Preservation of Supermodularity in Two Dimensional Parametric Optimization Problems and its Applications

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Abstract: This paper establishes a new preservation property of supermodularity in a class of two dimensional parametric optimization problems, where the constraint set may not be a lattice. This property and its extensions include several existing results in the literature as special cases, and provide powerful tools as we illustrate their applications to several operations models.

Key words: Preservation of supermodularity, parametric optimization problems, inventory and pricing

1. Introduction

The concept of supermodularity provides a convenient tool in deriving monotone comparative statics in parameterized optimization problems. In many Markovian decision processes, one is concerned whether supermodularity can be preserved under dynamic programming recursions to derive structural results of optimal polices. One of the key preservation properties states that if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are lattices, $\boldsymbol{D}$ is a sublattice of $\boldsymbol{X} \times \boldsymbol{Y}$, and $g(\boldsymbol{x}, \boldsymbol{y})$ is supermodular in $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D}$, then the function $f(\boldsymbol{x}) = \max_{\boldsymbol{y}}[g(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D}]$ is supermodular on the set of $\boldsymbol{x}$ for which the maximization is well defined (see Topkis 1998, Theorem 2.7.6). Under the above conditions, one can also show that the optimal solution set is increasing in $\boldsymbol{x}$. The above preservation property is powerful and widely used in many problems. However, to apply it, the set $\boldsymbol{D}$ is required to be a sublattice. Relaxing the lattice requirement has been proven a significant challenge. Indeed, without the lattice condition, the analysis becomes much more complicated even in some very simple settings in which supermodularity can be preserved.

The objective of this paper is to establish a new preservation property of supermodularity under optimization operations when the constraint set may not be a lattice. Specifically, consider the following optimization problem parameterized by *two dimensional* vectors $\boldsymbol{x} \in \boldsymbol{S} = \{A\boldsymbol{y} : \boldsymbol{y} \in \boldsymbol{D}\}$,

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \left\{g(\boldsymbol{y}) : A\boldsymbol{y} = \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}\right\}, \tag{1}$$

where A is a $2 \times n$ matrix, $\boldsymbol{D}$ is a closed convex sublattice of $\Re^n$ and g is an n-dimensional function defined on $\boldsymbol{D}$. Throughout of this paper, we assume that the maximization is well defined whenever $\boldsymbol{x} \in \boldsymbol{S}$. Our main result Theorem 1 shows that f is concave and supermodular on $\boldsymbol{S}$ if A is non-negative and g is concave and supermodular on $\boldsymbol{D}$. Several extensions are also presented.

The significance of the above result is that the constraint set $\{(\boldsymbol{x}, \boldsymbol{y}) : A\boldsymbol{y} = \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}\}$ is not a lattice in general and may not be mapped to become one by a variable transformation. Of course, by relaxing the lattice requirement, we have to assume concavity of the objective function and impose a requirement on the dimension of the parameter vector. In addition, in general the optimal solution set is not monotone in $\boldsymbol{x}$. Though it may appear restrictive, relaxing the above assumptions even slightly may render the preservation property invalid. More importantly, the property and its extensions include several existing results in the literature as special cases, and they prove quite powerful as we illustrate their applications to several operations models.

Our first application is a two-product coordinated pricing and inventory control problem with cross-price effects over a finite planing horizon. In this model, the retailer observes the initial inventories of two products at the beginning of each period, and then simultaneously decide their prices and the ordering quantities. The demand of each product during a period is stochastic and depends not only on its own price but also the price of the other product. The objective is to maximize the total expected profit over the planning horizon assuming zero lead time and backlogging of unfilled demand.

In the second application, we consider a two-stage coordinated dynamic pricing and inventory control problem over a finite planning horizon. In the model, the firm observes the initial raw material inventory level and the finished product inventory level at the beginning of each period, and then decides the amount of raw material to be purchased, the amount of product produced from the raw material, and the selling price of product. Demand of the product is stochastic and depends on its price. There is no lead time for delivery and unused inventory is carried over to the next period. The objective is to maximize the total profit over the whole horizon.

In the third application, we consider a self-financing retailer who sells a single product over a finite planning horizon with its operational decisions limited by its cash flow. At the beginning of each period, the retailer observes the initial inventory level of the product and its available capital on hand, and then decides the amount of product to be ordered such that the ordering costs do not exceed the available capital. The delivery is immediate, unused capital is deposited to the savings account. After demand during the period is realized, unused inventory is carried over to the next period and unsatisfied demand is lost. The retailer obtains its profit by either depositing the unused capital or selling the product. The objective is to maximize the total profit over the planning horizon.

Our first and second applications fall into the fast growing literature on integrated inventory and pricing models, for which we refer to Chen and Simchi-Levi (2011) for an up-to-date survey. Papers directly related to our first application include Zhu and Thonemann (2009), Song and Xue (2007) and Ceryan et al. (2009), who analyze models with substitutable products and develop structural properties of the optimal policies. Our second and third applications are extension of Yang (2004) and Chao et al. (2008), respectively.

Compared with these papers, our approach based on the results developed in this paper is significantly simpler and provides additional insights to these applications. For instance, for our first application, Zhu and Thonemann (2009), Song and Xue (2007) and Ceryan et al. (2009) prove the submodularity of the profit-to-functions by analyzing the first-order optimality condition (the KKT condition) of the optimization problems resulted from the dynamic programming recursion. Their proofs are lengthy and unfortunately not very insightful. They also require smoothness assumptions on objective functions and can only deal with simple feasible sets. In fact, for tractability, all these three papers ignore the lower and upper bound constraints on prices when they analyze the KKT conditions, even though such constraints are indispensable in particular for linear demand models. Our approach allows us to treat integrated inventory and pricing models with complementary products and substitutable products in a unified framework and derives new structural results. Yang (2004) analyzes a model related to our second application without pricing decisions. Again, his approach is also based on complicated analysis on the first-order optimality condition of the optimization problems resulted from the dynamic programming recursion.

This paper is organized as follows. In Section 2, we present our main result, its special cases and extensions. In Section 3, we apply our main result to the three mentioned applications. Section 4 summarizes this paper and provides some future research problems. Throughout this paper, many proofs are provided in the appendix unless otherwise specified.

Before we proceed, we introduce the notations and basic concepts used in this paper. Sets are expressed by boldface capital letters (e.g., $\boldsymbol{D}$ and $\boldsymbol{S}$), matrices by regular capital letters (e.g., A and B), vectors by boldface lowercase letters (e.g., $\boldsymbol{x}$ and $\boldsymbol{y}$) and real numbers by regular lowercase letters. We also write $A=[a_{i,j}]_{m,n}$ or $\boldsymbol{x}=[x_i]_n$ sometimes to emphasize entries of A or components of $\boldsymbol{x}$, where subscripts outside the bracket indicate the size of A or the dimension of $\boldsymbol{x}$. All vectors are column vectors, and $\mathbf{0}, \boldsymbol{e}$ are the vectors with all components $0, 1$, respectively.

Given any $m\times n$ matrix A, subset $\boldsymbol{D}$ of $\Re^n$, vectors $\boldsymbol{x}=[x_i]_n$ and $\boldsymbol{y}=[y_i]_n$, denote by $A\geq 0$ if all its entries are non-negative, $|A|$ the determinant of A when $m=n$, $A(\boldsymbol{D})=\{A\boldsymbol{x}:\boldsymbol{x}\in\boldsymbol{D}\}\subset\Re^m$, $\boldsymbol{x}\leq\boldsymbol{y}$ if $x_i\leq y_i$ for all i, $\boldsymbol{x}\vee\boldsymbol{y}=[\max(x_i,y_i)]_n$ and $\boldsymbol{x}\wedge\boldsymbol{y}=[\min\{x_i,y_i\}]_n$. $\boldsymbol{D}$ is called a *convex set* if $\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}\in\boldsymbol{D}$ for all $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{D}$ and $0\leq\lambda\leq 1$, and a *sublattice* (of $\Re^n$) if $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{D}$ implies that $\boldsymbol{x}\wedge\boldsymbol{y},\boldsymbol{x}\vee\boldsymbol{y}\in\boldsymbol{D}$. For example, $[\boldsymbol{l},\boldsymbol{u}]=\{\boldsymbol{x}:\boldsymbol{l}\leq\boldsymbol{x}\leq\boldsymbol{u}\}$ forms a convex sublattice of $\Re^n$, where some components of vectors $\boldsymbol{l},\boldsymbol{u}$ could be $-\infty,+\infty$, respectively (if $u_i=+\infty$, for example, $x_i\leq u_i$ is to be understood as $x_i<+\infty$).

Given a function f defined on a subset $\boldsymbol{S}$ of $\Re^n$ (in case $\boldsymbol{S}$ is not specified, we implicitly assume $\boldsymbol{S}=\Re^n$), we say f is *increasing* if $\boldsymbol{x}\leq\boldsymbol{y}\in\boldsymbol{S}$ implies that $f(\boldsymbol{x})\leq f(\boldsymbol{y})$, *supermodular* if $\boldsymbol{S}$ is a sublattice and $f(\boldsymbol{x})+f(\boldsymbol{y})\leq f(\boldsymbol{x}\wedge\boldsymbol{y})+f(\boldsymbol{x}\vee\boldsymbol{y})$ for all $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{S}$, and *concave* if $\boldsymbol{S}$ is convex and $\lambda f(\boldsymbol{x})+(1-\lambda)f(\boldsymbol{y})\leq f(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y})$ for all $0\leq\lambda\leq 1$ and $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{S}$. We say f is *decreasing*, *convex* or *submodular* if $-f$ is increasing, concave or supermodular. f is *monotone* if it is either increasing or decreasing, *bimonotone* if it is a bivariate function increasing in one variable and decreasing in the other, a *valuation* if it is submodular and supermodular. When referring to a convex function, we assume it is well-behaved (i.e., closed, proper and lower semi-continuous) and may take the value $+\infty$. For details on these concepts, we refer to Rockafellar (1970), Topkis (1998) and Simchi-Levi et al. (2005).

2. Main results

In this section, we first show in Theorem 1 that concavity and supermodularity can be preserved in problem (1) if $A\geq 0$. We then present a preservation property of components-wise concavity and supermodularity in Proposition 1 on a special case of problem (1) and an extension of Theorem 1 by replacing the constraint $A\boldsymbol{y}=\boldsymbol{x}$ with $A\boldsymbol{y}=B\boldsymbol{x}$ for some matrix B with two columns in Proposition 2. From Corollary 2 to Corollary 3 we discuss another special case of problem (1) and show several preservation properties. Finally, the applicability and limitation of our results are demonstrated on several examples including linear programs and quadratic programs. We point out that several results in the literature can be directly derived from ours as we go along.

THEOREM 1. *Assume that A is a non-negative $2\times n$ matrix in problem* (1). *If $\boldsymbol{D}$ is a closed convex sublattice, then so is $\boldsymbol{S}$; if g is concave and supermodular on $\boldsymbol{D}$, then so is f on $\boldsymbol{S}$.*

Proof: It is straightforward to see $\boldsymbol{S}=A(\boldsymbol{D})$ is closed and convex. Concavity of f on $\boldsymbol{S}$ follows from Theorem 5.4, Rockafellar (1970). It remains to prove that $\boldsymbol{S}$ is a sublattice and f is supermodular on $\boldsymbol{S}$, i.e., $\boldsymbol{x}\wedge\tilde{\boldsymbol{x}},\boldsymbol{x}\vee\tilde{\boldsymbol{x}}\in\boldsymbol{S}$ and $f(\boldsymbol{x})+f(\tilde{\boldsymbol{x}})\leq f(\boldsymbol{x}\wedge\tilde{\boldsymbol{x}})+f(\boldsymbol{x}\vee\tilde{\boldsymbol{x}})$ for any $\boldsymbol{x},\tilde{\boldsymbol{x}}\in\boldsymbol{S}$.

