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Cartesian P-property and Its Applications to the Semidefinite Linear Complementarity Problem*

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Abstract. We introduce a Cartesian P-property for linear transformations between the space of symmetric matrices and present its applications to the *semidefinite linear complementarity problem* (SDLCP). With this Cartesian P-property, we show that the SDLCP has GUS-property (i.e., globally unique solvability), and the solution map of the SDLCP is locally Lipschitzian with respect to input data. Our Cartesian P-property strengthens the corresponding P-properties of Gowda and Song [15] and allows us to extend several numerical approaches for monotone SDLCPs to solve more general SDLCPs, namely SDLCPs with the Cartesian P-property. In particular, we address important theoretical issues encountered in those numerical approaches, such as issues related to the stationary points in the merit function approach, and the existence of Newton directions and boundedness of iterates in the non-interior continuation method of Chen and Tseng [6].

Keywords: Cartesian P-property – SDLCP – Globally unique solvability – Merit functions – Non-interior continuation method

1. Introduction

There recently has been growing interest in searching for solution methods for the *semidefinite linear complementarity problem* (SDLCP), ranging from the general theoretical framework [36, 39, 34, 5, 29] to concrete numerical methods including interior-point methods [23, 32], path-following methods [37], non-interior continuation methods [6], and smoothing Newton methods [22, 35, 7], to name a few. Most of these numerical methods are extended from their counterparts for linear complementarity problems (LCPs), which might be nonmonotone.

Unlike LCPs, numerical methods proposed so far focus solely on monotone SDLCPs. However, as we will see below, there are SDLCPs which may not possess the monotone property. Thus a natural question is whether one can identify a class of nonmonotone SDLCPs which are suitable for algorithmic approaches.

To motivate this research, we start by introducing the mathematical formulation of SDLCPs to be analyzed in this paper. Let $\mathcal{X}$ denote the space of $n \times n$ block-diagonal real matrices with m blocks of sizes $n_1, \ldots, n_m$ ($n = \sum_{i=1}^{m} n_i$), respectively (the

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blocks are fixed). Thus, $\mathcal{X}$ is closed under matrix addition $X + Y$, multiplication XY, and transposition X^T, where $X, Y \in \mathcal{X}$. Furthermore, if $X \in \mathcal{X}$ is invertible, then its inverse $X^{-1} \in \mathcal{X}$. We endow $\mathcal{X}$ with the inner product and norm:

$$\langle X, Y\rangle := \mathrm{tr}[X^T Y], \qquad \|X\| := \sqrt{\langle X, X\rangle},$$

where $X, Y \in \mathcal{X}$ and $\mathrm{tr}[\cdot]$ denotes the matrix trace (i.e., $\mathrm{tr}[X] = \sum_{i=1}^n X_{ii}$). ($\|X\|$ is the Frobenius-norm of X). Let $\mathcal{S}$ denote the subspace comprising those $X \in \mathcal{X}$ that are symmetric, i.e., $X^T = X$. We denote by $\mathcal{S}_+(\mathcal{S}_{++}$, respectively) the cone of symmetric positive semidefinite (positive definite, respectively) matrices in $\mathcal{S}$. We use the symbol $X \succeq (\succ) 0$ to say that $X \in \mathcal{S}_+(\mathcal{S}_{++})$.

Given a function $F : \mathcal{S} \mapsto \mathcal{S}$, the *semidefinite complementarity problem* (SDCP(F)), is to find a matrix $X \in \mathcal{S}$ such that

$$X \succeq 0,\ \ Y \succeq 0, \ \text{ and } \ \langle X, Y\rangle = 0, F(X) - Y = 0. \tag{1}$$

When F is an affine function, i.e. $F(X) := L(X) + Q$ for a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$ and a matrix $Q \in \mathcal{S}$, the SDCP(F) reduces to the *semidefinite linear complementarity problem* (SDLCP(L, Q)), of finding a matrix $X \in \mathcal{S}$ such that

$$X \succeq 0,\ \ Y := L(X) + Q \succeq 0, \ \text{ and } \ \langle X, Y\rangle = 0. \tag{2}$$

Finally, when $\mathcal{X}$ contains only the diagonal matrices (i.e., $n_1 = n_2 = \cdots = n_m = 1$ and $m = n$), then SDLCP(L, Q) becomes the *linear complementarity problem*, denoted by LCP(M, q), of finding a vector $x \in \Re^n$ such that

$$x \geq 0,\ \ y := Mx + q \geq 0, \ \text{ and } \ \langle x, y\rangle = 0 \tag{3}$$

where $M \in \Re^{n\times n}$ and $q \in \Re^n$ (in fact, M is the matrix representation of L and $q = \mathrm{diag}(Q)$).

It is well known (see [8]) that a class of nonmonotone LCPs which has been well solved is the P-LCPs (i.e., the matrix M in LCP(M, q) is a P-matrix). Apparently, in this case, the corresponding SDLCPs are not monotone anymore and hence it is not clear whether the numerical methods proposed so far are still appropriate. Thus, it is important to identify a class of nonmonotone SDLCPs, called Cartesian P-SDLCPs, which can be solved by some of these well-developed algorithms.

Earlier attempts along this direction have been made by Gowda and Song [15] in identifying more general SDLCPs which enjoy certain properties, the so-called P-type properties generalized from LCPs (see also [16, 17, 30, 18] for more discussions on these P-type properties). Unfortunately, it is not known whether SDLCPs with their P-type properties can be solved by any of the above well-developed numerical methods.

Recall that for the LCP(M, q), $M \in \Re^{n\times n}$ is a P-matrix if one of the following equivalent characterizations holds:

(i) For each nonzero $x \in \Re^n$, there exists an index $i \in \{1, \ldots, n\}$ such that $x_i(Mx)_i > 0$.

(ii) The implication

$$x \in \Re^n, \quad x * (Mx) \le 0 \implies x = 0$$

holds, where $x * (Mx)$ is the componentwise product of vectors x and Mx, and the inequality is defined componentwise.

It is known that if M is a P-matrix then (see [28] for (iii) and [14] for

(iii) LCP(M, q) is globally uniquely solvable for all $q \in \Re^n$ (i.e., GUS-property of M).
(iv) The solution map of LCP(M, q) is locally Lipschitzian with respect to data (M, q).

