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TECHNICAL NOTE

Preservation of Quasi-*K*-Concavity and Its Applications

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In this paper, we establish a new preservation property of quasi-K-concavity under certain optimization operations. One important application of the result is to analyze joint inventory-pricing models for single-product periodic-review inventory systems with concave ordering costs. At each period, an ordering quantity and a selling price of the product are determined simultaneously. Demand is random but sensitive to the price. The objective is to maximize the total expected discounted profit over a finite planning horizon. Assuming that demand is a deterministic function of the selling price plus a random perturbation with a positive Pólya or uniform distribution, we show that a generalized (s, S, p) policy is optimal.

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1. Introduction

The concept of quasi-K-concavity was introduced by Porteus (1971) to prove the optimality of a generalized (s, S) policy for inventory systems with concave ordering costs. To apply this concept to characterize optimal inventory policies, one relies heavily on some preservation properties under certain optimization operations. In this paper, we provide a new preservation property of quasi-K-concavity, which says that under mild technical conditions, $\max_d[\alpha(d) + \beta(y - d)]$ is quasi-K-concave if the one-dimensional functions $\alpha(\cdot)$ and $\beta(\cdot)$ are concave and quasi-K-concave, respectively.

The preservation property plays a critical role in analyzing joint inventory-pricing models with concave ordering costs. Specifically, consider a firm managing an inventory system with concave ordering cost, which may arise when the firm replenishes from a single supplier providing incremental quantity discount or multiple suppliers with different fixed costs and variable costs. Demand is random and depends on the selling price. Unsatisfied demand in each period is fully backlogged. At the beginning of each period, the firm makes pricing and inventory replenishment decisions simultaneously so as to maximize the total expected discounted profit over a finite planning horizon.

For such a model, we show by employing the preservation property of quasi-K-concavity that when demand is a deterministic function of the selling price plus a random perturbation with a positive Pólya or uniform distribution, the value functions belong to the class of quasi-K-concave functions, and therefore a generalized (s, S, p) policy is optimal. Under such a policy, inventory is managed based on a generalized (s, S) policy. That is, there is a sequence of reorder points s_i and order-up-to levels S_i (both are increasing in i) such that if the starting inventory level is lower than the reorder point s_i but higher than s_{i+1}, the firm places an order to raise its inventory level to S_i. The optimal price is set according to the inventory level after replenishment. For the special case with two suppliers, one with only variable cost, whereas the other has both fixed and variable costs, we prove that the generalized (s, S, p) policy is still optimal when the additive random component in the demand function has a strongly unimodal density.

Our model falls within the growing research stream on inventory and pricing coordination. Recently, significant progress has been made on analyzing integrated inventory and pricing models with fixed ordering cost and stochastic demand for both backlog (see Chen and Simchi-Levi 2004a, b; Huh and Janakiraman 2008) and lost sales (see Chen et al. 2005, Huh and Janakiraman 2008, Song et al. 2009) cases. For a recent review of this literature, readers are referred to Chen and Simchi-Levi (2008). However, we are not aware of any paper analyzing inventory and pricing models with concave ordering cost, which may be partly due to the technical complexity involved.

Our paper is closely related to classical stochastic inventory models with general concave ordering costs analyzed by Porteus (1971, 2002), who introduced the concept of quasi-K-concavity to prove the optimality of generalized (s, S) inventory policies when demand is a positive Pólya or uniform random variable. Recently, Fox et al. (2006) analyzed a special case of Porteus's model with two suppliers, one with only variable cost and the other with both fixed and variable costs. Using the concepts of K-concavity introduced by Scarf (1960) and quasiconcavity (equivalently, quasi-0-concavity), they prove that the generalized (s, S) policy (indeed, a bit simplified policy) is optimal when demand has a strongly unimodal density. Our results and analysis, building upon the new preservation property of quasi-K-concavity as well as preservation properties of K-concavity and quasiconcavity, extend those in Porteus (1971, 2002) and Fox et al. (2006) to include pricing decision.

The rest of this paper is organized as follows. In §2, we present our major technical results, which are then applied to characterize the optimal policy for our inventory and pricing model with concave ordering cost in §3.

2. Main Technical Results

In this section, we present our preservation property of quasi-K-concavity. Quasi-K-concavity was introduced by Porteus (1971) to prove the optimality of a generalized (s, S) policy for inventory systems with general concave ordering costs. By definition, a one-dimensional function f is quasi-K-concave if for any $x_1 \leqslant x_2$ and $\lambda \in [0, 1]$, $f((1-\lambda)x_1 + \lambda x_2) \geqslant \min\{f(x_1), f(x_2) - K\}$. For brevity, readers are referred to Porteus (2002) for properties of this class of functions.