Let $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$ be the optimal solutions associated with $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ in problem (1), respectively. Because $\boldsymbol{D}$ is a sublattice, $\boldsymbol{y}\wedge\tilde{\boldsymbol{y}},\boldsymbol{y}\vee\tilde{\boldsymbol{y}}\in\boldsymbol{D}$ and $\boldsymbol{a}=A(\boldsymbol{y}\wedge\tilde{\boldsymbol{y}}),\boldsymbol{b}=A(\boldsymbol{y}\vee\tilde{\boldsymbol{y}})\in\boldsymbol{S}$. Since $A\geq 0$, $\boldsymbol{a}\leq\boldsymbol{x}\wedge\tilde{\boldsymbol{x}}$ hence $\boldsymbol{x}\wedge\tilde{\boldsymbol{x}}$ belongs to the convex hull of $\{\boldsymbol{a},\boldsymbol{x},\tilde{\boldsymbol{x}}\}$ (see Figure 1 for the illustration). We know from the convexity of $\boldsymbol{S}$ that $\boldsymbol{x}\wedge\tilde{\boldsymbol{x}}\in\boldsymbol{S}$. Similarly, we also have $\boldsymbol{x}\vee\tilde{\boldsymbol{x}}\in\boldsymbol{S}$.

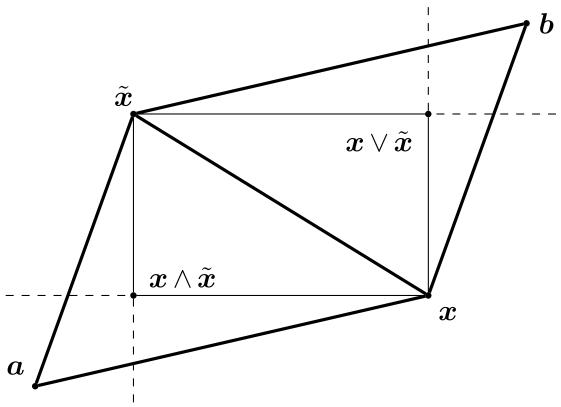

Figure 1 Relative positions of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{x}$ and $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}, \boldsymbol{x} \vee \tilde{\boldsymbol{x}}$ in the proof of Theorem 1

Denote $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}} = \lambda \boldsymbol{a} + \mu \boldsymbol{x} + \nu \tilde{\boldsymbol{x}}$ for some $0 \le \mu, \nu, \lambda \le \lambda + \mu + \nu = 1$. Then $\boldsymbol{x} \vee \tilde{\boldsymbol{x}} = \lambda \boldsymbol{b} + \mu \tilde{\boldsymbol{x}} + \nu \boldsymbol{x}$ by $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{x} \wedge \boldsymbol{y} + \boldsymbol{x} \vee \boldsymbol{y} = \boldsymbol{x} + \tilde{\boldsymbol{x}}$. The concavity of f implies that

$$\lambda[f(\boldsymbol{a}) + f(\boldsymbol{b})] + (1-\lambda)[f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}})] \le f(\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}) + f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}}).$$

In addition, the definition of f and the supermodularity of g lead to

$$f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}}) = g(\boldsymbol{y}) + g(\tilde{\boldsymbol{y}}) \le g(\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}) + g(\boldsymbol{y} \vee \tilde{\boldsymbol{y}}) \le f(\boldsymbol{a}) + f(\boldsymbol{b}).$$

By combining the above two inequalities, we conclude that $f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}}) \le f(\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}) + f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}})$. ☐

REMARK 1. Theorem 1 still holds when the equality constraints $A\boldsymbol{y} = \boldsymbol{x}$ in (1) are replaced by inequality constraints. Indeed, it suffices to add non-negative slack or surplus variables to the inequality constraints and apply Theorem 1 in the current format to establish the same result.

REMARK 2. The statement of Theorem 1 remains valid for some discrete cases. Specifically, suppose all entries of A are integers, $\boldsymbol{D} = [\boldsymbol{l}, \boldsymbol{u}] \cap \mathcal{Z}^n$, where $\mathcal{Z}$ denotes the set of all integers and $\boldsymbol{l}, \boldsymbol{u} \in \mathcal{Z}^n$. We can show that if g is supermodular on $\boldsymbol{D}$ and *integrally concave* (see Section 3.4, Murota 2003) then so is f on $\boldsymbol{S}$. The proof is almost identical except that we now deal with the concave extensions of g and f instead.

Note that in the proof of Theorem 1, we only need the supermodularity of g and the concavity of f (not the concavity of g) to ensure the supermodularity of f. The concavity of g does provide a sufficient condition for the concavity of f though. One may ask whether the concavity of g can be replaced by component-wise concavity such that supermodularity can still be preserved. Though the answer is negative in general as we illustrate later in this section on an unconstrained quadratic program, the concavity of g can be weakened for a special case of problem (1). The key is to observe that in the proof of Theorem 1, we construct $\boldsymbol{a}$ such that $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ can be expressed as a convex combination of $\boldsymbol{a}, \boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$, and then apply the concavity of f on $\boldsymbol{S}$. If one can guarantee that $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ and $\boldsymbol{a}$ lie on the same vertical line, then the concavity of $f(x_1, x_2)$ in x_2 is sufficient to complete the proof. This observation motivates the proposition below.

Proposition 1. *Consider the optimization problem parameterized by $\boldsymbol{x} = [x_1, x_2] \in \boldsymbol{S}$:*

$$f(x_1, x_2) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{g(x_1, \boldsymbol{y}) : \alpha_1 x_1 + \boldsymbol{\alpha}' \boldsymbol{y} = x_2, (x_1, \boldsymbol{y}) \in \boldsymbol{D}\},$$

where $\boldsymbol{D} \subset \Re^n$, $\boldsymbol{S} = \{(y_1, a_1 y_1 + \boldsymbol{\alpha}'\boldsymbol{y}) : (y_1, \boldsymbol{y}) \in \boldsymbol{D}\}$, $\alpha_1 \geq 0$ and $\boldsymbol{\alpha} \geq \boldsymbol{0}$. If $\boldsymbol{D}$ is a sublattice and $\boldsymbol{D}(y_1) = \{\boldsymbol{y} : [y_1, \boldsymbol{y}] \in \boldsymbol{D}\}$ is convex for any y_1, then $\boldsymbol{S}$ is a sublattice and $\boldsymbol{S}(x_1) = \{x_2 : [x_1, x_2] \in \boldsymbol{S}\}$ is convex for any x_1. Moreover, if $g(y_1, \boldsymbol{y})$ is supermodular on $\boldsymbol{D}$ and concave in $\boldsymbol{y}$ on $\boldsymbol{D}(y_1)$ for all y_1, then $f(x_1, x_2)$ is supermodular on $\boldsymbol{S}$ and concave in x_2 on $\boldsymbol{S}(x_1)$ for all x_1.

When $n = 2$, $A \geq 0$ and $|A| > 0$, Theorem 1 implies that if g on $\boldsymbol{D}$ is concave and supermodular, then so is $g(P\boldsymbol{x})$ on $A(\boldsymbol{D})$, where $P = A^{-1}$. Notice that the matrix P has non-negative diagonal entries and non-positive off-diagonal entries (any matrix with this property will be referred to as an L_0-matrix thereafter). A stronger result can be obtained from Proposition 1, which provides a sufficient condition such that supermodularity is preserved under linear variable transformations.

Corollary 1. *For any 2×2 L_0-matrix P, if $\boldsymbol{D}$ is a convex sublattice in $\Re^2$, then so is the set $\boldsymbol{S} = \{\boldsymbol{x} : P\boldsymbol{x} \in \boldsymbol{D}\}$; if a function g on $\boldsymbol{D}$ is component-wise concave and supermodular, then so is the function $g(P\boldsymbol{x})$ on $\boldsymbol{S}$.*

The following proposition presents an extension of problem (1).

Proposition 2. *Given some $m \times n$ matrix A and $m \times 2$ matrix B such that $B'A \geq 0$ and $B'B$ is an L_0-matrix, closed convex sublattice $\boldsymbol{D}$ of $\Re^n$, and concave and supermodular function g on $\boldsymbol{D}$, define $f(\boldsymbol{x})$ as below on $\boldsymbol{S} = \{\boldsymbol{x} : \exists\, \boldsymbol{y} \in \boldsymbol{D}$ such that $A\boldsymbol{y} = B\boldsymbol{x}\}$:*

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \big\{g(\boldsymbol{y}) : A\boldsymbol{y} = B\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}\big\}.$$

Then $\boldsymbol{S}$ is a closed convex sublattice and f is concave and supermodular on $\boldsymbol{D}$.

Next we consider a special case of problem (1) below. Given convex sublattices $\boldsymbol{D}_n$ of $\Re^2$ and real-valued functions f_n defined on $\boldsymbol{D}_n$ for $n = 1, 2, \cdots, N$, let $\boldsymbol{S}$ be the *Minkowski sum* of $\boldsymbol{D}_n$, i.e., $\boldsymbol{S} = \{\sum_{n=1}^N \boldsymbol{y}_n : \boldsymbol{y}_n \in \boldsymbol{D}_n, \forall n\}$, and for any $\boldsymbol{x} \in \boldsymbol{S}$,

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_N}{\text{maximize}} \ \left\{\sum\nolimits_{n=1}^N f_n(\boldsymbol{y}_n) : \sum\nolimits_{n=1}^N \boldsymbol{y}_n = \boldsymbol{x}, \boldsymbol{y}_n \in \boldsymbol{D}_n, \ \forall n\right\}. \tag{2}$$

When all $\boldsymbol{D}_n = \Re^2$, $-f$ is called the *infimal convolution* of $-f_n$ (see Rockafellar 1970, Section 5).

The following result is an immediate corollary of Theorem 1. We omit its proof.

Corollary 2. *Suppose P is a non-singular 2×2 matrix. For problem (2), if all $P^{-1}(\boldsymbol{D}_n)$ are convex sublattices of $\Re^2$, and $f_n(P\boldsymbol{x})$ are concave and supermodular on $P^{-1}(\boldsymbol{D}_n)$, then $P^{-1}(\boldsymbol{S})$ forms a convex sublattice of $\Re^2$, and $f(P\boldsymbol{x})$ is concave and supermodular on $P^{-1}(\boldsymbol{S})$.*

When P is the identity matrix, Corollary 2 states the preservation of concavity and supermodularity in problem (2). In general, we may have some flexibility of choosing the matrix P depending on applications. Three interesting instances of P are listed below:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad J_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Note that all these linear transformations are projections, i.e., $J(J\boldsymbol{x}) = J_1(J_1\boldsymbol{x}) = J_2(J_2\boldsymbol{x}) = \boldsymbol{x}$.

The linear transformation J maps a vector $[x_1, x_2]$ to $[x_1, -x_2]$. Geometrically, $J(\boldsymbol{S})$ is the reflection of the set $\boldsymbol{S}$ at the horizontal axis. Interestingly the transformation shows a simple but useful relation between two dimensional submodular functions and supermodular functions, whose proof follows directly from the definitions of supermodularity and submodularity and thus is omitted.

Lemma 1. *When both $\boldsymbol{S}$ and $J(\boldsymbol{S})$ are sublattices of $\Re^2$, $f(\boldsymbol{x})$ is supermodular on $\boldsymbol{S}$ if and only if $f(J\boldsymbol{x})$ is submodular on $J(\boldsymbol{S})$.*

This above lemma allows us to convert a statement on supermodularity to the related statement on submodularity. For example, we know from Corollary 1 that if g is component-wise concave and submodular on $\Re^2$ then so is the function $g(B\boldsymbol{x})$ in $\boldsymbol{x}$ for any 2×2 non-negative matrix B.

The other two transformations J_1 and J_2 map a vector $[x_1, x_2]$ to $[x_1 - x_2, -x_2]$ and $[-x_1, x_2 - x_1]$, respectively. We will use them in the following to analyze a closely related concept, *$L^\natural$-concavity*, which finds applications in inventory models (see, for instance, Zipkin 2008).

Definition 1. The function f is $L^\natural$-concave on $\Re^n$ if $f(\boldsymbol{x} - \xi \boldsymbol{e})$ is supermodular in $(\boldsymbol{x}, \xi) \in \Re^{n+1}$.

Our definition of $L^\natural$-concavity follows from Murota (2003). As pointed out by Murota (2003), an $L^\natural$-concave function f is also concave and supermodular. Moreover, its Hessian matrix $\nabla^2 f(\boldsymbol{x})$, provided the existence, has non-positive diagonal entries and non-negative off-diagonal entries, and possesses the diagonal dominance property, i.e., the summation of entries in each row is non-positive. It should be mentioned that depending on applications, one may also assume $\xi \le 0$ or $\xi \ge 0$ in the definition of $L^\natural$-concavity. For example, Zipkin (2008) uses $\{\xi : \xi \le 0\}$ as the domain of ξ when he applies it to inventory models with lost sales. However, our definition allows us to easily characterize $L^\natural$-concavity through supermodularity in $\Re^2$.