The P-property introduced by Gowda and Song is a natural extension of (ii) to linear transformations $L : \mathcal{S} \mapsto \mathcal{S}$ and has close relations to the Lyapunov stability theorem [15, Theorem 5]. Unfortunately, even though SDLCP(L, Q) is globally solvable for all $Q \in \mathcal{S}$ when L has the P-property, it may admit more than one solution (i.e., GUS-property of L may not hold). To achieve the GUS-property of L, they further introduced the P_2-property (see Section 3 for definitions of those concepts). In some sense the P_2-property seems to be a proper extension of P-matrix notion to its counterpart in symmetric-matrix spaces. However, its definition presents itself a great deal of difficulty in analyzing it, see [30]. Furthermore, it is not clear whether the P_2-property implies the locally Lipschitzian property of the solution map of SDLCP(L, Q) with respect to data (L, Q), neither is known if the P-type SDLCPs can be numerically solved by any methods mentioned above. Thus there is a need in identifying more appropriate P-type transformations so that properties like (iii) and (iv) hold and such P-type SDLCPs can be solved by certain efficient numerical methods.

In this paper, we introduce a new concept, called Cartesian P-property, which is a natural extension of definition (i) of P-matrix and is motivated by P properties on Cartesian products in $\Re^n$ introduced by Facchinei and Pang in their recent book [10]. This Cartesian P-property guarantees not only the GUS-property of L (see Proposition 3 and Corollary 3), but also the locally Lipschitzian property of the solution map of SDLCP(L, Q) with respect to data (L, Q) (see Proposition 4 and Proposition 6). This makes a complete extension of (iii) and (iv) for P-LCPs to their counterparts in SDLCPs. In some extreme cases, the Cartesian P-property reduces to well-known concepts. For example, when $\mathcal{X}$ contains only the diagonal matrices (i.e., $m = n$), the Cartesian P-property reduces to the P-property of a matrix. If $\mathcal{X}$ contains only one diagonal block (i.e., $m = 1$), the Cartesian P-property of L is equivalent to the strong monotonicity of L (i.e., $\langle X, L(X)\rangle > 0$ for all $0 \neq X \in \mathcal{S}$). However, when $\mathcal{X}$ contains non-trivial diagonal blocks (i.e., $1 < m < n$), this equivalent characterization is no long valid. We also note that non-trivial diagonal blocks of $\mathcal{X}$ may arise from certain SDLCP reformulations of semidefinite linear programming problems, see Kojima et.al. [23], and also [16, Appendix A].

Our new concept of Cartesian P-property allows us to generalize two important numerical approaches, namely the merit function approach and the continuation/smoothing approach, to more general SDLCPs. The first approach was initiated by Tseng [36] and has a close relation to the second, see [6, 22, 35, 7, 5]. An important issue in the merit function approach is when the stationary points of merit functions are the solutions of the original problem, a question raised by Tseng [36, Q1]. We answer this question by focusing on two commonly used merit functions: the implicit Lagrangian function and the Fischer-Burmeister function (their original appearance in the context of nonlinear complementarity problems (NCP) can be found in Mangasarian and Solodov [26] and Fischer [12, 13]). In particular, we show, in Proposition 8, that if L has the Cartesian P (respectively, P_0)-property then the stationary points of the implicit Lagrangian function (respectively, the Fischer-Burmeister function) are the solutions of the original problem. This provides an answer to the question raised by Tseng [36, Q1]. For the second approach, we show that the non-interior continuation method of Chen and Tseng [6] for solving general monotone semidefinite complementarity problems is suitable for the solution of Cartesian P-SDLCPs. We prove its global as well as its local superlinear convergence by addressing two important issues of the algorithm: nonsingularity of Jacobian matrices (which define Newton's equations) and boundedness of neighborhoods of iterates (see Proposition 9 and Corollary 6). This part of investigation not only verifies that Chen-Tseng's algorithm is suitable for nonmonotone problems, but also opens up the possibility of applying several other numerical methods to problems having the Cartesian P-property. Indeed, careful study reveals that similar results also hold for the smoothing methods studied by Kanzow and Nagel [22] and by Sun, Sun, and Qi [35].

The paper is organized as follows. In Section 2, we introduce the new concept of Cartesian P-property and study its equivalent characterization. In Section 3, we show that the Cartesian P-property implies both the GUS-property and locally Lipschitzian property of the solution map. The GUS-property actually follows from the fact that the Cartesian P-property implies the P_2-property. The converse of this fact is not true. In Section 4, we analyze the merit function approach for SDLCPs with the Cartesian P- or P_0-property. In Section 5, we show that Chen-Tseng's algorithm can be applied to solve P-SDLCPs by addressing important issues of nonsingularity of Jacobian matrices and boundedness of neighborhoods of iterates, and we conclude Section 5 by remarking that Chen-Tseng's algorithm is globally as well as locally superlinearly convergent for the Cartesian P-SDLCP. We draw our conclusion in Section 6.

Notation: For a matrix $A \in \mathcal{X}$, A_{ij} denotes its (i, j)th element; diag(A) denotes a vector in $\Re^n$ whose ith component is A_{ii}. Conversely, for a vector u in $\Re^n$, Diag(u) denotes a diagonal matrix whose ith diagonal element is u_i. The p-norm of A is defined by $\|A\|_p := \sup_{\|x\|_p=1} \|Ax\|_p$. Two frequently used inequalities are

$$\max_{i,j} |A_{ij}| \le \|A\|_2 \quad \text{and} \quad \|A\|_2 \le \max\{\sqrt{n_1}, \ldots , \sqrt{n_m}\}\|A\|_\infty. \tag{4}$$

