a stationary (s, S) policy is not optimal. In fact, assume that a stationary (s, S) policy is optimal and c^1_η is the average profit per period associated with the optimal stationary (s, S) policy. Since $\rho\eta$ is sufficiently small, there are four solutions for the equation $Q^1_\eta(x) = \hat{c}$, which are denoted by s_η, a_η, b_η, and d_η with $s_\eta \le a_\eta \le b_\eta \le d_\eta$. Also notice that Q^1_η and Q^1 are piecewise linear functions and the difference between Q^1_η and Q^1 is $O(\rho\eta)$, which implies that the differences between $s_\eta, a_\eta, b_\eta, d_\eta$ and s_0, a_0, b_0, d_0 are $O(\rho\eta)$, respectively. Following a proof similar to the one of Lemma 4, one can see that there exist optimal s and S such that $Q^1_\eta(S) \ge Q^1_\eta(s) = c^1_\eta \ge \hat{c}$. Therefore, $s, S \in [s_\eta, a_\eta] \cup [b_\eta, d_\eta]$.

Notice that the inventory level will drop from S to a level below s in exactly one period with probability no less than $1-\eta-\eta^2$. Thus, from the definitions of $I^\gamma(s,x,\mathbf{d})$, $M^\gamma(s,x,\mathbf{d})$, and $c^\gamma(s,S,\mathbf{d})$ in (9), (10), and (12), we have $c^1(s,S)=-k+Q^1_\eta(S)+O(\rho\eta)$. Since $S_0=4$ is the global maximizer of Q^1, $S=S_0+O(\rho\eta)$ and at the inventory level S, the expected demand associated with the best-pricing strategy is $\mathbf{d}(S)=4$. Let

$$x_1=S-0.25\mathbf{d}(S)=S-1 \quad \text{and} \quad x_2=S-0.2\mathbf{d}(S)=S-0.8.$$

Since $\rho\eta$ is sufficiently small, $s_\eta<x_1<a_\eta<x_2<b_\eta<S<d_\eta$ and $x_1=3+O(\rho\eta)$.

We now argue that $s\in[s_\eta,x_1]$. This is done by distinguishing between three cases.

Case (a). $s\in[b_\eta,d_\eta]$. In this case, $M^1(s,S,\mathbf{d})=1$ and $I^1(s,S,\mathbf{d})\le Q^1_\eta(S)$ for any feasible $\mathbf{d}$. Hence, $c^1(s,S,\mathbf{d})\le -k+Q^1_\eta(S)$.

Case (b). $s\in(x_1,a_\eta]$. The inventory will drop from S to x_2 with probability η^2 and to a level less than s with probability $1-\eta^2$ in just one period. Thus, from the definitions of $I^\gamma(s,x,\mathbf{d})$, $M^\gamma(s,x,\mathbf{d})$, and $c^\gamma(s,S,\mathbf{d})$ in (9), (10), and (12),

$$c^1(s,S)=\frac{-k+Q^1_\eta(S)+\eta^2Q^1_\eta(x_2)}{1+\eta^2}.$$

Case (c). $s\in[s_\eta,x_1]$. The inventory will drop from S to x_1 with probability η, to x_2 with probability η^2, and to a level less than s with probability $1-\eta-\eta^2$ in just one period, and within one additional period, the inventory level will drop to below s from inventory level x_1 or x_2. Again, from the definitions of $I^\gamma(s,x,\mathbf{d})$, $M^\gamma(s,x,\mathbf{d})$, and $c^\gamma(s,S,\mathbf{d})$ in (9), (10), and (12), it is easy to see that

$$c^1(s,S)=\frac{-k+Q^1_\eta(S)+\eta Q^1_\eta(x_1)+\eta^2Q^1_\eta(x_2)}{1+\eta+\eta^2}.$$

Since $Q^1_0(x_1)>-k+Q^1_0(S)>Q^1_0(x_2)$ and $O(\rho\eta)$ is sufficiently small,

$$\frac{-k+Q^1_\eta(S)+\eta Q^1_\eta(x_1)+\eta^2Q^1_\eta(x_2)}{1+\eta+\eta^2}>-k+Q^1_\eta(S)>\frac{-k+Q^1_\eta(S)+\eta^2Q^1_\eta(x_2)}{1+\eta^2}.$$

Thus, $s\in[s_\eta,x_1]$, $c^1_\eta=[-k+Q^1_\eta(S)+\eta Q^1_\eta(x_1)+\eta^2Q^1_\eta(x_2)]/(1+\eta+\eta^2)$, and $S=S_0+O(\rho\eta)$.

We now show that the stationary (s,a_η,b_η,S) inventory policy and its associated best-pricing strategy yields an average profit per period strictly greater than c^1_η. In such a policy, we raise the inventory level to S when the initial inventory level is less than s or in $[a_\eta,b_\eta]$; otherwise, no order is placed. To compute the average profit per period associated with the stationary (s,a_η,b_η,S) inventory policy, we define I and M similarly to the definitions $I^\gamma(s,S,\mathbf{d})$ and $M^\gamma(s,S,\mathbf{d})$ in (9) and (10). In particular, for the $(s_\eta,a_\eta,b_\eta,S,\mathbf{d})$ policy with $\mathbf{d}(S)=4$, $\mathbf{d}(x_1)=4$, $\mathbf{d}(x_2)=3$, let $I(x)$ be the expected profit incurred during a horizon that starts with initial inventory level x and ends, at this period or a later period, with an inventory level less than s, and let $M(x)$ be the expected time to drop from initial inventory level x to a level below s. Therefore, $I(x)=M(x)=0$ for $x<s$,

$$I(S)=Q^1_\eta(S)+\eta I(x_1)+\eta^2I(x_2), \qquad I(x_1)=Q^1_\eta(x_1), \qquad I(x_2)=-k+Q^1_\eta(S),$$

and

$$M(S)=1+\eta M(x_1)+\eta^2M(x_2), \qquad M(x_1)=1, \qquad M(x_2)=M(S).$$

Therefore, the average profit per period associated with the stationary $(s,a_\eta,b_\eta,S,\mathbf{d})$ policy is

$$\tilde{c}=\frac{-k+I(S)}{M(S)}=\frac{-k+Q^1_\eta(S)+\eta Q^1_\eta(x_1)}{1+\eta}.$$

It is easy to see that $\tilde{c}>c^1_\eta$. This is a contradiction, which implies that a stationary (s,S) policy is not optimal.

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