Lemma 2. *Suppose the function f is defined on $\Re^2$. The four statements are equivalent: (a) f is $L^\natural$-concave; (b) $f(J_1\boldsymbol{x})$ is $L^\natural$-concave; (c) $f(J_2\boldsymbol{x})$ is $L^\natural$-concave; (d) $f(\boldsymbol{x}), f(J_1\boldsymbol{x})$ and $f(J_2\boldsymbol{x})$ are supermodular.*

By Lemmas 1, 2 and all above discussions, we have the following result on problem (2) from Corollary 2, where the proof is omitted.

Corollary 3. *Assume in problem (2) that all $\boldsymbol{D}_n$ are convex sublattices of $\Re^2$.*

(a) If all f_n are concave and supermodular on $\boldsymbol{D}_n$, then so is f on the convex sublattice $\boldsymbol{S}$.

(b) If all $J(\boldsymbol{D}_n)$ are sublattices of $\Re^2$ and all f_n are concave and submodular on $\boldsymbol{D}_n$, then f is concave and submodular on the convex sublattice $\boldsymbol{S}$.

(c) If all $\boldsymbol{D}_n = \Re^2$ and all f_n are $L^\natural$-concave, then so is f on $\boldsymbol{S} = \Re^2$.

It should be mentioned that in Corollary 3(b) the condition on $J(\boldsymbol{D}_n)$ is indispensable. Actually it may fail if $J(\boldsymbol{D}_n)$ are not sublattices. For example, consider the problem below for all $x_1, x_2 \geq 0$:

$$f(x_1,x_2) = \text{maximize } \{y_1 : \ y_1 + z_1 = x_1, \ y_2 + z_2 = x_2, \ 0 \leq y_1 \leq y_2, \ z_1, z_2 \geq 0\}.$$

Note that the objective is a valuation and the set $\boldsymbol{D}_1 = \{[y_1, y_2] : 0 \leq y_1 \leq y_2\}$ forms a sublattice. Solving this optimization problem gives us $f(x_1,x_2) = \min\{x_1,x_2\}$, which is supermodular as is consistent with Corollary 3(a). However, we cannot apply Corollary 3(b) because $J(\boldsymbol{D}_1)$ is not a sublattice. In fact, f is not submodular since $f(0,0)+f(1,1) = 0+1 > 0+0 = f(0,1)+f(1,0)$.

We demonstrate the applicability and limitation of our results on a few examples.

Example 1. The following optimization problem is presented in Chao et al. (2009) when they develop and analyze dynamic capacity expansion models:

$$f_0(x_1,x_2) = \text{maximize } \{g_0(x_1,y_2) : x_2 \leq y_2 \leq x_1 + x_2\}.$$

They prove that if $g_0(x_1,x_2)$ is submodular and concave in x_2, then so is $f_0(x_1,x_2)$, which serves as the key technical tool in their analysis. Chao et al. (2009) comment that it is usually challenging to prove the preservation of submodularity under maximization.

We now show that it follows directly from our results. Define $g(\boldsymbol{x}) = g_0(J\boldsymbol{x})$, $f(\boldsymbol{x}) = f_0(J\boldsymbol{x})$ and rewrite the above problem as

$$f(x_1,x_2) = \text{maximize } \left\{g(x_1,y_2) : y_1 + y_2 = x_2, 0 \leq y_1 \leq x_1\right\}.$$

Since the submodularity of g_0, f_0 is equivalent to the supermodularity of g, f, we immediately obtain the key technical result of Chao et al. (2009) from Proposition 1.

Example 2 (Linear Programs). Zipkin (2003) considers the linear programming problem

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{\boldsymbol{p}'\boldsymbol{y} : A\boldsymbol{y} \leq \boldsymbol{x}, \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{u}\},$$

where $\boldsymbol{p}, \boldsymbol{u}$ are two given n-dimensional vectors and A is a $2 \times n$ matrix. Using intricate geometrical argument, he shows that $f(\boldsymbol{x})$ is supermodular in $\boldsymbol{x}$ over $\boldsymbol{x} \geq 0$ if A is non-negative. Interestingly, this result immediately follows form Remark 1 of Theorem 1.

Zipkin (2003) also proves that as long as the maximization below is well defined for all $\boldsymbol{x} \geq \boldsymbol{0}$,

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{\boldsymbol{p}'\boldsymbol{y} : A\boldsymbol{y} \leq \boldsymbol{x}, C\boldsymbol{y} \leq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}\},$$

$f(\boldsymbol{x})$ is supermodular over $\boldsymbol{x} \geq 0$ for arbitrary matrices A and C with proper sizes. Unfortunately, our result does not cover this case. As we show later in Example 4, it does not work even for the case of quadratic objective. It is interesting to observe that $f(\boldsymbol{x})$ is not necessarily supermodular if $\boldsymbol{x}$ is not restricted in the non-negative orthant. Here is an example:

$$\boldsymbol{p} = [-1, 0, 0, -1], \qquad C = \mathbf{0}, \qquad A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -2 & -1 & 2 & 1 \end{bmatrix}.$$

Calculation shows $f(x_1, x_2) = \min\{0, x_1 + x_2, 2x_1 + x_2, 3x_1 + 2x_2\}$. One can verify the submodularity of f. However, f is not supermodular since $f(0,0) + f(1,-1) = 0 + 0 > -2 + 0 = f(0,-1) + f(1,0)$. This example also indicates that without the condition $A \geq 0$, Theorem 1 may fail even if the objective function in problem (1) is linear.

Example 3 (quadratic programs I). Suppose P, Q are $n \times n$ symmetric matrices such that $P + Q$ is negative definite. Define $g(\boldsymbol{y}, \boldsymbol{z}) = \frac{1}{2}\boldsymbol{y}'P\boldsymbol{y} + \frac{1}{2}\boldsymbol{z}'Q\boldsymbol{z}$ and for all $\boldsymbol{x} \in \Re^n$,

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}, \boldsymbol{z}}{\text{maximize}} \ \{g(\boldsymbol{y}, \boldsymbol{z}) : \boldsymbol{y} + \boldsymbol{z} = \boldsymbol{x}\}.$$

Calculation shows that $\boldsymbol{y}(\boldsymbol{x}) = (P+Q)^{-1}Q\boldsymbol{x}$ solves the problem, and f is quadratic associated with the Hessian matrix $\nabla^2 f(\boldsymbol{x}) = P(P+Q)^{-1}Q$. When $n = 2$, we further have

$$\nabla^2 f(\boldsymbol{x}) = P(P+Q)^{-1}Q = \tfrac{|Q|}{|P+Q|}P + \tfrac{|P|}{|P+Q|}Q.$$

There are some interesting observations on Example 3. First, it is a special case of problem (1) when $n = 2$. From the expression of $\nabla^2 f(\boldsymbol{x})$, we know that if g is supermodular then so is f, which is consistent with the statement of Theorem 1. This result does not seem to follow directly from Theorem 2.7.6, Topkis (1998) since the constraint set does not form a sublattice. One may simplify the example by eliminating $\boldsymbol{z}$ and the constraints. However, even when all entries of P are zero and Q has positive off-diagonal entries, we know that $g(\boldsymbol{y}, \boldsymbol{z})$ is supermodular in $(\boldsymbol{y}, \boldsymbol{z})$ but $g(\boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y})$ is neither submodular nor supermodular in $(\boldsymbol{x}, \boldsymbol{y})$.

Second, we can not weaken the concavity assumption on g in Theorem 1 to component-wise concavity. Consider Example 3 with P, Q and $\nabla^2 f(\boldsymbol{x})$ given below.

$$P = \begin{bmatrix} -9 & 4 \\ 4 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 4 \\ 4 & -9 \end{bmatrix}, \quad \nabla^2 f(\boldsymbol{x}) = \frac{7}{18}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

In this instance, g is component-wise concave and supermodular. However, f is submodular.

Third, Theorem 1 does not hold in higher dimensional spaces. Consider Example 3 with P, Q and the related Hessian matrix of f given below.

$$P = \begin{bmatrix} -11 & 8 & 0 \\ 8 & -16 & 5 \\ 0 & 5 & -10 \end{bmatrix}, \quad Q = \begin{bmatrix} -7 & 4 & 0 \\ 4 & -14 & 5 \\ 0 & 5 & -9 \end{bmatrix}, \quad \nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} -4.25 & 2.79 & -0.01 \\ 2.79 & -7.27 & 2.49 \\ -0.01 & 2.49 & -4.73 \end{bmatrix}.$$

In the instance, g is $L^\natural$-concave. However, f is neither supermodular nor submodular. Extending our results to higher dimensional spaces is interesting and challenging.

Finally, the optimal solution may not be monotone or may not have a clear monotonicity pattern even in cases in which we do have monotonicity. To see this, consider Example 3 with P, Q given below and their related optimal solutions $\boldsymbol{y}(\boldsymbol{x}) = [y_1(x_1,x_2), y_2(x_1,x_2)]$.

$$P = \begin{bmatrix} -6 & 5 \\ 5 & -6 \end{bmatrix}, \quad Q = \begin{bmatrix} -5 & 2 \\ 2 & -1 \end{bmatrix}, \quad \boldsymbol{y}(\boldsymbol{x}) = \frac{1}{28}\begin{bmatrix} 21 & -7 \\ 13 & -3 \end{bmatrix}\boldsymbol{x};$$

$$P = \begin{bmatrix} -6 & 3 \\ 3 & -4 \end{bmatrix}, \quad Q = \begin{bmatrix} -3 & 2 \\ 2 & -6 \end{bmatrix}, \quad \boldsymbol{y}(\boldsymbol{x}) = \frac{1}{65}\begin{bmatrix} 20 & 10 \\ -3 & 44 \end{bmatrix}\boldsymbol{x}.$$

In both instances, g are supermodular. However, in the first instance, for both $i = 1,2$ $y_i(x_1,x_2)$ are increasing in x_1 but decreasing in x_2. In the second instance, $y_1(x_1,x_2)$ is increasing in both x_1 and x_2 but $y_2(x_1,x_2)$ is increasing in x_2 and decreasing in x_1.

Remark 3. It is interesting to observe that if the equality constraints $A\boldsymbol{y} = \boldsymbol{x}$ in problem (1) are replaced by $A\boldsymbol{y} \le \boldsymbol{x}$, then the optimal solution sets do possess certain monotonicity properties. Specifically, denote $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ as the optimal solution sets associated with $\boldsymbol{x}_1 \le \boldsymbol{x}_2 \in \boldsymbol{S}$, respectively. We can verify that $\boldsymbol{Y}_2$ is greater than $\boldsymbol{Y}_1$ in the so-called *$\mathcal{C}$-flexible set order* Quah (2007). Moreover, Proposition 3 in Quah (2007) states that the $\mathcal{C}$-flexible set order is stronger than the *weak induced set order* (Section 2.4, Topkis 1998). Therefore, for any $\boldsymbol{y}_1 \in \boldsymbol{Y}_1$ there exists $\boldsymbol{y}_2 \in \boldsymbol{Y}_2$ such that $\boldsymbol{y}_1 \le \boldsymbol{y}_2$, and for any $\boldsymbol{y}_2 \in \boldsymbol{Y}_2$ there exists $\boldsymbol{y}_1 \in \boldsymbol{Y}_1$ such that $\boldsymbol{y}_1 \le \boldsymbol{y}_2$. Note that if g is strictly concave, then we know that the optimal solution is unique and increasing in $\boldsymbol{x}$.