For $X \in \mathcal{S}$, let X_ν denote its νth block and $[X]_+$ denote its projection to $\mathcal{S}_+$. Let $\mathcal{S}_\nu$ denote the subspace of $n_\nu \times n_\nu$ symmetric matrices. Then it follows from [37, Lemma 2.1] that $X_+ = \text{Diag}([X_1]_+, \cdots , [X_m]_+)$, where $[X_\nu]_+$ is the orthogonal projection of X_ν to $(\mathcal{S}_\nu)_+$, the cone of positive semidefinite matrices of $n_\nu \times n_\nu$. For $\nu \in \{1, \ldots , m\}$, let $\mathcal{I}_\nu$ contain all the indices belonging to the νth block of $\mathcal{X}$, $\mathcal{O}_\nu$ denotes the set of orthogonal matrices of size $n_\nu \times n_\nu$, and $\mathcal{O}$ denote the set of orthogonal $P \in \mathcal{X}$ (i.e., $P^T = P^{-1}$, $P \in \mathcal{X}$). Given two matrices $A, B \in \mathcal{X}$, $A \circ B$ denotes the Hadamard product of A and B, i.e., $A \circ B = [A_{ij}B_{ij}]_{i,j=1}^n$. For a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$, we denote its operator norm $|||L||| := \max_{\|X\|=1} \|L(X)\|$. For $\nu \in \{1, \ldots, m\}$, $X \in \mathcal{S}$, we denote $L_\nu(X)$ the νth block of $L(X)$. Given a matrix $A \in \mathcal{X}$, the Lyapunov transformation $L_A : \mathcal{S} \mapsto \mathcal{S}$ is defined by $L_A(X) := AX + XA^T$. Suppose $F : \mathcal{S} \mapsto \mathcal{S}$ is differentiable, we let $\nabla F(X)$ denote the Jacobian of F at X.

2. Cartesian *P* properties

We first state definitions of Cartesian P properties for a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$.

Definition 1. *A linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$ is said to have*

(i) the Cartesian P-property if for any $0 \neq X \in \mathcal{S}$

$$\max_{1 \le \nu \le m} \langle X_\nu, L_\nu(X)\rangle > 0; \tag{5}$$

(ii) the Cartesian P_0-property if for any $0 \neq X \in \mathcal{S}$ there exists $\nu \in \{1, \ldots, m\}$ such that

$$X_\nu \neq 0 \ \textit{and} \ \langle X_\nu, L_\nu(X)\rangle \ge 0. \tag{6}$$

Definition 1 is motivated by P properties on Cartesian products in $\Re^n$ introduced by Facchinei and Pang in their recent book [10, Section 3.5.2]. It is easy to see that when $\mathcal{X}$ contains only one block (i.e., $m = 1$), the Cartesian (P) P_0-property becomes the (strong) monotonicity of L, i.e., $\langle X, L(X)\rangle(>) \ge 0$ for all $X \in \mathcal{S}$, and when $\mathcal{X}$ contains only diagonal matrices (i.e., $m = n$), it becomes the (P) P_0 properties of matrices. Both cases have been well studied, see [23, 37, 6, 22, 35] for the former and the recent book [10] for the latter. The following equivalent characterization is very useful.

Proposition 1. *For a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$, it holds*

(i) L has the Cartesian P-property if and only if for any $0 \neq X \in \mathcal{S}$ and any $P \in \mathcal{O}$, there exists an index $i \in \{1, \ldots, n\}$ such that

$$\left(PXL(X)P^T\right)_{ii} > 0, \tag{7}$$

and

(ii) L has the Cartesian P_0-property if and only if $L + \epsilon I$ has the property (7) for every $\epsilon > 0$, where I is the identity transformation on $\mathcal{S}$.

When $\mathcal{X}$ contains only one block, Proposition 1 gives an interesting equivalent characterization of strong monotone linear tranformations as formally stated below.

Corollary 1. *When $\mathcal{X}$ contains only one block (i.e., $m = 1$), a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$ is strong monotone if and only for any $0 \neq X \in \mathcal{S}$ and any $P \in \mathcal{O}$, there exists an index $i \in \{1, \ldots, n\}$ such that (7) holds.*

The significance of characterization (7) is with the fact that it involves all orthogonal matrices in $\mathcal{X}$, which quite often are nonsymmetric, while definition (5) only involves symmetric matrices. This fact is particular useful as our analysis is often conducted on spectral decomposition of symmetric matrices, rather than on symmetric matrices themselves. In order to prove this equivalence, we need some lemmas.

Lemma 1. *Let A be a 2×2 symmetric real matrix. Then the following statements hold:*

(i) There exists an orthogonal matrix P in $\Re^{2\times 2}$ such that the nonzero diagonal entries of PAP^T have the same sign.
(ii) If $\mathrm{tr}[A] \neq 0$, there exists an orthogonal matrix P in $\Re^{2\times 2}$ such that both the diagonal entries of PAP^T have the same sign as $\mathrm{tr}[A]$.

Proof. Since A is symmetric, there exists an orthogonal matrix Q such that QAQ^T is a diagonal matrix. So without loss of generality, we assume that A is a diagonal matrix and $A = \mathrm{Diag}(\lambda_1, \lambda_2)$.

(i) If A is positive semidefinite (i.e., $\lambda_1\lambda_2 \geq 0$), then we are done. Now we assume that $\lambda_1\lambda_2 < 0$. Let the orthogonal matrix P be

$$P := \frac{1}{\sqrt{|\lambda_1| + |\lambda_2|}} \begin{pmatrix} \sqrt{|\lambda_1|} & \sqrt{|\lambda_2|} \\ -\sqrt{|\lambda_2|} & \sqrt{|\lambda_1|} \end{pmatrix}.$$

Then it is straightforward to show that the two diagonal entries of PAP^T are $\lambda_1 + \lambda_2$ and 0.

(ii) Let the orthogonal matrix P be

$$P := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then both of the two diagonal entries of PAP^T are $(\lambda_1 + \lambda_2)/2$. The proof is completed by noticing that the product of any two orthogonal matrices is also an orthogonal matrix. □

A direct but very useful consequence of the proof of Lemma 1 is

Corollary 2. *Let A be a 2×2 symmetric real matrix. If A is not definite, then there exists an orthogonal matrix P such that PAP^T has at least one zero diagonal entry.*

Now we extend the results above to the general case. The proof involves repeating use of Lemma 1 and Corollary 2.