Among all quasi-K-concave functions, we mainly consider one class called *quasi-K-concave function with changeover a*. A function f is quasi-K-concave with changeover a if it is increasing on $(-\infty, a]$ and non-K-increasing on $[a, \infty)$ (non-K-increasing means that for $x_1 < x_2$, $f(x_1) \geqslant f(x_2) - K$). An important property for this class of functions is that the quasi-K-concavity is preserved under integral convolution with respect to a positive Pólya or a positive uniform random variable. Positive Pólya (also called one-sided Pólya) distribution includes, among others, all finite convolutions of exponentially distributed random variables. Thus, as a special case, Erlang distribution is positive Pólya. Although the positive Pólya distribution appears to be restrictive, Cox (1962) notes that for any given μ and $\sigma^2 \in [\mu^2/n, \mu^2]$ for some natural number n, a random variable with mean μ and variance σ^2 can be generated through a convolution of n exponential random variables. We refer to Porteus (1971) for more details on this class of random variables and its relationship with quasi-K-concave functions.

We now present our major result of this section, which says that quasi-K-concavity can be preserved under a maximization operation. Let $\alpha(\cdot)$ and $\beta(\cdot)$ be one-dimensional continuous functions defined in a bounded interval $\mathcal{D} = [\underline{d}, \bar{d}]$ and in the real line, respectively. Define a new function

$$\Gamma(y) = \max_{d \in \mathcal{D}} [\alpha(d) + \beta(y - d)]. \tag{1}$$

Theorem 1. *If $\alpha(\cdot)$ is a differentiable concave function and $\beta(\cdot)$ is a continuously differentiable quasi-K-concave function with some finite changeover ξ^0, then the function $\Gamma(\cdot)$ defined in problem* (1) *is quasi-K-concave with a finite changeover no less than ξ^0.*

Because the proof is quite involved and long, it is provided in the appendix. An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/. Here we only briefly sketch the main idea of the proof. First, we show that $y - d(y)$ is nonincreasing in y, where $d(y)$ is the smallest maximizer for problem (1). Second, we show that $d(y_0)$ is a maximizer of function $\alpha(\cdot)$, where y_0 is the largest point such that $\Gamma(\cdot)$ is nondecreasing in $(-\infty, y_0]$. Third, we show that y_0 is no less than the largest changeover of $\beta(\cdot)$. Finally, we use the results from the previous steps to prove that $\Gamma(\cdot)$ is non-K-increasing for $y \geqslant y_0$, and thus is quasi-K-concave with y_0 as its changeover.

In the next section, we will use Theorem 1 to analyze an inventory and pricing model with concave ordering cost. For a special case of the model involving two suppliers, one with only variable cost and the other with both fixed and variable costs, we can prove that a generalized (s, S, p) policy is optimal under a bit more relaxed conditions using similar preservation properties of quasiconcavity (equivalently, quasi-0-concavity) and K-concavity.

Theorem 2. *Let $\alpha(\cdot)$ and $\beta(\cdot)$ be two continuous functions and $\Gamma(\cdot)$ be defined in problem* (1). *We have the following results*: (a) *if $\alpha(\cdot)$ and $\beta(\cdot)$ are both quasiconcave, $\Gamma(\cdot)$ is also quasiconcave*; (b) *if $\alpha(\cdot)$ is concave and $\beta(\cdot)$ is K-concave, $\Gamma(\cdot)$ is also K-concave.*

Note that, different from Theorem 1, part (a) of the above result only requires the quasiconcavity of α. We also comment that K-concavity was first introduced by Scarf (1960) to show that an (s, S) policy is optimal for stochastic inventory models with fixed ordering costs. Chen and Simchi-Levi (2004a) implicitly use Theorem 2, part (b) to prove the optimality of (s, S, p) policy for an inventory and pricing problem with fixed ordering cost and additive demand.

3. Applications: Optimality of Generalized (s, S, p) Policy

In this section, we show how to apply our preservation properties in the previous section to analyze joint inventory and pricing models with concave ordering costs. It is worthwhile mentioning that a similar approach can be used to analyze another important application, namely, inventory models incorporating sales effort/promotion decisions and concave ordering costs.