Example 4 (quadratic programs II). Consider the problem below for all $[x_1,x_2] \ge \boldsymbol{0}$:

$$\begin{aligned} f(x_1,x_2) = \text{maximize} \;\; & \tfrac{1}{2}(\boldsymbol{y}-\boldsymbol{e})'Q(\boldsymbol{y}-\boldsymbol{e}) \\ \text{subject to} \;\; & \boldsymbol{\alpha}_1'\boldsymbol{y} \le x_1, \quad \boldsymbol{\alpha}_2'\boldsymbol{y} \le x_2, \quad \boldsymbol{y} \ge \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{\alpha}_1 = [1, -\frac{1}{2}]$, $\boldsymbol{\alpha}_2 = [-\frac{1}{2}, 1]$ and $Q = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$. Let $A = \begin{bmatrix} \boldsymbol{\alpha}_1' \\ \boldsymbol{\alpha}_2' \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$. Depending on whether each constraint $\boldsymbol{\alpha}_i'\boldsymbol{y} \le x_i$ is active or not, we have

$$f(x_1,x_2) = \begin{cases} 0, & \text{if } [x_1,x_2] \in \boldsymbol{S}_0, \\ -(x_1 - \frac{1}{2})^2, & \text{if } [x_1,x_2] \in \boldsymbol{S}_1, \\ -(x_2 - \frac{1}{2})^2, & \text{if } [x_1,x_2] \in \boldsymbol{S}_2, \\ \frac{1}{2}\left(A^{-1}\boldsymbol{x} - \boldsymbol{e}\right)' Q \left(A^{-1}\boldsymbol{x} - \boldsymbol{e}\right), & \text{if } \boldsymbol{x} = [x_1,x_2] \in \boldsymbol{S}_3, \end{cases}$$

where $\boldsymbol{S}_0 = \{\boldsymbol{x} : 2\boldsymbol{x} \ge \boldsymbol{e}\}$, $\boldsymbol{S}_i = \{[x_1,x_2] : 0 \le 2x_i \le 1, 3 \le 2x_i + 4x_{3-i}\}$ for $i = 1,2$ and $\boldsymbol{S}_3 = \{\boldsymbol{x} \ge \boldsymbol{0} : \boldsymbol{x} \notin \boldsymbol{S}_i, i = 0,1,2\}$. That is, neither constraint is active when $\boldsymbol{x} \in \boldsymbol{S}_0$, only the constraint $\boldsymbol{\alpha}_i'\boldsymbol{y} \le x_i$ is active when $\boldsymbol{x} \in \boldsymbol{S}_i$ for each $i = 1,2$, and both constraints are active when $\boldsymbol{x} \in \boldsymbol{S}_3$.

Calculation shows that $\frac{\partial^2}{\partial x_1 \partial x_2} f(x_1,x_2) = -\frac{4}{3}$ for any interior point $\boldsymbol{x} \in \boldsymbol{S}_3$. Hence, unlike the linear programming problems analyzed in Zipkin (2003), f is not supermodular over $\boldsymbol{x} \geq \boldsymbol{0}$. It also provides another instance to demonstrate that Theorem 1 may fail without the condition $A \geq 0$.

3. Applications

In this section we apply results in Section 2 to three finite horizon periodic-review operational models. In all these models, we will deal with parameterized optimization problems with the following mathematical structure.

$$
\begin{aligned}
v(x_1,x_2) = \underset{y_1,y_2}{\text{maximize}} \quad & [f(z_1,z_2) + \bar{v}_+(y_1,y_2)] \\
\text{subject to} \quad & y_1 = x_1 + z_1, \quad y_2 = x_2 + z_2, \\
& a_1 \leq y_1 \leq b_1, \quad a_2 \leq y_2 \leq b_2, \quad h_n(y_1,y_2) \leq 0, \ \forall\, 1 \leq n < M, \\
& l_1 \leq z_1 \leq u_1, \quad l_2 \leq z_2 \leq u_2, \quad h_n(z_1,z_2) \leq 0, \ \forall M \leq n < N,
\end{aligned}
$$

where some components l_i, a_i could be $-\infty$ and b_i, u_i could be $+\infty$. Define

$$
\boldsymbol{Y} = \{\boldsymbol{y} \in [\boldsymbol{a},\boldsymbol{b}] : h_n(\boldsymbol{y}) \leq 0, \ \forall 1 \leq n < M\}, \quad \boldsymbol{Z} = \{\boldsymbol{z} \in [\boldsymbol{l},\boldsymbol{u}] : h_n(\boldsymbol{z}) \leq 0, \ \forall M \leq n < N\},
$$

where $\boldsymbol{l} = [l_1,l_2], \boldsymbol{u} = [u_1,u_2], \boldsymbol{a} = [a_1,a_2]$ and $\boldsymbol{b} = [b_1,b_2]$. We can rewrite the above problem as

$$
v(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{[f(\boldsymbol{y}-\boldsymbol{x}) + \bar{v}_+(\boldsymbol{y})] : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{y}-\boldsymbol{x} \in \boldsymbol{Z}\}. \tag{3}
$$

Note that the domain of v is $\boldsymbol{X} = \{\boldsymbol{y}-\boldsymbol{z} : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{z} \in \boldsymbol{Z}\}$. We have the following theorem on (3).

THEOREM 2. *(a) If all h_n are convex and bimonotone, then $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{X}$ are convex sublattices in $\Re^2$. In addition, if $\bar{v}_+$ on $\boldsymbol{Y}$ and f on $\boldsymbol{Z}$ are concave and supermodular, then so is v on $\boldsymbol{X}$.*
(b) If all h_n are convex and monotone, then $J(\boldsymbol{Y})$, $J(\boldsymbol{Z})$ and $J(\boldsymbol{X})$ are convex sublattices in $\Re^2$. In addition, if $\bar{v}_+(J\boldsymbol{y})$ on $J(\boldsymbol{Y})$ and $f(J\boldsymbol{z})$ on $J(\boldsymbol{Z})$ are concave and supermodular, and $\boldsymbol{X}$ also forms a sublattice, then v on $\boldsymbol{X}$ is concave and submodular.

Proof: Observe that problem (3) can be converted to the format amicable to (2) as

$$
v(\boldsymbol{x}) = \underset{\boldsymbol{y},\boldsymbol{z}}{\text{maximize}} \ \{f(-\boldsymbol{z}) + \bar{v}_+(\boldsymbol{y}) : \boldsymbol{y}+\boldsymbol{z} = \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{z} \in \boldsymbol{Z}_-\},
$$

where $\boldsymbol{Z}_- = \{\boldsymbol{z} \in \Re^2 : -\boldsymbol{z} \in \boldsymbol{Z}\}$. Part (a) follows from Theorem 1 since that conditions on h_n ensure that $\boldsymbol{Y}$ and $\boldsymbol{Z}_-$ are convex sets (Theorem 4.6, Rockafellar 1970) and sublattices (Example 2.2.7, Topkis 1998). Part (b) follows from Lemma 1 and part (a). □

If additional conditions are imposed on f and $\boldsymbol{Z}$ in problem 3, the optimal solution to problem (3) exhibits certain monotonicity properties. Note that in the following theorem we say $f(x_1,x_2)$ is separable if it can be expressed as $f_1(x_1) + f_2(x_2)$ for two univariate functions f_1 and f_2.

THEOREM 3. *Suppose that $\boldsymbol{Z} = [\boldsymbol{l},\boldsymbol{u}]$ and f is separable in problem* (3).

(a) If $\bar{v}_+$ is supermodular, all h_n are continuous and bimonotone, and f is concave, then there exists an optimal solution $[y_1(x_1,x_2), y_2(x_1,x_2)]$ to problem (3) *such that $y_i(x_1,x_2)$ is increasing in both x_1 and x_2 for $i=1,2$.*

(b) If $\bar{v}_+$ is submodular, all h_n are continuous and monotone, and f is linear, then there exists an optimal solution $[y_1(x_1,x_2), y_2(x_1,x_2)]$ to problem (3) *such that $y_i(x_1,x_2)$ is increasing in x_i and decreasing in x_j for $i,j=1,2$ and $i \neq j$.*

The managerial interpretation and intuition of the above characterization on the optimal solution will become clear when we talk about the concrete applications. Notice that we introduce no concavity/convexity assumptions on $\bar{v}_+$ and h_n in Theorem 3. However, they will be required in all the following applications to inductively show the supermodularity/submodularity of profit-to-go functions. Moreover, with these concavity/convexity assumptions, if f_t is linear then more refined characterization of optimal solution $\boldsymbol{y}(\boldsymbol{x})$ is possible by partitioning the space of the parameter $\boldsymbol{x}$ into several regions, which is provided in the appendix.

3.1. Coordinated pricing and inventory control with cross-price effects

Consider a retailer who decides the ordering quantities and prices of two products over a finite planning horizon with T periods. At the beginning of each period, the retailer observes the initial inventory levels x_i and then simultaneously decides the selling prices p_i and the order-up-to-levels y_i for products $i=1,2$. The demand of product i during a period is given by $d_i(p_1,p_2)+\varepsilon_i$, where ε_i is a random variable with expected value 0, $d_i(p_1,p_2)$ is the expected demand of product i depending on the prices of both products. Denote $\boldsymbol{x}=[x_1,x_2], \boldsymbol{y}=[y_1,y_2]$, $\boldsymbol{p}=[p_1,p_2]$, $\boldsymbol{\varepsilon}=[\varepsilon_1,\varepsilon_2]$ and $\boldsymbol{d}^{\varepsilon}=\boldsymbol{d}(\boldsymbol{p})+\boldsymbol{\varepsilon}$. The demand function can be time dependent but we drop the time index for simplicity. We assume that random vectors are independent across time, there is no lead time for delivery, unsatisfied demand is backlogged and unused inventory is carried over to the next period.

As common in the literature, the expected demand $\boldsymbol{d}(\boldsymbol{p})$ is assumed to be linear as $\boldsymbol{d}(\boldsymbol{p})=\boldsymbol{b}-A\boldsymbol{p}$ for some vector $\boldsymbol{b}\geq \boldsymbol{0}$ and the price sensitivity coefficient matrix $A=[a_{i,j}]_{2,2}$. Suppose $\boldsymbol{p}\in[\boldsymbol{l},\boldsymbol{u}]$, where $\boldsymbol{l}\leq\boldsymbol{u}$ are lower and upper bounds on the prices such that $\boldsymbol{d}^{\varepsilon}\geq\boldsymbol{0}$ almost surely. For product i, coefficients $a_{i,i}$, $a_{i,j}$ respectively denote its own price sensitivity and the cross price sensitivity to the other product j $(j\neq i)$. We assume that $a_{i,i}\geq 0$, that is, the demand of a product is decreasing in its own price. Depending on the nature of products, we focus on two cases: (a) the two products are *complements*, i.e., an increase in the price of one product will decrease the demanded amount of the other product, or equivalently $a_{i,j}\geq 0$; (b) the two products are *substitutes*, i.e., an increase in the price of one product will increase the demanded amount of the other product, or equivalently $a_{i,j}\leq 0$. In addition, we assume that the price change of one product has a stronger effect on its own demand than on the other product's demand, i.e., $a_{i,i}\geq|a_{i,j}|$. Note that A is positive semi-definite under these assumptions. For our purpose, we assume A is positive definite. In this case, there is a one-to-one correspondence between the expected demands and the prices.

It will be convenient to use the expected demands instead of prices as the decision variables. Denote the realized demand vector as $\boldsymbol{d}^{\varepsilon} = \boldsymbol{d} + \boldsymbol{\varepsilon}$ and the corresponding price vector as $\boldsymbol{p}(\boldsymbol{d}) = A^{-1}(\boldsymbol{b} - \boldsymbol{d})$. The expected one-period revenue is given by $r(\boldsymbol{d}) = \boldsymbol{d}'\boldsymbol{p}(\boldsymbol{d})$, which can be easily verified to be concave. Moreover, in the complementary product case, $r(\boldsymbol{d})$ is supermodular and $r(A\boldsymbol{d})$ is submodular; in the substitutable product case, $r(\boldsymbol{d})$ is submodular and $r(A\boldsymbol{d})$ is supermodular.