Lemma 2. *Let A be a matrix of size $r \times r$ and $\mathcal{O}_r$ be the set of orthogonal matrices of size $r \times r$. Then, there exists $P \in \mathcal{O}_r$ such that all nonzero diagonal entries of the matrix PAP^T have the same sign. Furthermore, if $\mathrm{tr}[A] \neq 0$, then there exists $P \in \mathcal{O}_r$ such that all of the diagonal entries of PAP^T have the same sign as $\mathrm{tr}[A]$.*

Proof. Without loss of generality, let $r \geq 2$. First we assume A is symmetric. Suppose there exist $i, j \in \{1, \ldots, r\}$ such that $i < j$ and $A_{ii}A_{jj} < 0$, we consider the 2×2 minor of A, denoted $A(i, j)$ with the entries being the intersections of ith, jth rows and ith, jth columns, i.e.,

$$A(i, j) = \begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{pmatrix}.$$

From Corollary 2, there exists an orthogonal matrix $P \in \Re^{2\times 2}$, say

$$P := \begin{pmatrix} P_{ii} & P_{ij} \\ P_{ji} & P_{jj} \end{pmatrix}$$

such that at least one diagonal entry of $PA(i, j)P^T$ equals 0. Now define an "almost" diagonal matrix $Q \in \Re^{r\times r}$ by

$$Q_{st} = \begin{cases} 1 & s = t \neq i, j \\ P_{rs} & s, t \in \{i, j\} \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that Q is an orthogonal matrix, and the diagonal entries of A and QAQ^T are different in only two entries. Moreover, QAQ^T has one more zero diagonal entry than A. Repeating this process on the new matrix QAQ^T until all nonzero diagonal entries have the same sign. Since there is one more zero diagonal entry at each step, the process will end in a finite number of steps. Let B denote the final matrix of such process. This proves the first part of Lemma 2.

If $\mathrm{tr}[A] \neq 0$, it is easy to see that $\mathrm{tr}[A] = \mathrm{tr}[B]$. Then at least one diagonal entry of B is nonzero. By Lemma 1 (ii), and using the similar process above on B, we can obtain a matrix whose all diagonal entries have the same sign, which of course is the same as $\mathrm{tr}(A)$.

If A is not symmetric, we consider the symmetric matrix $\hat{A} := (A + A^T)/2$. We observe that for any orthogonal P, PAP^T and $P\hat{A}P^T$ have the same diagonal entries. Then our result follows from the proof for the symmetric case. □

Proof of Proposition 1. (i) Suppose that L has the Cartesian P-property. Then for any $0 \neq X \in \mathcal{S}$ there exists $\nu \in \{1, \ldots, m\}$ such that $\langle X_\nu, L_\nu(X)\rangle > 0$. Let $P \in \mathcal{O}$ be arbitrary but fixed and write $P = \mathrm{Diag}[P_1, \ldots, P_m]$ where $P_\nu \in \mathcal{O}_{n_\nu}$. Then it follows the identity

$$\mathrm{tr}[P_\nu X_\nu L_\nu(X) P_\nu^T] = \langle X_\nu, L_\nu(X)\rangle > 0$$

there must exist an $i \in \mathcal{I}_\nu$ such that $(P_\nu X_\nu L_\nu(X) P_\nu^T)_{ii} > 0$. This proves that the property (7) holds.

Now suppose the property (7) holds and L does not satisfy the Cartesian P-property. Then there exists $0 \neq X \in \mathcal{S}$ such that $\langle X_\nu, L_\nu(X)\rangle \leq 0$ for all $\nu \in \{1, \ldots, m\}$. It follows from Lemma 2 that there exists for each ν an orthogonal matrix $P_\nu \in \mathcal{O}_{n_\nu}$ such that all the diagonal elements of $P_\nu X_\nu L_\nu(X) P_\nu^T$ are nonpositive. Let $P = \mathrm{Diag}[P_1, \ldots, P_m]$. Then $P \in \mathcal{O}$ and at the same time all the diagonal elements of the matrix $PXL(X)P^T$ are nonpositive, contradicting the property (7). This establishes the Cartesian P-property of L.

(ii) Suppose that L has the Cartesian P_0-property, then for any but fixed $0 \neq X \in \mathcal{S}$ there exists $\nu \in \{1, \ldots, m\}$ such that $X_\nu \neq 0$ and $\langle X_\nu, L_\nu(X)\rangle \geq 0$. Therefore, for any $\epsilon > 0$

$$\langle X_\nu, L_\nu(X) + \epsilon X_\nu\rangle \geq \epsilon \|X_\nu\|^2 > 0.$$

That is, $L + \epsilon I$ has the Cartesian P-property. The first part (i) implies that $L + \epsilon I$ has the property (7).
Suppose now that $L + \epsilon I$ has the property (7). Then, (i) implies that it also has the Cartesian P-property. That is, for any given $0 \neq X \in \mathcal{S}$, there exists $\nu \in \{1, \dots, m\}$ (depending on ϵ) such that

$$0 < \langle X_\nu, L_\nu(X) + \epsilon X_\nu \rangle = \langle X_\nu, L_\nu(X) \rangle + \epsilon \|X_\nu\|^2.$$

We may choose a subsequence $\{\epsilon\}$ as ϵ goes to *zero* such that the index ν satisfying the above inequality be fixed (due to the finite many choices of ν). It is necessary that $\mathcal{X}_\nu \neq 0$ and $\langle X_\nu, L_\nu(X) \rangle \geq 0$. Hence, we proved that L has the Cartesian P_0-property. □

An obvious consequence of the proof above is that L has the Cartesian P_0-property if and only if $L + \epsilon I$ has the Cartesian P-property. The following result is an equivalent characterization of the Cartesian P-property and will be useful later on. Its proof is trivial and is hence omitted.

Proposition 2. *A linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$ satisfies the Cartesian P-property if and only if for any $0 \neq X \in \mathcal{S}$, any orthogonal matrix $P \in \mathcal{O}$, and any nonsingular diagonal matrix $D \in \mathcal{X}$, there exists an index $i \in \{1, \dots, n\}$ such that*

$$(DPXL(X)P^T D^{-1})_{ii} > 0.$$

3. Lipschitz continuity of the solution map

In this section, we study existence, uniqueness, and continuity of solutions of SDLCP(L, Q) when L has the Cartesian P-property. We simply call such a problem the Cartesian P-SDLCP. Let $\varphi(L, Q)$ denote the solution set of SDLCP(L, Q). The main results in this section are:

(a) $\varphi(L, Q)$ contains a unique solution for any $Q \in \mathcal{S}$ (i.e., L has the GUS-property), and (b) $\varphi(L, Q)$ is locally Lipschitz continuous with respect to data L and Q.