3.1. The Model

Consider a single-product periodic-review inventory system in which a firm needs to replenish its inventory and set the selling price simultaneously at the beginning of each period over a finite planning horizon with length T. Customer demand is random but depends on the price. Unsatisfied demand is fully backlogged. The firm faces a concave piecewise-linear ordering cost, which can be viewed as ordering from different (say M) suppliers with different fixed and variable ordering costs. Delivery lead times from all suppliers are assumed to be zero, as is common in the literature of joint inventory and pricing optimization. Ordering from supplier i incurs a fixed cost K_i and a unit cost c_i. Without loss of generality, assume that $c_1 > c_2 > \cdots > c_M \geqslant 0$, and $0 \leqslant K_1 < K_2 < \cdots < K_M$.

The remaining inventory at the end of each period t incurs a unit holding cost h_t, whereas the unsatisfied demand incurs a unit backlog cost b_t. We use $L_t(x) = h_t \max\{x, 0\} + b_t \max\{-x, 0\}$ to denote the inventory holding and customer backlog cost given the ending inventory x. Let γ be the discount factor, $0 \leqslant \gamma \leqslant 1$. Similar to Porteus (2002) (Assumption B1, Chapter 9.4), we assume that $(c_1 - \gamma c_M) \leqslant b_t$, which implies that it is more cost effective to fill an order now from a more expensive supplier than delaying it until the next period using a cheaper supplier (in terms of only variable cost). The selling price of the product in period t is $p_t \in [\underline{p}_t, \bar{p}_t]$, and the demand has the following additive form

$$D_t(p_t, \epsilon_t) = D_t(p_t) + \epsilon_t,$$

in which ϵ_t is a continuous random variable with cdf $F_t(\cdot)$ and mean μ_t. We also make the following assumption on the function $D_t(p_t)$.

Assumption 1. *For all $t = 1, 2, \ldots, T$, $D_t(p)$ has an inverse $D_t^{-1}(d)$, which is continuous and strictly decreasing. Furthermore, the expected revenue*

$$R_t(d) := (d + \mu_t) D_t^{-1}(d)$$

is differentiable and concave in d.

Assumption 1 implies that there is a one-to-one correspondence between the selling price $p_t \in [\underline{p}_t, \bar{p}_t]$ and $d_t = D_t(p_t) \in \mathscr{D}_t \equiv [\underline{d}_t, \bar{d}_t]$, where $\underline{d}_t = D_t(\bar{p}_t)$ and $\bar{d}_t = D_t(\underline{p}_t)$. Therefore, in what follows, to facilitate the analysis, we will use d instead of p as the decision variable. The concavity requirement for $R_t(d)$ is standard in the literature. Demand functions that satisfy this requirement include, among others, linear demand $D_t(p) = a - bp$ and exponential demand $D_t(p) = ae^{-bp}$.

We seek an optimal ordering and pricing policy for the firm so as to maximize its total expected discounted profit over the entire planning horizon. Let $v_t(x)$ be the optimal total expected discounted profit from period t to T. Note that $v_t(x)$ is a maximization over possible ordering from all M available suppliers:

$$v_t(x) = \max_{1 \leqslant i \leqslant M} \Big\{ -c_i(y - x) + \sup_{y \geqslant x} [\widehat{H}_t(y) - K_i \delta(y - x)] \Big\},$$

where $\delta(q) = 1$ if $q > 0$ and 0 otherwise, $\widehat{H}_t$ is given as

$$\widehat{H}_t(y) = \max_{d \in \mathscr{D}_t} \{ R_t(d) + \mathsf{E}[\widehat{G}_t(y - d - \epsilon_t)] \},$$

and $\widehat{G}_t(x)$, including inventory-holding and customer backlog cost as well as the discounted profit from next period, is given by

$$\widehat{G}_t(x) = -L_t(x) + \gamma v_{t+1}(x).$$

To facilitate our sequel analysis, we define

$$G_{it}(x) = \widehat{G}_t(x) - c_i x,$$
$$H_{it}(y) = \widehat{H}_t(y) - c_i y,$$
and
$$\hat{R}_{it}(d) = R_t(d) - c_i d.$$