The ordering cost is proportional to the ordering quantity specified by $c(\boldsymbol{z}) = \boldsymbol{c}'\boldsymbol{z}$ for an ordering quantity vector $\boldsymbol{z} = [z_1, z_2]$. For an amount $\boldsymbol{x} = [x_1, x_2]$ of inventory carried over from one period to the next, an inventory holding and backorder cost $h(\boldsymbol{x}) = h_1(x_1) + h_2(x_2)$ is incurred, where $h_i(x_i)$, assumed to be convex, represents the inventory holding cost when $x_i > 0$ and the shortage penalty cost when $x_i < 0$. To avoid technicality, we assume that $\mathbb{E}[h(\boldsymbol{y} - \boldsymbol{\varepsilon})]$ is strictly convex, where $\mathbb{E}$ is the expectation operator corresponding to random variables $\boldsymbol{\varepsilon}$. The objective of the retailer is to find an ordering and pricing decision so as to maximize its expected total profit over the planning horizon. Let $v_t(\boldsymbol{x})$ be the profit-to-go function of period t starting with an inventory level $\boldsymbol{x}$. The dynamic program can be formulated as

$$\begin{aligned} v_t(\boldsymbol{x}) = \underset{\boldsymbol{y},\boldsymbol{d}}{\text{maximize}} \quad & \{r(\boldsymbol{d}) - \boldsymbol{c}'\boldsymbol{z} + g_t(\boldsymbol{y} - \boldsymbol{d})\} \\ \text{subject to} \quad & \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{z}, \quad \boldsymbol{0} \le \boldsymbol{z} \le \boldsymbol{k}, \quad \boldsymbol{l} \le A^{-1}(\boldsymbol{b} - \boldsymbol{d}) \le \boldsymbol{u}, \end{aligned}$$

where $g_t(\boldsymbol{x}) = \mathbb{E}[v_{t+1}(\boldsymbol{x} - \boldsymbol{\varepsilon}) - h(\boldsymbol{x} - \boldsymbol{\varepsilon})]$, the ordering quantity $\boldsymbol{z}$ is non-negative and bounded above by $\boldsymbol{k}$ and without loss of generality, assume $v_{T+1}(\boldsymbol{x}) = \boldsymbol{c}'\boldsymbol{x}$. For any given nonsingular 2×2 matrix P, the above problem can be equivalently reformulated as

$$v_t(P\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \left\{f_t(P\boldsymbol{y}) - \boldsymbol{c}'P\boldsymbol{y}\right\} + \boldsymbol{c}'P\boldsymbol{x} \tag{4a}$$

$$\text{subject to } \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{z}, \quad \boldsymbol{0} \le P\boldsymbol{z} \le \boldsymbol{k}$$

$$f_t(P\boldsymbol{y}) = \underset{\boldsymbol{d}}{\text{maximize}} \ \left\{r(P\boldsymbol{d}) + g_t(P\tilde{\boldsymbol{x}})\right\}, \tag{4b}$$

$$\text{subject to } \boldsymbol{y} = \boldsymbol{d} + \tilde{\boldsymbol{x}}, \ \tilde{\boldsymbol{l}} \le A^{-1}P\boldsymbol{d} \le \tilde{\boldsymbol{u}},$$

where $\tilde{\boldsymbol{l}} = A^{-1}\boldsymbol{b} - \boldsymbol{u}$ and $\tilde{\boldsymbol{u}} = A^{-1}\boldsymbol{b} - \boldsymbol{l}$.

Because $r(\boldsymbol{d})$ and $\mathbb{E}[-h(\boldsymbol{y} - \boldsymbol{\varepsilon})]$ are strictly concave, problem (4a) has unique optimal solution, denoted by $\boldsymbol{y}(\boldsymbol{x}) = [y_1(x_1, x_2), y_2(x_1, x_2)]$, when P is the identity matrix. We have the following proposition. It is worth mentioning that Proposition 3 remains valid if all functions (e.g., r and h), system inputs (e.g., $\boldsymbol{l}$ and $\boldsymbol{u}$) except A, and the random variables $\boldsymbol{\varepsilon}$ are time-dependent.

PROPOSITION 3. *In all periods, v_t and f_t are concave.*

(a) In the complementary product case, $v_t(\boldsymbol{x})$ and $f_t(\boldsymbol{y})$ are supermodular, $v_t(A\boldsymbol{x})$ and $f_t(A\boldsymbol{y})$ are submodular, and $y_i(x_1, x_2)$ is increasing in either x_1 or x_2.

(b) *In the substitutable product case, $v_t(\boldsymbol{x})$ and $f_t(\boldsymbol{y})$ are submodular, $v_t(A\boldsymbol{x})$ and $f_t(A\boldsymbol{y})$ are supermodular, and $y_i(x_1,x_2)$ is increasing in x_i and decreasing in x_j for $i,j=1,2$ and $i\neq j$.*

Proof: Let $\mathcal{L}$ be the collection of all 2×2 matrices $L=[\ell_{i,j}]_{2,2}$ such that $\ell_{i,1}\ell_{i,2}\leq 0$ for $i=1,2$. Notice that if $P, A^{-1}P\in\mathcal{L}$, then sets $\{\boldsymbol{z}:\boldsymbol{0}\leq P\boldsymbol{z}\leq\boldsymbol{k}\}$ and $\{\boldsymbol{d}:\tilde{\boldsymbol{l}}\leq A^{-1}P\boldsymbol{d}\leq\tilde{\boldsymbol{u}}\}$ are sublattices (Example 2.2.7, Topkis 1998); moreover, $-h(P\boldsymbol{x})$ is concave and supermodular in $\boldsymbol{x}$ (Lemma 2.6.2, Topkis 1998). We next verify these statements by selecting proper matrices P.

(a) Let P be the identity matrix and AJ in (4), respectively. It is straightforward to see $P, A^{-1}P\in\mathcal{L}$ and $r(P\boldsymbol{d})$ is concave and supermodular. Since v_{T+1} is linear as assumed, we can inductively prove that in all periods $v_t(P\boldsymbol{x})$ and $f_t(P\boldsymbol{y})$ are concave and supermodular by Theorem 2. That is, $v_t(\boldsymbol{x})$, $f_t(\boldsymbol{y})$, $v_t(AJ\boldsymbol{x})$ and $f_t(AJ\boldsymbol{y})$ are concave and supermodular. We then conclude the properties of v_t and f_t by Lemma 1. In addition, Theorem 3(a) implies that $y_i(x_1,x_2)$ is increasing in either x_1 or x_2.

(b) Let $P=J$ and A in (4), respectively. Similarly by Theorem 2 and Lemma 1, we can verify properties of v_t and f_t. In addition, Theorem 3(a) implies that $J\boldsymbol{y}(J\boldsymbol{x})=[y_1(x_1,-x_2),-y_2(x_1,-x_2)]$ is increasing in $[x_1,x_2]$, that is, $y_i(x_1,x_2)$ is increasing in x_i and decreasing in x_j for $i\neq j$. □

The above proposition implies that in the complementary product case, the optimal order-up-to-levels are increasing in the initial inventory levels of both products, and in the substitutable product case, the optimal order-up-to-level of a product is increasing in its own initial inventory level while decreasing in the other product's initial inventory level.

A simpler version of our model was analyzed by Zhu and Thonemann (2009), which deals with only the substitutable product case without the constraint $\boldsymbol{z}\leq\boldsymbol{k}$. Song and Xue (2007) consider a more general setting with more than two substitutable products and derive structural results of the optimal order-up-to levels similar to Zhu and Thonemann (2009) for the two product case. Ceryan et al. (2009) extend Zhu and Thonemann (2009) by introducing the constraint $\boldsymbol{z}\leq\boldsymbol{k}$ and an additional resource capacity constraint $z_1+z_2\leq k_0$. Notice that for Ceryan et al. (2009)'s model, we can characterize functions v_t, f_t and optimal solutions $\boldsymbol{y}(\boldsymbol{x})$ as the same as Proposition 3(b) by using the similar argument. It is appropriate to point out that all the structure results on the optimal inventory decision in these papers can be easily derived by our approach (we present in Figure 3 in the appendix to illustrate the structure of $\boldsymbol{y}(\boldsymbol{x})$ for Ceryan et al. (2009)'s model).

Compared with these three papers, we deal with both the complementary product case and the substitutable product case in a unified framework. We are not aware of any paper which analyzes integrated inventory and pricing models with complementary products. Moreover, in the substitutable product case, we prove the supermodularity of $v_t(A\boldsymbol{x})$, which is new in the literature. Even though these three papers present results on $v_t(\boldsymbol{x})$ almost identical to Proposition 3, our approach is significantly simpler. In fact, all these papers establish the submodularity of $v_t(\boldsymbol{x})$ recursively by analyzing the first-order optimality condition (the KKT condition) of problem (3).

Their approaches can only handle simple feasible set and require some technical conditions on the objective functions (e.g., smoothness almost everywhere). And all three papers ignore the bound constraints $\boldsymbol{p} \in [\boldsymbol{l}, \boldsymbol{u}]$ on prices, though such constraints are imposed in their models. For example, Zhu and Thonemann (2009) discuss the range of optimal prices *after* deriving their structural results. Song and Xue (2007) mention that the price vector $\boldsymbol{p}$ belongs to some compact set in their introduction section but does not explicitly analyze it when proving the related theorem.

REMARK 4. In general, the optimal prices may not be monotone as illustrated in Zhu and Thonemann (2009). However, when the matrix A is symmetric, Ceryan et al. (2009) and Zhu and Thonemann (2009) prove that $\boldsymbol{p}(\boldsymbol{x})$ is decreasing in $\boldsymbol{x}$ (again by analyzing the KKT condition and ignoring the bound constraints on prices).

REMARK 5. Proposition 3 could fail when demand follows the multiplicative model $\boldsymbol{d}^{\varepsilon} = \boldsymbol{b} - A^{\varepsilon}\boldsymbol{p}$, where entries of the price sensitivity coefficient matrix A^{ε} are random variables. To see it, consider a special case when $\boldsymbol{c} = \boldsymbol{0}, h(\boldsymbol{x}) = 0$ and $v_{t+1}(\boldsymbol{x}) = l_0(\boldsymbol{x}) - \boldsymbol{x}'B\boldsymbol{x}$ in (4b) for some linear term $l_0(\boldsymbol{x})$. Here the quadratic term $\boldsymbol{x}'B\boldsymbol{x}$ in v_{t+1} can be treated as a perturbation which is relatively small comparing to $l_0(\boldsymbol{x})$. Suppose $\mathbb{E}A^{\varepsilon} = A$ for some positive definite A and $\mathbb{E}(A^{\varepsilon} - A)B(A^{\varepsilon} - A) = Q$. Let $\boldsymbol{d} = \boldsymbol{b} - A\boldsymbol{p}$ be the decision variable. In this case problem (4b) becomes

$$\begin{aligned} f_t(\boldsymbol{y}) = \underset{\boldsymbol{d}}{\text{maximize}} \quad & -\boldsymbol{d}'(A^{-1} + A^{-1}QA^{-1})\boldsymbol{d} - (\boldsymbol{y} - \boldsymbol{d})'B(\boldsymbol{y} - \boldsymbol{d}) + l(\boldsymbol{y}, \boldsymbol{d}) \\ \text{subject to} \quad & \boldsymbol{l} \le A^{-1}(\boldsymbol{b} - \boldsymbol{d}) \le \boldsymbol{u}, \end{aligned}$$

where $l(\boldsymbol{y}, \boldsymbol{d})$ is some linear function in terms of $\boldsymbol{y}$ and $\boldsymbol{d}$. Let us consider the instance

$$A^{-1} = \begin{bmatrix} 0.54 & 0 \\ 0 & 0.98 \end{bmatrix}, \quad B = \begin{bmatrix} 0.97 & 0.07 \\ 0.07 & 0.98 \end{bmatrix}, \quad A^{\varepsilon} = A + \varepsilon \begin{bmatrix} 0.54 & -0.48 \\ -0.48 & 0.64 \end{bmatrix},$$

where ε is some random variable with the expected value 0 and variance 1. It is no hard to see the objective is quadratic and strictly concave. Moreover, by properly selecting $\boldsymbol{l}$ and $\boldsymbol{u}$ one can expect that the constraint is inactive when $\boldsymbol{y}$ belongs to some nonempty open subset of $\Re^2$. For these $\boldsymbol{y}$ the above problem reduces to a special case of Example 3. In this situation calculation shows that

$$\nabla^2 v_{t+1}(\boldsymbol{x}) = -2B = \begin{bmatrix} -1.94 & -0.14 \\ -0.14 & -1.96 \end{bmatrix}, \quad \nabla^2 f_t(\boldsymbol{y}) = \begin{bmatrix} -0.38 & 0.05 \\ 0.05 & -0.59 \end{bmatrix}.$$

In this example, v_{t+1} is submodular but f_t is not.