We furnish result (a) by investigating the relations of the Cartesian P-property with other P-type properties introduced by Gowda and Song [15]. Notice that Gowda and Song define these properties for $\mathcal{S}$ with $m = 1$. However, it is straightforward to extend their definitions to the general (block) space $\mathcal{S}$.

Definition 2. *Given a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$, we say that L has the*

(i) P-property if

$$X \text{ and } L(X) \text{ commute}, \quad XL(X) \preceq 0 \Longrightarrow X = 0;$$

and

(ii) P_2-property if

$$X \succeq 0,\ Y \succeq 0,\ (X - Y)[L(X) - L(Y)](X + Y) \preceq 0 \Longrightarrow X = Y.$$

The commutativity of X and $L(X)$ in the P-property makes the analysis of the P-property simpler, since X and $L(X)$ are simultaneously diagonalizable. It is known that the P-property of L implies the global solvability of SDLCP(L, Q), i.e., $\varphi(L, Q) \neq \emptyset$ for any $Q \in \mathcal{S}$. However, an example in Gowda and Song [15] shows that $\varphi(L, Q)$ may contain more than one element. Thus, they propose the P_2-property, which implies the uniqueness of $\varphi(L, Q)$ for any $Q \in \mathcal{S}$ (i.e., L has the GUS-property). We point out that Gowda and Song show that the P_2-property implies the GUS-property for $\mathcal{S}$ with $m = 1$ (i.e, $\mathcal{S}$ contains only one back). However, their proof carries over to the general (block) space $\mathcal{S}$. Few results for the P_2-property (compared to the P-property) has been known, partly because the concept of P_2-property is not amenable to analysis. Recently, Parthasarathy et.al. showed that if L is strongly monotone, then L has the P_2-property [30, Theorem 4]. This motivates them to ask whether the converse is true.

In the following we show that the Cartesian P-property implies the P_2-property of L. This result in turn implies that the converse of [30, Theorem 4] is not true for the general (block) space $\mathcal{S}$, i.e., a linear transformation having the P_2-property might not be strongly monotone, because a linear transformation L having the Cartesian P-property might not be strongly monotone.

Proposition 3. *If a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$ has the Cartesian P-property, then it has the P_2-property.*

Proof. Assume that L has the Cartesian P-property. If $m = 1$, i.e., $\mathcal{X}$ contains only one diagonal block, then L must be strongly monotone, which in turn implies the P_2-property of L [30, Theorem 4].

Now assume that $m > 1$, i.e., each element of $\mathcal{X}$ contains more than one diagonal block. For simplicity, we only consider the case where $\mathcal{X}$ contains two diagonal blocks. The general case follows a similar argument. Suppose there exist two matrices $X \succeq 0$ and $Y \succeq 0$ with

$$X = \begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_1 & \\ & Y_2 \end{pmatrix}$$

such that

$$(X - Y)[L(X) - L(Y)](X + Y) \preceq 0, \tag{8}$$

we shall show that $X = Y$. The proof below is motivated by the proof of [30, Theorem 4] where X and Y contain only one block (i.e., corresponding to $X_1 = Y_1 = 0$ in our case).

Assume that $X \neq Y$. Certainly $0 \neq X + Y \succeq 0$, since $X \succeq 0$ and $Y \succeq 0$. Then there exist $P \in \mathcal{O}$ and positive numbers $\lambda_1, \ldots, \lambda_{r_1}$ $(0 \leq r_1 \leq n_1)$ and $\beta_1, \ldots, \beta_{r_2}$ $(0 \leq r_2 \leq n_2)$, $r_1 + r_2 \geq 1$, such that the following spectral decomposition holds:

$$P(X + Y)P^T = D \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} D,$$

where

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \; D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \; E_1 = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_1 := \mathrm{Diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{r_1}}, 1, \ldots, 1) \in \Re^{n_1 \times n_1},$$

$$D_2 := \mathrm{Diag}(\sqrt{\beta_1}, \ldots, \sqrt{\beta_{r_2}}, 1, \ldots, 1) \in \Re^{n_2 \times n_2},$$

and I_r is the identity matrix of size $r \times r$.

Let $A = D^{-1} P X P^T D^{-1}$ and $B = D^{-1} P Y P^T D^{-1}$, then $A \neq B$,

$$A + B = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \tag{9}$$

and

$$X = P^T D A D P, \quad Y = P^T D B D P. \tag{10}$$

Since $A \succeq 0$ and $B \succeq 0$, we must have

$$A = \begin{pmatrix} A_{r_1} & 0 & & \\ 0 & 0 & & \\ & & A_{r_2} & 0 \\ & & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} B_{r_1} & 0 & & \\ 0 & 0 & & \\ & & B_{r_2} & 0 \\ & & 0 & 0 \end{pmatrix},$$

where A_r and B_r are symmetric positive semidefinite matrices of size $r \times r$. It follows from (8) and (10) that

$$D(A - B)[\hat{L}(A) - \hat{L}(B)](A + B)D \preceq 0, \tag{11}$$

where $\hat{L}(Z) = DPL(P^T DZDP)P^T D$. Writing

$$\hat{L}(A) - \hat{L}(B) = \begin{pmatrix} C_1 & C_3 & & \\ C_3 & C_4 & & \\ & & C_2 & C_5 \\ & & C_5 & C_6 \end{pmatrix},$$

we get from (11) and (9) that $D_{r_i}(A_{r_i} - B_{r_i})C_i D_{r_i} \preceq 0, i = 1, 2$, where

$$D_{r_1} := \mathrm{Diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{r_1}}) \text{ and } D_{r_2} := \mathrm{Diag}(\sqrt{\beta_1}, \ldots, \sqrt{\beta_{r_2}}).$$