With these definitions, we easily rewrite the above equations as

$$v_t(x) = \max_{1 \leqslant i \leqslant M} \Big\{ c_i x + \sup_{y \geqslant x} [H_{it}(y) - K_i \delta(y - x)] \Big\} \tag{2}$$

$$H_{it}(y) = \max_{d \in \mathscr{D}_t} \{ \hat{R}_{it}(d) + \mathsf{E}[G_{it}(y - d - \epsilon_t)] - c_i \mu_t \} \tag{3}$$

$$G_{it}(x) = -c_i x - L_t(x) + \gamma v_{t+1}(x). \tag{4}$$

We assume without loss of generality that $v_{T+1}(\cdot) = 0$. Note that

$$\lim_{|x| \to \infty} G_{it}(x) = \lim_{|x| \to \infty} H_{it}(y) = \lim_{|x| \to \infty} v_t(x) = -\infty \quad \text{as}$$
$$\lim_{|x| \to \infty} L_t(x) = +\infty \quad \text{and} \quad b_t \geqslant c_1 - \gamma c_M.$$

We end this section with the definition of *generalized* (s, S) *policy*.

Definition 1. A policy π is called generalized (s, S) if there exists an m and a sequence of parameters

$$s_m \leqslant s_{m-1} \leqslant \cdots \leqslant s_1 \leqslant S_1 \leqslant S_2 \leqslant \cdots \leqslant S_m,$$

such that, given starting inventory level x, the optimal order-up-to level $\pi(x)$ is given by S_m if $x < s_m$, S_i if $s_{i+1} \leqslant x < s_i$ for $i = 1, 2, \ldots, m-1$, and x otherwise.

3.2. Analysis

In this section, we analyze the optimization problem (2)–(4) and characterize the optimal policies.

Let V^* denote the set of continuous functions v: $\mathbb{R} \to \mathbb{R}$ such that $-c_M x + v(x)$ is nondecreasing on $(-\infty, 0]$ and that $-c_i x + v(x)$ is non-K_i-increasing for each i on $\mathbb{R}$. The following result provides a characterization of the optimal policy of problem (2)–(4).

Theorem 3. *If $v_{t+1} \in V^*$ and ϵ_t is a positive Pólya or a positive uniform random variable, then* (a) *H_{it} is quasi-K_i-concave with changeover at some $a_{it} \geqslant 0$ for each i;* (b) *there exists a generalized (s, S, p) policy that is optimal in period t; and* (c) *$v_t \in V^*$. Thus, for our joint inventory and pricing problem* (2)–(4), *a generalized (s, S, p) policy is optimal.*

Proof. For part (a), we first rewrite

$$G_{it}(y) = -\big[(c_i - \gamma c_M)y + L_t(y)\big] + \gamma[v_{t+1}(y) - c_M y\big].$$

The property of $v_{t+1}(y)$, together with the assumptions $(c_1 - \gamma c_M) \leqslant b_t$ and $c_i < c_1$, implies that each term in $G_{it}(y)$ is increasing in $(-\infty, 0]$, and thus $G_{it}(y)$ is increasing in $(-\infty, 0]$. Moreover, for $y > 0$, we rewrite $G_{it}(y)$ as

$$G_{it}(y) = -\big[(1-\gamma)c_i y + L_t(y)\big[+\gamma[-c_i y + v_{t+1}(y)].$$

Because $-[(1-\gamma)c_i y + L_t(y)[$ is decreasing and hence non-$(1-\gamma)K_i$-increasing and $\gamma[-c_i y + v_{t+1}(y)]$ is non-γK_i-increasing as $v_{t+1} \in V^*$, $G_{it}(y)$ is non-K_i-increasing for $y > 0$. Thus, $G_{it}(y)$ is quasi-K_i-concave with changeover 0. Because ϵ_t has positive Pólya distribution, $\mathsf{E}[G_{it}(y - d - \epsilon_t)]$ is quasi-K_i-concave in y with a positive changeover. $\lim_{|x|\to\infty} G_{it}(x) = -\infty$ implies that this changeover is finite. In addition, because ϵ_t is a continuous random variable, $\mathsf{E}[G_{it}(y - d - \epsilon_t)]$ is continuously differentiable. Thus, by Theorem 1, $H_{it}(y)$ is quasi-K_i-concave with a changeover at some $a_{it} \geqslant 0$, and part (a) is proven.

For part (b), because $H_{it}(y)$ is quasi-K_i-concave, it is optimal to replenish inventory following a generalized (s, S) policy, which follows directly from Lemma 9.13 in Porteus (2002). Moreover, there exists an optimal $d^*_{it}(y)$, such that

$$d^*_{it}(y) = \arg\max_{d \in [\underline{d}_t, \bar{d}_t]} \big\{\hat{R}_{it}(d) + \mathsf{E}[G_{it}(y - d - \epsilon_t)]\big\}.$$

Note that the optimal $d^*_{it}(y)$ is set based on the resulting inventory level y after the replenishment decision, and we can find the optimal price p^* through $D_t^{-1}(d^*_{it}(y)) = p^*$ given i is the supplier being ordered from.