3.2. Two-stage inventory control

Consider a two-stage coordinated dynamic pricing and inventory control problem with random supply and demand over a finite planning horizon. At the beginning of each period, the firm observes the initial raw material inventory level x_1^0 and the finished product inventory level x_2, and then decides the amount z_1^0 of raw material to be purchased (i.e., $z_1^0 \ge 0$) or sold (i.e., $z_1^0 \le 0$). Assume the is no lead time for delivery. With $x_1^0 + z_1^0$ amount of raw material on hand, the firm simultaneously determines the amount z_2 of raw material to be converted into finished product, and the selling price p of the finished product in the period. Suppose one unit of the finished product consumes one unit of the raw material and $0 \le z_2 \le x_1^0 + z_1^0$. Right after the production, the inventory level of raw material becomes $x_1^0 + z_1^0 - z_2$ and that of finished product becomes $x_2 + z_2$. By the end of this period, an additional amount ε_1^0 of raw material arrives and brings the raw material inventory level up to $x_1^0 + z_1^0 - z_2 + \varepsilon_1^0$, where ε_1^0 is a non-negative random variable. Moreover, an amount $d(p) - \varepsilon_2$ of demand for the finished product arrives and brings its inventory level down to $x_2 + z_2 - d(p) + \varepsilon_2$, where unsatisfied demand is backlogged, ε_2 is a random variable independent on ε_1^0 with expected value 0 and $d(p)$ denotes the expected demand given the price p. Assume that $d(p)$ is strictly deceasing in p, which implies that there is a one-to-one correspondence between expected demand and selling price. For convenience, we use $d = d(p)$ as the decision variable and denote the selling price by $p = p(d)$, where $d \in [l, u]$ for some $0 \le l \le u$ with $p(u) \ge 0$.

Following the literature (e.g., Zipkin 2000), we use *echelon inventory levels* $\boldsymbol{x} = [x_1, x_2]$ as system states, where $x_1 = x_1^0 + x_2$ is the total inventory of raw material and finished product. Suppose it incurs the costs $c_1(z_1^0)$, $c_2(z_2)$, $h_1(x_1^0)$ and $h_2(x_2)$ if z_1^0 units of raw material is purchased/sold, z_2 units of the finished product is produced, x_1^0 units of raw material inventory and x_2 units of product inventory are carried over to the next period, where $h_2(x_2)$ is to be understood as the shortage penalty cost if $x_2 < 0$. To avoid technicality, we assume that the expected one-period revenue $r(d) = dp(d)$ is strictly concave, and $c_1(z_1) + c_2(z_2)$ and $\mathbb{E}[h_1(x_1 - x_2 + \varepsilon_1^0) + h_2(x_2 + \varepsilon_2)]$ are strictly convex, where $\mathbb{E}$ denotes the expectation operator corresponding to ε_1^0 and ε_2. The firm's objective is to maximize the expected total profit over the T-period planning horizon.

Let $v_{T+1}(x_1, x_2) = 0$ and $v_t(x_1, x_2)$ be the profit-to-go functions with respect to the echelon inventory levels $[x_1, x_2]$ at the beginning of period $t = 1, \cdots, T$. We can formulate the problem as

$$\begin{aligned} v_t(x_1, x_2) = \underset{y_1, y_2, d}{\text{maximize}} \quad & [r(d) - c_1(z_1) - c_2(z_2) + g_t(y_1 - d, y_2 - d)] \\ \text{subject to} \quad & y_1 = x_1 + z_1,\ y_2 = x_2 + z_2,\ y_2 \le y_1,\ z_2 \ge 0,\quad l \le d \le u \end{aligned}$$

where $[y_1, y_2]$ denotes the echelon inventory levels right after the production, the constraint $y_1 \le y_2$ indicates that the amount of finished product produced from raw material can not exceed the amount of on-hand raw material, and

$$g_t(x_1, x_2) = \mathbb{E}[v_{t+1}(x_1 + \varepsilon_1^0 + \varepsilon_2, x_2 + \varepsilon_2) - h_1(x_1 - x_2 + \varepsilon_1^0) - h_2(x_2 + \varepsilon_2)].$$

Note that system inputs can be time-dependent which will not affect our later analysis.

The problem can be equivalently reformulated as

$$v_t(x_1, x_2) = \underset{y_1, y_2}{\text{maximize}} \ [f_t(y_1, y_2) - c_1(z_1) - c_2(z_2)], \tag{5a}$$

$$\text{subject to} \quad y_1 = x_1 + z_1,\ y_2 = x_2 + z_2,\ y_2 \le y_1,\ z_2 \ge 0$$

$$f_t(y_1, y_2) = \underset{d \in [l,u]}{\text{maximize}} \ [r(d) + g_t(y_1 - d, y_2 - d)], \tag{5b}$$

Since functions $r(d), -c_1(z_1), -c_2(z_2), -h_1(z_1^0)$ and $-h_2(z_2)$ are strictly concave as assumed, there exist unique optimal solutions $[y_1(x_1,x_2), y_2(x_1,x_2)]$ and $d(y_1,y_2)$ respectively to problems (5a) and (5b). Observe that the both problems are special cases of problem (3). We have the following proposition from Theorems 2 and 3.

PROPOSITION 4. *In all periods, v_t and f_t are $L^\natural$-concave, and $y_i(x_1,x_2)$ are increasing in x_1 and x_2 for $i = 1,2$. Moreover, $d(y_1,y_2) \le d(y_1+\delta, y_2+\delta) \le d(y_1,y_2) + \delta$ for any $\delta \ge 0$.*

Proof: Suppose v_{t+1} is $L^\natural$-concave, which is true in the last period $t = T$. By

$$g_t(x_1 - \xi, x_2 - \xi) = \mathbb{E}[v_{t+1}(x_1 - \xi, x_2 - \xi) - h_1(x_1 - x_2) - h_2(x_2 - \xi)],$$

where h_1, h_2 are convex as assumed, one can easily verify the $L^\natural$-concavity of g_t. Because (5b) is a special case of (2), it is no hard to see from Corollary 3(a) that $f_t(\boldsymbol{y}), f_t(J_1\boldsymbol{y})$ and $f_t(J_2\boldsymbol{y})$ are supermodular. Therefore f_t is $L^\natural$-concave by Lemma 2, and so is v_t by Corollary 3(c).

The monotonicity of $y_i(x_1,x_2)$ follows from Theorem 3(a). To characterize $d(y_1,y_2)$, we need the results of Lemma 3 in Zipkin (2008), which claims that there exists $d_0(y_1,y_2)$ solving the unconstrained problem $\max_d [r(d) + g_t(y_1 - d, y_2 - d)]$ such that for any $\delta \ge 0$,

$$d_0(y_1,y_2) \le d_0(y_1+\delta, y_2+\delta) \le d_0(y_1,y_2) + \delta.$$

Observe that $d(y_1,y_2) = \max\{l, \min[d_0(y_1,y_2), u]\}$ and

$$\max\{l, \min[d_0+\delta, u] = \max\{l-\delta, \min[d_0, u-\delta] + \delta.$$

We then conclude the inequality on $d(y_1,y_2)$. $\square$

REMARK 6. Though Zipkin (2008) uses a slightly different definition of $L^\natural$-concavity by restricting $\xi \le 0$, one can exactly follow his proof to see Lemma 3, Zipkin (2008) holds under our definition.

The structure of $\boldsymbol{y}(\boldsymbol{x})$ is consistent with the intuition that higher initial inventory level leads to higher order-up-to-levels. Moreover, the two inequalities of $d(y_1,y_2)$ imply that lower price should be charged so as to reduce the inventory level of finished product; however, the reduction has bounded sensitivity. Furthermore, when c_1 and c_2 are linear, refined structure of $y(x_1,x_2)$ can derived as the problem 3. For simplicity, we omit the details here.

Yang (2004) considers a similar problem without pricing. He assumes that $c_1(z_1)$ is either strictly convex or linear, and $c_2(z_2)$ is linear. Different from our model in which the echelon inventory levels play the role of system states, he models the *minus cost-to-go function v_t* as below in the inventory levels of raw material and finished product.

$$v_t(x_1^0, x_2) = \underset{y_1,y_2}{\text{maximize}} \ [-c_1(z_1^0 + z_2) - c_2(z_2) + \mathbb{E}g_t(y_1 + \varepsilon_1^0, y_2 - \varepsilon_2)]$$

$$\text{subject to} \quad y_1 = x_1 + z_1, \quad y_2 = x_2 + z_2, \quad y_1 \ge 0, \quad z_2 \ge 0,$$

where $g_t(x_1, x_2) = -h_1(x_1^0) - h_2(x_2) + v_{t+1}(x_1, x_2)$. Yang (2004) then analyzes the related KKT conditions and inductively prove that all v_t are concave, supermodular and their Hessian matrices are diagonal dominant. Since that $L^\natural$-concavity implies concavity and the diagonal dominance property for smooth functions, our results immediately lead to the same concavity and diagonal dominance properties on v_t as Yang (2004). In addition, because $c_1(z_1^0+z_2)+c_2(z_2)$ is supermodular in $[z_1^0, z_2]$, the supermodularity of v_t can also be obtained from Theorem 2.

3.3. Inventory control with self-financing

Consider a self-financing retailer who sells a single product over a finite planning horizon with the operational decisions limited by its cash flow. At the beginning of each period, the retailer observes the initial inventory level x_1 of the product and his/her capital level s on hand, and then places an order of size z_1 to raise the inventory level up to $y_1 = x_1 + z_1$. The order is received right away which incurs an ordering cost c per unit. We assume that the total ordering cost cz_1 can not exceed the available capital s. Unused capital $s - cz_1$ is deposited to a savings account and the earning is $r(s - cz_1)$ at the end of the period, where $r \geq 1$ and $r - 1$ is the interest rate. A demand d^ε arrives during the period. The retailer fills the demand from his/her available inventory with a unit price p and receives a revenue $p\min\{y_1, d^\varepsilon\}$ from sales. The revenue increases to $rp\min\{y_1, d^\varepsilon\}$ at the end of the period. Unused inventory is carried over to the next period and unsatisfied demand is lost, which incurs the inventory holding and shortage penalty cost $h(y_1 - d^\varepsilon)$. Assume that h is convex and $p \geq c$ (i.e., profit increases as the amount of sold product increases).

Define $x_2 = s + px_1$ as the current capital plus the revenue if all inventory on hand is sold out. It will be convenient to use $[x_1, x_2]$ as system states. Under this setting, the state $[\tilde{x}_1, \tilde{x}_2]$ in the next period satisfies $\tilde{x}_1 = (y_1 - d^\varepsilon)^+$ and $\tilde{x}_2 = r(y_2 - p\tilde{x}_1)$, where $y_2 = s - cz_1 + py_1 = x_2 + pz_1 - cz_1$.

Let $v_{T+1}(x_1, x_2) = 0$ and $v_t(x_1, x_2)$ be the profit-to-go functions in period $t = 1, \cdots, T$. The retailer's objective is to maximize the expected ending profit and faces the dynamic recursion

$$
\begin{aligned}
v_t(x_1, x_2) = \underset{y_1, y_2}{\text{maximize}} \;\; & \mathbb{E}\left[f_t\big(y_1 - d^\varepsilon, ry_2\big) - h(y_1 - d^\varepsilon)\right] \\
\text{subject to} \;\; & y_1 = x_1 + z_1, \quad y_2 = x_2 + z_2, \\
& py_1 \leq y_2, \quad z_1 \geq 0, \quad z_2 = (p - c)z_1,
\end{aligned}
\tag{6}
$$

where the expectation operator $\mathbb{E}$ associates with random variables d^ε, the constraint $py_1 \leq y_2$ corresponds to the cash flow limitation, and $f_t(x_1, x_2) = v_{t+1}(x_1^+, x_2 - px_1^+)$ with $x_1^+ = \max(x_1, 0)$. We assume that $\mathbb{E}[h(\boldsymbol{y} - \boldsymbol{\varepsilon})]$ is strictly convex to avoid technicality, which ensures the uniqueness of the optimal solution, denoted by $\boldsymbol{y}(\boldsymbol{x}) = [y_1(x_1, x_2), y_2(x_1, x_2)]$, to problem (6).

Apparently (6) is a special case of problem (3). We have the following results on problem (6).

Proposition 5. *All $v_t(x_1, x_2)$ are decreasing in x_1, increasing in x_2, jointly concave and supermodular. Moreover, the optimal solution $\boldsymbol{y}(\boldsymbol{x})$ is increasing in $\boldsymbol{x}$.*

Proof: Suppose these statements are true in period $t+1$, which are obvious in the last period $t=T$. From the definition of f_t, we know $f_t(x_1,x_2)$ is decreasing in x_1 and increasing in x_2. Then one can verify the monotonicity of v_t from the expression (6).