Therefore, we have

$$\begin{cases} ((A_{r_1} - B_{r_1})C_1)_{jj} \leq 0 \ \ \forall\, j \in \{1, \ldots, r_1\} \text{ and} \\ ((A_{r_2} - B_{r_2})C_2)_{jj} \leq 0 \ \ \forall\, j \in \{1, \ldots, r_2\}. \end{cases} \tag{12}$$

We note that

$$\begin{aligned} & D^{-1} P(P^T DZDP)L(P^T DZDP)P^T D \\ & = ZDPL(P^T DZDP)P^T D \\ & = Z\hat{L}(Z). \end{aligned} \tag{13}$$

Since L has the Cartesian P-property, it follows from Proposition 2 that for any nonzero $Z \in \mathcal{S}$, there exists $i \in \{1, \ldots, n\}$ such that

$$(D^{-1} P(P^T DZDP)L(P^T DZDP)P^T D)_{ii} > 0.$$

Thus (13) implies that for such i, we have

$$(Z\hat{L}(Z))_{ii} > 0, \tag{14}$$

i.e., $\hat{L}$ has the Cartesian P-property.

Letting $Z = A - B \neq 0$, we have

$$Z\hat{L}(Z) = \begin{pmatrix} (A_{r_1} - B_{r_1})C_1 & (A_{r_1} - B_{r_1})C_3 & & \\ 0 & 0 & & \\ & & (A_{r_2} - B_{r_2})C_2 & (A_{r_2} - B_{r_2})C_5 \\ & & 0 & 0 \end{pmatrix}.$$

Hence (14) implies that there exists $i \in \{1, \ldots, r_1\}$ or $i \in \{1, \ldots, r_2\}$ such that

$$((A_{r_1} - B_{r_1})C_1)_{ii} > 0 \quad \text{or} \quad ((A_{r_2} - B_{r_2})C_2)_{ii} > 0.$$

This strict inequality contradicts the established fact (12). Thus we must have $A = B$, or equivalently $X = Y$. That is, L has the P_2-property. □

Since the P_2-property implies the GUS-property of L [15], the next result follows immediately.

Corollary 3. *If a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$ satisfies the Cartesian P-property, then $\varphi(L, Q)$ contains unique solution for any $Q \in \mathcal{S}$.*

The following example shows that the converse of Proposition 3 is not true in general, i.e., the P_2-property does not necessarily imply the Cartesian P-property.

Example 1. Given a matrix $A \in \mathcal{X}$, the associated two-sided multiplication transformation $M_A : \mathcal{S} \to \mathcal{S}$ is defined by $M_A(X) := AXA^T$. According to [18, Corollary 6], M_A has the P_2-property if and only if A is positive definite or negative definite. Now let

$$A := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that M_A has the P_2-property. However,

$$XM_A(X) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

that is, M_A does not satisfy the Cartesian P-property.

Given a matrix $A \in \mathcal{X}$, in terms of the Lyapunov transformation L_A, Parthasarathy et. al. [30, Theorem 5] asserts that

$$\begin{aligned} A \text{ is positive definite} &\iff L_A \text{ has the } P_2\text{-property} \\ &\iff L_A \text{ is strongly monotone.} \end{aligned}$$

For any linear transformation L, the strong monotonicity of L implies the Cartesian P-property. Considering Proposition 3 we have the following corollary.

Corollary 4. *Given a matrix $A \in \mathcal{X}$, the following statements are equivalent:*

(i) A is positive definite.
(ii) L_A has the strong monotonicity property.

(iii) L_A *has the* P_2*-property.*
(iv) L_A *has the Cartesian* P*-property.*

When $A + A^T$ is nonsingular, we have the following equivalence, which is an extension of [30, Corollary 1] to the Cartesian P-property.

Corollary 5. *Suppose* $\det(A + A^T) \neq 0$ *where* $A \in \mathcal{X}$*, then the following are equivalent:*

(i) L_A *has the GUS-property.*
(ii) L_A *satisfies the* P_2*-property.*
(iii) L_A *satisfies the Cartesian* P*-property.*

Proof. We already know that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) (see, Corollary 3 and [30, Corollary 1]). So it suffices to prove (i) $\Longrightarrow$ (iii). Suppose L_A has the GUS-property, according to [15, Theorem 9] A is positive semidefinite. It follows that $A + A^T$ is positive definite because it is nonsingular. This implies that

$$\mathrm{tr}[XL_A(X)] = \mathrm{tr}[X(A + A^T)X] > 0 \ \text{ for any } 0 \neq X \in \mathcal{S}. \tag{15}$$

That is, A is positive definite, which together with Corollary 4 immediately implies that L_A satisfies the Cartesian P-property. □

Not only does the Cartesian P-property imply the GUS-property of L, but also it implies the local Lipschitz continuity of the solution map $\varphi(L, Q)$ with respect to the data (L, Q). We shall prove the latter below.

For a linear transformation $L : \mathcal{S} \mapsto \mathcal{S}$ having the Cartesian P-property, we define

$$\alpha(L) := \min_{\|X\|=1} \max_{1 \le \nu \le m} \langle X_\nu, L_\nu(X) \rangle.$$

It is easy to see that $\alpha(L) > 0$.

Proposition 4. *Assume that the linear transformation* $L : \mathcal{S} \mapsto \mathcal{S}$ *has the Cartesian* P*-property. Let* $Q, \hat{Q} \in \mathcal{S}$ *and* $X, \hat{X}$ *be the unique solution of SDLCP*(L, Q) *and SDLCP*$(L, \hat{Q})$ *respectively. Then*

$$\|\hat{X} - X\| \le \alpha(L)^{-1} \|\hat{Q} - Q\|. \tag{16}$$

Proof. Let

$$Y := L(X) + Q \ \text{ and } \ \hat{Y} := L(\hat{X}) + \hat{Q}.$$

We note that $X, Y, \hat{X}, \hat{Y}$ are all positive semidefinite. It follows that

$$\begin{aligned}(\hat{X} - X)L(\hat{X} - X) &= (\hat{X} - X)\{\hat{Y} - Y - (\hat{Q} - Q)\} \\ &= -\hat{X}Y - X\hat{Y} - (\hat{X} - X)(\hat{Q} - Q).\end{aligned}$$

The second inequality uses the relation $\hat{X}\hat{Y} = XY = 0$.