For the proof of part (c), readers are referred to Porteus (2002, pp. 147–148) for detailed steps. □

We now focus on a special case of the model presented above. Specifically, we assume that there are only two suppliers: suppliers H and L, where supplier H charges a variable cost c_1 per unit but no fixed cost ($K_1 = 0$), whereas supplier L charges a variable cost c_2 ($c_1 > c_2$) per unit plus a fixed cost $K_2 = K > 0$. Such a cost structure is commonly seen in the practice of a dual sourcing strategy, as demonstrated by Fox et al. (2006). Similar to the general model, we assume $b_t \geqslant c_1 - \gamma c_2$.

Given this cost structure, Fox et al. (2006) prove for a corresponding inventory model without pricing decisions the optimality of generalized (s, S)-type policies when demand has *strongly unimodal densities*. The class of strongly unimodal density functions is a broader class of random variables and includes many commonly used probability distributions such as normal, uniform, or gamma distribution with shape parameter $p \geqslant 1$. A salient property of strongly unimodal density functions is the preservation of quasiconcavity, i.e., $\mathsf{E}[f(x - \epsilon)]$ is still quasiconcave if f is and ϵ has a strongly unimodal density (for more discussion on strongly unimodal density functions, see Dharmadhikari and Joag-Dev 1988).

The result in Fox et al. (2006) can be extended to our setting with pricing decisions. Specifically, the following result implies that the optimal inventory policy is a hybrid version of a base-stock policy plus an (s, S) policy.

Theorem 4. *For our joint inventory and pricing problem with two suppliers, under Assumption* 1 *with ϵ_t having a strongly unimodal density, there exist parameters s_t, S_t^L, S_t^H for period t such that the optimal order-up-to level y_t^* takes one of the two forms: If $S_t^H \leqslant s_t$, order from supplier L based on the following (s_t, S_t^L) policy,*

$$y_t^* = \begin{cases} S_t^L, & \text{if } x \leqslant s_t \\ x_t, & \text{if } x > s_t; \end{cases}$$

otherwise, follow an (s_t, S_t^H, S_t^L) mixed-ordering policy,

$$y_t^* = \begin{cases} S_t^L \ (\text{order from supplier } L) & \text{if } x \leqslant s_t \\ S_t^H \ (\text{order from supplier } H) & \text{if } s_t < x \leqslant S_t^H \\ x & \text{if } x > S_t^H. \end{cases}$$

Finally, set the optimal price $p^* = D_t^{-1}(d_t^*(y_t^*))$ based on the inventory level after replenishment.

The proof of the above result is almost parallel to the one by Fox et al. (2006), who essentially show that both quasiconcavity and K-concavity can be preserved under dynamic programming recursions. Thus, rather than presenting the complete proof, we will only sketch the key steps to prove the preservation of quasiconcavity and K-concavity under dynamic programming recursions (2)–(4) while highlighting the major differences with Fox et al. (2006). The main idea of the proof is to show by induction that $v_t(x)$ is K-concave and $G_{1t-1}(x)$ is quasiconcave in two steps. In the first step, one can prove that if $H_{1t}(y)$ is quasiconcave with nonnegative changeover and $H_{2t}(y)$ is K-concave, then the policy described in Theorem 4 is optimal, and in addition, $v_t(x)$ is K-concave and $G_{1t-1}(x)$ is quasiconcave with nonnegative changeover. This step can be proven by following an argument similar to the one in Fox et al. (2006).

In the second step, we prove that if $v_{t+1}(x)$ is K-concave and $G_{1t}(x)$ is quasiconcave with nonnegative changeover, then $H_{1t}(y)$ is quasiconcave with nonnegative changeover and $H_{2t}(y)$ is K-concave. Observe that quasiconcavity is preserved under integral convolution with a strongly unimodal densities, whereas K-concavity is preserved under integral convolution with general densities. Thus, to complete the proof of the second step, it suffices to use Theorem 2 to show that quasiconcavity and K-concavity are preserved under the optimization operation (1), which constitutes the major difference between our proof and the one in Fox et al. (2006).

4. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

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