By Corollary 1, $v_{t+1}(x_1,x_2-px_1)$ is concave and supermodular, which together with the monotonicity of v_{t+1} implies that $f_t(x_1,x_2)$ is concave and supermodular. Because $(p-c)z_1$ is increasing in z_1, properties on v_t and $\boldsymbol{y}(\boldsymbol{x})$ follow from Theorems 2 and 3. □

In Proposition 5, The monotonicity of $v_t(x_1,x_2)$ obeys the intuition that lower initial inventory level x_1 for higher initial total value $x_2=s+px_1$ brings more flexibility for retailer's operations and hence leads to higher ending profit. Moreover, though omitted here, one can obtain some refined characterization of $\boldsymbol{y}(\boldsymbol{x})$ by similar arguments as problem (3).

A simpler version of the problem without the inventory holding and shortage penalty cost is analysed by Chao et al. (2008), where all parameters (including the cumulative distribution function of demand) are time-independent. The major difference between our model and the one in Chao et al. (2008) is the definition of system states. Chao et al. (2008) model the profit-to-go functions $v^0(x_1,s)$ in terms of the initial inventory level x_1 and capital s. Unlike our results, they only prove that $v^0(x_1,s)$ is jointly concave and increasing in s, and characterize the structure of optimal solution under some specific conditions.

4. Conclusion

In this paper we study a class of two dimensional parameterized optimization problems, and establish the preservation of supermodularity together with concavity, where the constraint set may not be a lattice and may not be mapped to become one by a variable transformation. We also present several variations in Section 2 including the preservation of supermodularity together with the component-wise concavity, submodularity together with concavity and $L^\natural$-concavity.

Our results include several results in the literature as special cases. They significantly simplify the proofs of several operational models, some of which have not been treated rigorously, and shed new insights on these models. We believe our results can be applied in many other models.

Our results also bring up several interesting issues that need further research. First, as we comment in Example 3, our results can not be directly extended to higher dimensional space. A natural question is under what conditions the preservation of supermodularity in problem (1) holds when we have more than two parameters.

The second question is whether we can say anything about the structure of the optimal solution to problem (1). As we notice in Example 3, the optimal solution may fail to be monotone in general. It would be interesting to identify conditions under which the optimal solution is monotone.

Acknowledgements: This research is partly supported by NSF Grants CMMI-0653909, CMMI-0926845 ARRA and CMMI-1030923. The authors also thank Professor Paul Zipkin for his constructive comments and suggestions on the preliminary draft of this paper.

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Appendix

Proof of Proposition 1

Apparently the concavity of g in term of $\boldsymbol{y}$ implies that $f(x_1, x_2)$ is concave in x_2. Rewrite the problem as below, which is a special case of (1) with $[1, \mathbf{0}]$ being the first row of the matrix A.

$$f(x_1, x_2) = \underset{(y_1, \boldsymbol{y}) \in \boldsymbol{D}}{\text{maximize}} \; \{g(y_1, \boldsymbol{y}) : y_1 = x_1, a_1 y_1 + \boldsymbol{\alpha}'\boldsymbol{y} = x_2\}.$$

For any $\boldsymbol{y}_+ = (y_1, \boldsymbol{y}), \tilde{\boldsymbol{y}}_+ = (\tilde{y}_1, \tilde{\boldsymbol{y}}) \in \boldsymbol{D}$, note that $(A\boldsymbol{y}_+) \wedge (A\tilde{\boldsymbol{y}}_+) = A(\boldsymbol{y}_+ \wedge \tilde{\boldsymbol{y}}_+)$. Following the same proof of Theorem 1, we define $\boldsymbol{a}$ from any two $\boldsymbol{x}, \tilde{\boldsymbol{x}}$. It is easy to verify that $\boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}} = [s_1, s_2]$ satisfy $a_1 = s_1$. Hence the concavity of $f(x_1, x_2)$ in x_2 completes the proof, too.

Proof of Corollary 1

The basic idea is to de composite P as $P = LU$ for some triangle matrices L and U, then sequentially discuss $g_1(\boldsymbol{x}) = g(L\boldsymbol{x})$ and $g_2(\boldsymbol{x}) = g_1(U\boldsymbol{x}) = g(P\boldsymbol{x})$. The statement is straightforward when both diagonal entries of P are zero. Without loss of generality we assume that the first diagonal entry of P is 1. Two cases are considered respectively depending on the sign of $|P|$.

If $|P| \geq 0$ then we can express $P = LU$ as below:

$$P = \begin{bmatrix} 1 & -p \\ -\bar{p} & p_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\bar{p} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -p \\ 0 & |P| \end{bmatrix} = LU,$$

where $p, \bar{p}$ and p_2 are some non-negative real numbers. Denote $g_1(\boldsymbol{x}) = g(L\boldsymbol{x})$, i.e., $g_1(x_1, x_2) = g(x_1, x_2 - \bar{p}x_1)$. Apparently g_1 is component-wise concave. Moreover, we have

$$g_1(x_1, x_2) = \text{maximize } \{g(x_1, y) : \bar{p}x_1 + y = x_2, [x_1, y] \in \boldsymbol{D}\}.$$

Therefore g_1 is supermodular on $\{\boldsymbol{x} : L\boldsymbol{x} \in \boldsymbol{S}\}$ by Proposition 1. Following a similar argument, we can verify the component-wise concavity and supermodularity of $g_2(\boldsymbol{x}) = g_1(U\boldsymbol{x})$ on $\{\boldsymbol{x} : LU\boldsymbol{x} \in \boldsymbol{S}\}$, i.e., $g(P\boldsymbol{x})$ on $\{\boldsymbol{x} : P\boldsymbol{x} \in \boldsymbol{S}\}$.

If $|P| < 0$, consider $g(PJ_0\boldsymbol{x})$ for the linear transformation J_0 mapping a vector $[x_1, x_2]$ to $[x_2, x_1]$. Because $|PJ_0| = -|P| > 0$, and that $g(P\boldsymbol{x})$ on $\{\boldsymbol{x} : P\boldsymbol{x} \in \boldsymbol{S}\}$ is component-wise concave and supermodular if and only of so is $g(PJ_0\boldsymbol{x})$ on $\{\boldsymbol{x} : PJ_0\boldsymbol{x} \in \boldsymbol{S}\}$, we conclude this proof immediately.

Proof of Proposition 2

Recall that $B'B$ has non-negative diagonal entries and non-positive off-diagonal entries. If $B'B$ is singular, then some real numbers λ_1, λ_2, vector $\boldsymbol{v}$ satisfy that $\lambda_1\lambda_2 \leq 0$ and $B\boldsymbol{x} = (\lambda_1 x_1 + \lambda_2 x_2)\boldsymbol{v}$ for all $\boldsymbol{x} = [x_1, x_2]$. In this case f depends on $\boldsymbol{x}$ through $\lambda_1 x_1 + \lambda_2 x_2$ hence its supermodularity follows from its concavity. It leads no loss of generality to assume $B'B$ is non-singular.

For any $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \boldsymbol{S}$, let $\boldsymbol{y}, \tilde{\boldsymbol{y}}$ be the corresponding optimal solutions. Since $\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}, \boldsymbol{y} \vee \tilde{\boldsymbol{y}} \in \boldsymbol{D}$, there exist $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{S}$ such that $A(\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}) = B\boldsymbol{a}$ and $A(\boldsymbol{y} \vee \tilde{\boldsymbol{y}}) = B\boldsymbol{b}$. By $B'A \geq 0$,

$$B'B\boldsymbol{a} = B'A(\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}) \leq (B'A\boldsymbol{y}) \wedge (B'A\tilde{\boldsymbol{y}}) = (B'B\boldsymbol{x}) \wedge (B'B\tilde{\boldsymbol{x}}).$$

Note that the inverse of $B'B$ is non-negative. We know $\boldsymbol{a} \leq \boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ and in a similar way, $\boldsymbol{x} \vee \tilde{\boldsymbol{x}} \leq \boldsymbol{b}$. By the same remaining part of the proof of Theorem 1, f is concave and supermodular on $\boldsymbol{S}$.

Proof of Lemma 2

At first we show (a) and (d) are equivalent. Let $\psi(x_1, x_2, \xi) = f(x_1 - \xi, x_2 - \xi)$ and observe that $f(J_1[x_1 - \xi, x_2 - \xi]) = \psi(x_1, \xi, x_2)$ and $f(J_2[x_1 - \xi, x_2 - \xi]) = \psi(\xi, x_2, x_1)$. On one hand from the definition we know f is $L^\natural$-concave if and only if ψ is supermodular. On the other hand, all the three functions are supermodular if and only if ψ is supermodular in any two of its variables with the other one fixed. We then conclude the equivalence between (a) and (d) by Theorems 2.6.1 and 2.6.2 in Topkis (1998).

By the equivalence between (a) and (d), we know that $f(J_1\boldsymbol{x})$ is $L^\natural$-concave if and only if $f(J_1\boldsymbol{x}), f(J_1^2\boldsymbol{x}) = f(\boldsymbol{x})$ and $f(J_2J_1\boldsymbol{x})$ are supermodular. Observe that $J_2J_1 = J_0J_2$ where J_0 is the linear transformation mapping a vector $[x_1, x_2]$ to $[x_2, x_1]$, and that a function $g(x_1, x_2)$ is supermodular if and only if so is $g(x_2, x_1)$. Therefore $f(J_2J_1\boldsymbol{x})$ is supermodular if and only if so is $f(J_2\boldsymbol{x})$. We then conclude the equivalence between (b) and (d). Similarly, (c) and (d) are equivalent, too.

Proof of Theorem 3

(a) Since f is separable, we can rewrite (3) as

$$\underset{\boldsymbol{y}=[y_1,y_2]}{\text{maximize}} \ \{[f_1(y_1 - x_1) + f_2(y_2 - x_2) + \bar{v}_+(\boldsymbol{y})] : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{y} - \boldsymbol{x} \in [\boldsymbol{l}, \boldsymbol{u}]\},$$

where the objective, regarded as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, is supermodular (Lemma 2.6.2, Topkis 1998) and the set $\{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{l} \le \boldsymbol{y} - \boldsymbol{x} \le \boldsymbol{u}\}$ forms a sublattice (Example 2.2.7, Topkis 1998). The monotonicity of $\boldsymbol{y}(\boldsymbol{x})$ follows from Theorem 2.8.1, Topkis (1998).

(b) Note that $J\boldsymbol{y}(J\boldsymbol{x})$ solves the problem

$$\underset{\boldsymbol{y}}{\text{maximize}} \ \{f(J\boldsymbol{y}) + \bar{v}_+(J\boldsymbol{y}) : \boldsymbol{y} \in J(\boldsymbol{Y}(J\boldsymbol{x}))\},$$

where the objective is supermodular by Lemma 1, and $J(\boldsymbol{Y}(J\boldsymbol{x}))$ forms a sublattice by Example 2.2.7, Topkis (1998) and that all h_n are monotone. Therefore $J\boldsymbol{y}(J\boldsymbol{x})$ is increasing in $\boldsymbol{x}$ as we proved in part(a). The monotonicity of $\boldsymbol{y}(\boldsymbol{x})$ then follows.

Characterization of the optimal solution to problem (3) for linear f_t

Recall the definition of function $v(\boldsymbol{y})$ in the proof of Theorem 3, which is concave and supermodular (submodular) if v_{t+1} is concave and supermodular (submodular). Suppose $\boldsymbol{y}_0(\boldsymbol{x})$ maximizes $v(\boldsymbol{y})$ over $\boldsymbol{Y}$. Notice that $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x})$ if $\boldsymbol{z}_0(\boldsymbol{x}) \in \boldsymbol{Z}$, where $\boldsymbol{z}_0(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}) - \boldsymbol{x}$. Otherwise $\boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{x}$ belongs to the boundary of $\boldsymbol{Z}$. We only need to characterize $\boldsymbol{y}(\boldsymbol{x})$ for the latter case $\boldsymbol{z}_0(\boldsymbol{x}) \notin \boldsymbol{Z}$.