It then follows from the definition of $\alpha(L)$ that

$$\begin{aligned}\alpha(L)\|\hat{X} - X\|^2 &\leq \max_{1\leq \nu\leq m} \langle \hat{X}_\nu - X_\nu, L_\nu(\hat{X} - X)\rangle \\ &\leq \max_{1\leq \nu\leq m} \langle \hat{X}_\nu - X_\nu, Q_\nu - \hat{Q}_\nu\rangle \\ &\leq \max_{1\leq \nu\leq m} \|\hat{X}_\nu - X_\nu\|\|Q_\nu - \hat{Q}_\nu\| \\ &\leq \|\hat{X} - X\|\|Q - \hat{Q}\|,\end{aligned}$$

where the second inequality holds since $X, Y, \hat{X}, \hat{Y}$ are all positive semidefinite. This gives (16). □

If we let $\hat{Q} = 0$, then (16) gives the following bound for the unique solution of SDLCP(L, Q)

$$\|X\| \leq \alpha(L)^{-1}\|Q\|. \tag{17}$$

In fact, we have a more tight bound as follows, which generalizes its LCP correspondence [27, Lemma 1].

Proposition 5. *Assume that L has the Cartesian P-property, and let X denote the unique solution of SDLCP(L, Q). Then*

$$\|X\| \leq \alpha(L)^{-1}\|[-Q]_+\|. \tag{18}$$

Proof. Because X is the solution of SDLCP(L, Q), $X \succeq 0$. From this inequality and the fact $X(L(X) + Q) = 0$, we have that

$$\begin{aligned}\alpha(L)\|X\|^2 &\leq \max_{1\leq \nu\leq m} \langle X_\nu, L_\nu(X)\rangle \\ &= \max_{1\leq \nu\leq m} \langle X_\nu, -Q_\nu\rangle \\ &\leq \max_{1\leq \nu\leq m} \langle X_\nu, [-Q_\nu]_+\rangle \\ &\leq \max_{1\leq \nu\leq m} \|X_\nu\|\|[-Q_\nu]_+\| \\ &\leq \|X\|\|[-Q]_+\|,\end{aligned}$$

where the second inequality follows from the positive semidefiniteness of X. This proves the bound (18). □

The bound (18) clearly says that *zero* is the only solution if Q is positive semidefinite. Recall from [27, Lemma 1] the LCP correspondence of the bound (18), in which the following fact is essential: For any $0 \leq x \in \Re^n$, it holds

$$x * q \leq x * q_+ \qquad \text{for all } q \in \Re^n$$

where $*$ means the componentwise product of vectors and $q_+ := \max\{0, q\}$. Unfortunately, we do not have the corresponding result for matrices:

$$(X(-Q))_{ii} \leq (X[-Q]_+)_{ii} \qquad \text{for all } Q \in \mathcal{S} \text{ and } i = 1, \ldots, n \tag{19}$$

where $X \succeq 0$ is given (the use of $-Q$ in the above inequality is only to keep consistence with the proof of (18)). The following example due to Lewis [24] disproves this possibility.

Example 2. Let

$$X := \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \in \mathcal{S}_+ \quad \text{and} \quad Q := \begin{bmatrix} 2 & -3 \\ -3 & 8 \end{bmatrix}$$

Then it is easy to see that $[-Q]_+ = 0$. We see $(X(-Q))_{11} = 2$. Hence, the inequality (19) cannot hold for this choices of X and Q.

Proposition 6. *Let $\mathcal{L}$ be a compact set of linear transformations which have the Cartesian P-property and $\mathcal{Q}$ be a bounded subset of $\mathcal{S}$. Then φ is Lipschitzian on $\mathcal{L} \times \mathcal{Q}$. In this case*

$$\|\varphi(L, Q) - \varphi(\hat{L}, \hat{Q})\| \le \kappa \left(|||L - \hat{L}||| + \|Q - \hat{Q}\| \right) \tag{20}$$

for all (L, Q) and $(\hat{L}, \hat{Q})$ in $\mathcal{L} \times \mathcal{Q}$, where

$$\kappa := \delta^{-1} \max\{1, \theta\delta^{-1}\}, \;\; \delta := \min_{L \in \mathcal{L}} \alpha(L), \;\; \theta := \max_{Q \in \mathcal{Q}} \|Q\|. \tag{21}$$

Proof. We first note that all constants in (21) are well defined. Assume that $\varphi(L, Q) = \{X\}$ and $\varphi(\hat{L}, \hat{Q}) = \{\hat{X}\}$. By putting $\hat{Q} = 0$ in (16), we have bound (17). We note that $\varphi(\hat{L}, (L - \hat{L})(X) + Q) = \{X\}$, we deduce from (16) that

$$\begin{aligned} \|X - \hat{X}\| &\le \alpha(\hat{L})^{-1} \|(L - \hat{L})(X) + Q - \hat{Q}\| \\ &\le \alpha(\hat{L})^{-1} \left\{ |||L - \hat{L}||| \|X\| + \|Q - \hat{Q}\| \right\} \\ &\le \alpha(\hat{L})^{-1} \left\{ \alpha(L)^{-1} \|L - \hat{L}\| || \|Q\| + \|Q - \hat{Q}\| \right\} \\ &\le \kappa \left(|||L - \hat{L}||| + \|Q - \hat{Q}\| \right). \end{aligned}$$

□

It is interesting to note that when $\mathcal{X}$ contains only diagonal matrices, bounds (16) and (20) reduce to their counterparts in LCPs; see [14, Theorem 10 and Theorem 11].

4. Stationary points of merit functions

Merit function approach for the solution of SDCPs was initiated by Tseng [36], which is also closely related to subsequent studies of several other approaches, see [34, 22, 6, 5, 7, 35]. Roughly speaking, the merit function approach reformulates the SDCP(F) defined in (1) as an equivalent unconstrained optimization problem

$$\min_{X \in \mathcal{S}} \; f(X), \tag{22}$$

where $f : \mathcal{S} \mapsto \Re_+$ is often differentiable and is called a merit function of SDCPs, in the sense that $f(X^*) = 0$ if and only if X^* is a solution of the SDCP. Thus, finding a solution for SDCP(F) is equivalent to solving the optimization problem (22). Unfortunately, numerical methods for (22) often stop at some stationary point X^* (i.e., $\nabla f(X^*) = 0$). It is hence of vital importance to identify conditions under which any stationary point of

$f(X)$ is a solution of SDCP(F). Some previously known conditions include the positive definiteness or positive semidefiniteness of $\nabla F(X^*)$, depending on the merit functions; see Tseng [36] or Proposition 7 below for more details.