We start from $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$. When $\boldsymbol{z}(\boldsymbol{x}) = [z_1(x_1, x_2), z_2(x_1, x_2)]$ belongs to the boundary of $\boldsymbol{Z}$, there are four possible cases: $z_1(x_1, x_2) = l_1$, $z_2(x_1, x_2) = l_2$, $z_1(x_1, x_2) = u_1$ and $z_2(x_1, x_2) = u_2$. We focus on the first case; the others can be discussed by similar arguments. If $z_1(x_1,x_2)=l_1$ or equivalently $y_1(x_1,x_2)=x_1+l_1$, $y_2(x_1,x_2)$ must solve the problem

$$\text{maximize } \{v(x_1+l_1,y_2):[x_1+l_1,y_2]\in \boldsymbol{Y},\ x_2+l_2\le y_2\le x_2+u_2\}.$$

Relax the constraint $x_2+l_2\le y_2\le x_2+u_2$ and denote $\bar{y}_2(x_1)$ as the related optimal solution. Because this is a concave maximization problem,

$$y_2(x_1,x_2)=\max\{x_2+l_2,\min[\bar{y}_2(x_1),x_2+u_2]\}.$$

Let $\gamma_1(x_1)=\bar{y}_2(x_1)-l_2$ and $\gamma_2(x_1)=\bar{y}_2(y_1)-u_2$, which are increasing (decreasing) functions by Theorem 3 if v is supermodular (submodular) and all h_n are bimonotone (monotone) in problem (3). Partition the state space of $\boldsymbol{x}$ by curves $x_2=\gamma_1(x_1)$ and $x_2=\gamma_2(x_1)$. Then it is optimal to let $y_2=x_2+l_2$ when $\boldsymbol{x}$ lies above the curve $x_2=\gamma_1(x_1)$, $y_2=x_2+u_2$ when $\boldsymbol{x}$ lies below the curve $x_2=\gamma_2(x_1)$, and $y_2=\bar{y}_2(x_1)$ otherwise.

The structure of $\boldsymbol{y}(\boldsymbol{x})$ is conceptually illustrated in Figure 2 when $\boldsymbol{Z}=[\boldsymbol{l},\boldsymbol{u}]$. The space of $\boldsymbol{x}$ is partitioned into nine areas by four curves $x_2=\gamma_k(x_1)$, $1\le k\le 4$, where all functions γ_k are increasing (decreasing) if v_{t+1} is supermodular (submodular) and all h_n are bimonotone (monotone). The

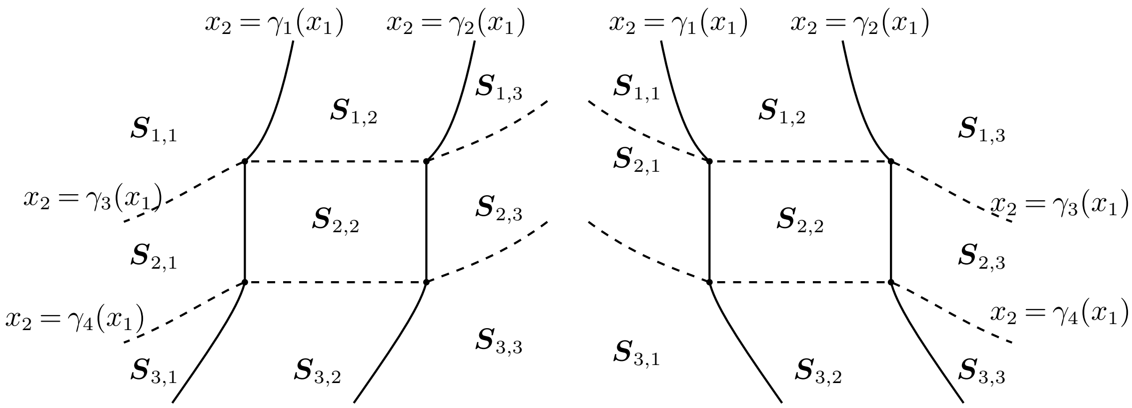

Figure 2 **Structure of $\boldsymbol{y}(\boldsymbol{x})$ when f_t is linear and $\boldsymbol{Z}=[\boldsymbol{l},\boldsymbol{u}]$. The left side corresponds to supermodular $\bar{v}_+$ and bimonotone h_n, the right side corresponds to submodular $\bar{v}_+$ and monotone h_n**

structure of $\boldsymbol{y}(\boldsymbol{x})$ is described as below:

1. If $\boldsymbol{x}=[x_1,x_2]$ lies above the curve $x_2=\gamma_3(x_1)$, then the constraint $y_2-x_2\ge l_2$ is active hence $y_2(x_1,x_2)=x_2+l_2$. If $\boldsymbol{x}$ lies below the curve $x_2=\gamma_4(x_1)$, then the constraint $y_2-x_2\le u_2$ is active hence $y_2(x_1,x_2)=x_2+u_2$. If $\boldsymbol{x}$ lies between the two curves, then it leads no loss of optimality to remove constraints $l_2\le y_2-x_2\le u_2$.
2. If $\boldsymbol{x}=[x_1,x_2]$ lies on the left side of the curve $x_2=\gamma_1(x_1)$, then the constraint $y_1-x_1\le u_1$ is active hence $y_1(x_1,x_2)=x_1+u_1$. If $\boldsymbol{x}$ lies on the right side of the curve $x_2=\gamma_2(x_1)$, then the constraint $y_1-x_1\ge l_1$ is active hence $y_1(x_1,x_2)=x_1+l_1$. If $\boldsymbol{x}$ lies between the two curves, then it leads no loss of optimality to remove constraints $l_1\le y_1-x_1\le u_1$.

We can characterize $\boldsymbol{y}(\boldsymbol{x})$ for $\boldsymbol{x}$ in each area. For example, if $\boldsymbol{x} \in \boldsymbol{S}_{1,2}$, i.e., $\boldsymbol{x}$ lies above $x_2 = \gamma_3(x_1)$ and between $x_2 = \gamma_1(x_1)$ and $x_2 = \gamma_2(x_1)$, then only the constraint $y_2 - x_2 \geq l_2$ is active. It is optimal to let $y_2(x_1, x_2) = x_2 + l_2$ and $y_1(x_1, x_2)$ maximizes $v(y_1, x_2 + l_2)$ over $\boldsymbol{Y}$. If $\boldsymbol{x} \in \boldsymbol{S}_{2,2}$, then it leads no loss of generality to remove the constraint $\boldsymbol{y} - \boldsymbol{x} \in [\boldsymbol{l}, \boldsymbol{u}]$. If $\boldsymbol{x} \in \boldsymbol{S}_{3,2}$, only the constraint $y_2 - x_2 \leq u_2$ is active therefore $y_2(x_1, x_2) = x_2 + u_2$ and $y_1(x_1, x_2)$ maximizes $v(y_1, x_2 + u_2)$ over $\boldsymbol{Y}$. Similar arguments can be made when $\boldsymbol{x}$ falls into other areas.

Next we consider $\boldsymbol{Z} = \{\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}] : h(\boldsymbol{z}) \leq 0\}$ for some convex h. Let $\bar{\boldsymbol{y}}(\boldsymbol{x})$ be the optimal solution associated with $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$, and $\bar{\boldsymbol{z}}(\boldsymbol{x}) = \bar{\boldsymbol{y}}(\boldsymbol{x}) - \boldsymbol{x}$. If $h(\bar{\boldsymbol{z}}(\boldsymbol{x})) \leq 0$, then $\boldsymbol{y}(\boldsymbol{x}) = \bar{\boldsymbol{y}}(\boldsymbol{x})$. Therefore we only needs to discuss these $\boldsymbol{x} \in \boldsymbol{\Omega} = \{\boldsymbol{x} : h(\bar{\boldsymbol{z}}(\boldsymbol{x})) \geq 0\}$.

Observe that $h(\boldsymbol{z}(\boldsymbol{x})) = 0$ for all $\boldsymbol{x} \in \boldsymbol{\Omega}$. If h is bimonotone (monotone) then $h(z_1, z_2) = 0$ determines some increasing (decreasing) function $z_2 = \alpha(z_1)$. Let $\boldsymbol{a} = [l_0, \alpha(l_0)]$ and $\boldsymbol{b} = [u_0, \alpha(u_0)]$ be the intersection points of the curve $h(\boldsymbol{z}) = 0$ and the boundary of $[\boldsymbol{l}, \boldsymbol{u}]$. Then

$$\boldsymbol{y}(\boldsymbol{x}) = \arg\max \{v(\boldsymbol{y}) : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{y} - \boldsymbol{x} = [\xi, \alpha(\xi)],\ l_0 \leq \xi \leq u_0\}.$$

Recall that $\boldsymbol{y}_0(\boldsymbol{x})$ maximizes v over $\boldsymbol{y}$ and $\boldsymbol{z}_0(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}) - \boldsymbol{x}$. Again, because it is a concave maximization problem, we can further partition the set $\boldsymbol{\Omega}$ into three parts depending on whether $\boldsymbol{z}_0(\boldsymbol{x}) < \xi$, $\xi \leq \boldsymbol{z}_0(\boldsymbol{x}) \leq \alpha(\xi)$ or $\boldsymbol{z}_0(\boldsymbol{x}) > \alpha(\xi)$.

Figure 3 conceptually illustrates how the addition constraint influences the structure of optimal solutions, where the left hand side shows the structure of $\bar{\boldsymbol{y}}(\boldsymbol{x})$, the optimal solution associated with $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$, and the right side shows that of $\boldsymbol{y}(\boldsymbol{x})$, the optimal solution associated with $\boldsymbol{Z} = \{\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}] : h(\boldsymbol{z}) \leq 0\}$ for some linear $h(z_1, z_2) = z_1 + z_2 - k_0$. Specifically, we partition the space of $\boldsymbol{x}$ by some curve $x_2 = \bar{\gamma}(x_1)$ such that the additional constraint is active if and only if $\boldsymbol{x} \in \boldsymbol{\Omega} = \{[x_1, x_2] : x_2 \leq \bar{\gamma}(x_1)\}$. $\boldsymbol{\Omega}$ is further partitioned into three parts $\boldsymbol{\Omega}_m, m = 1, 2, 3$, by two curves such that $\boldsymbol{z} = \boldsymbol{b}$ when $\boldsymbol{x} \in \boldsymbol{\Omega}_1$, the constraint $\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}]$ is inactive when $\boldsymbol{x} \in \boldsymbol{\Omega}_2$, and $\boldsymbol{z} = \boldsymbol{a}$ when $\boldsymbol{x} \in \Omega_3$. When $\boldsymbol{x} \notin \boldsymbol{\Omega}$, the characterization of $\boldsymbol{y}(\boldsymbol{x})$ is similar as $\bar{\boldsymbol{y}}(\boldsymbol{x})$.

When more constraints of the form $h_n(\boldsymbol{z}) \leq 0$ are involved in the expression of $\boldsymbol{Z}$, we can repeat the above discussions by adding constraints step by step, then characterize $\boldsymbol{y}(\boldsymbol{x})$ by further partitioning the space of $\boldsymbol{x}$.

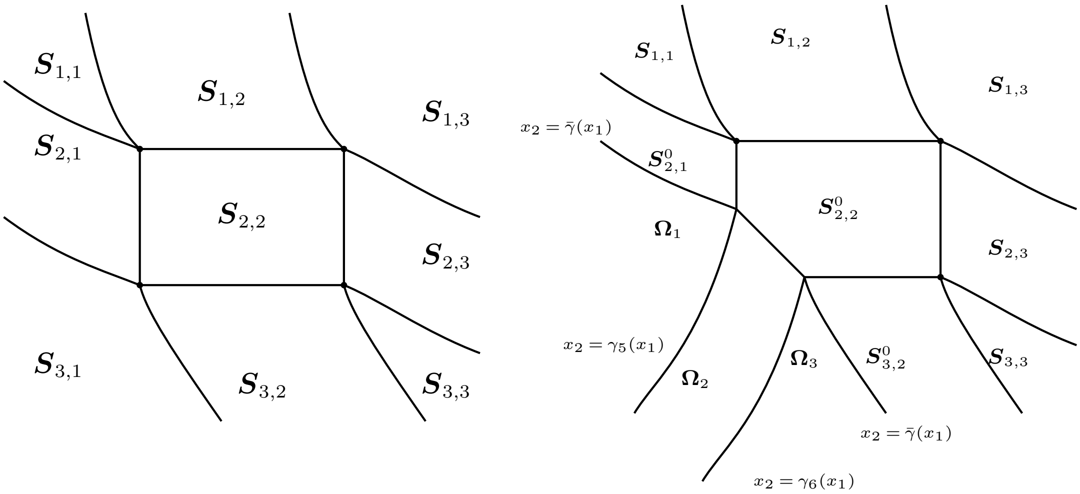

Figure 3 **Structure of $y(x)$ for linear f_t, submodular $\bar{v}_+$ and monotone linear h: The left side corresponds to $Z = [l, u]$ and the right side corresponds to $Z = \{z \in [l, u] : h(z) \geq 0\}$.**