In this section, we analyze two commonly used merit functions, namely the implicit Lagrangian function and the Fischer-Burmeister function (their first appearance in the context of NCPs can be found in [26] and [12, 13]), and show that the Cartesian P (respectively, P_0)-property provides the required sufficient conditions for the implicit Lagrangian function (respectively, the Fischer-Burmeister function). It is worth mentioning that those two functions play significant roles in the merit function approach for NCPs [10]. It is also appropriate to point out that similar results also apply to some other merit functions analyzed by Tseng [36] and Yamashita and Fukushima [39].

Recall from [36] the implicit Lagrangian function

$$f_\gamma(X) := \max_{Y,Z\in\mathcal{S}_+} \left\{ \langle F(X), X-Z\rangle - \langle Y, X\rangle - \frac{1}{2\gamma}(\|F(X)-Y\|^2 + \|X-Z\|^2) \right\}$$

where $\gamma > 1$ is a fixed constant, and the Fischer-Burmeister function

$$f_{FB}(X) := \frac{1}{2}\|\Phi(F(X), X)\|^2,$$

where $\Phi : \mathcal{S} \times \mathcal{S} \mapsto \mathcal{S}$ is the function

$$\Phi(A, B) := (A^2 + B^2)^{1/2} - (A + B).$$

The following results provide conditions under which any stationary point of f_γ or f_{FB} gives a solution of SDCP(F).

Proposition 7. *[36, Propositions 5.1, 6.1] Assume that $F : \mathcal{S} \mapsto \mathcal{S}$ is differentiable. Then the following holds:*

(i) f_γ is differentiable on $\mathcal{S}$ and any stationary point, X^*, of f_γ (i.e., $\nabla f_\gamma(X^*) = 0$) is a solution of SDCP(F) provided that $\nabla F(X^*)$ is positive definite.
(ii) f_{FB} is differentiable on $\mathcal{S}$ and any stationary point, X^*, of f_{FB} (i.e., $\nabla f_{FB}(X^*) = 0$) is a solution of SDCP(F) provided that $\nabla F(X^*)$ is positive semidefinite.

In the following, we extend Proposition 7 to more general SDCPs, namely to SDCP(F) in which $\nabla F(X)$ has Cartesian P-property or P_0-property. (Notice that corresponding results, in the context of NCPs, have been established in [9, 11, 19].) For this purpose, we give a brief overview on some known results from Tseng [36, Sections 5, 6]. The following result, regarding the gradient of the implicit Lagrangian function, is adapted from Proposition 5.1 and its proof in Tseng [36].

Theorem 1. *Assume that $F : \mathcal{S} \mapsto \mathcal{S}$ is differentiable. Then f_γ is differentiable and*

$$\nabla f_\gamma(X) = \nabla F(X)\left(R_\gamma(X) - \frac{1}{\gamma}S_\gamma(X)\right) + \left(S_\gamma(X) - \frac{1}{\gamma}R_\gamma(X)\right), \tag{23}$$

where

$$R_\gamma(X) := X - [X - \gamma F(X)]_+ \text{ and } S_\gamma(X) := F(X) - [F(X) - \gamma X]_+.$$

Furthermore,

$$\langle R_\gamma(X) - \frac{1}{\gamma}S_\gamma(X), S_\gamma(X) - \frac{1}{\gamma}R_\gamma(X)\rangle \geq 0 \textit{ for all } X \in \mathcal{S}. \tag{24}$$

The next result, regarding the gradient of the Fischer-Burmeister function, is adapted from Lemma 6.3 and Proposition 6.1 in Tseng [36]. Before presenting this result, we introduce some notation. Define a linear mapping $L_C : \mathcal{S}_C \mapsto \mathcal{S}_C$ by

$$L_C[X] = CX + XC,$$

where $C \succeq 0$ and $\mathcal{S}_C$ is the subspace of $\mathcal{S}$ comprising those $X \in \mathcal{S}_C$ whose nullspace contains the nullspace of C. As observed by Tseng [36], L_C is positive definite on $\mathcal{S}_C$ and hence has an inverse, L_C^{-1}, which is also positive definite on $\mathcal{S}_C$. Let $\Psi(A, B) := \frac{1}{2}\|\Phi(A, B)\|^2$. Tseng [36] shows that

$$\nabla_A \Psi(A, B) = \mathrm{sym}\left[L_C^{-1}[C - A - B](A - C)\right], \tag{25}$$

and

$$\nabla_B \Psi(A, B) = \mathrm{sym}\left[L_C^{-1}[C - A - B](B - C)\right], \tag{26}$$

where $C := (A^2 + B^2)^{1/2}$ and $\mathrm{sym}[X] := (X + X^T)/2$ for any $X \in \mathcal{X}$.

Theorem 2. *Assume that $F : \mathcal{S} \mapsto \mathcal{S}$ is differentiable. Then f_{FB} is differentiable and*

$$\nabla f_{FB}(X) = \nabla F(X)\nabla_A\Psi(F(X), X) + \nabla_B\Psi(F(X), X). \tag{27}$$

Furthermore,

$$\langle \nabla_A\Psi(F(X), X), \nabla_B\Psi(F(X), X)\rangle \geq \|(C - F(X) - X)G\|^2. \tag{28}$$

for $X \in \mathcal{S}$, where $C := (F(X) + X)^{1/2}$ and $G := L_C^{-1}(C - F(X) - X)$.

It is easy to verify that (24) and (28) also hold for each individual diagonal block.

Lemma 3. *For any $X \in \mathcal{S}$, we have, for any $\nu = 1, \ldots, m$, that*

(i)

$$\langle [R_\gamma(X) - \frac{1}{\gamma}S_\gamma(X)]_\nu, [S_\gamma(X) - \frac{1}{\gamma}R_\gamma(X)]_\nu\rangle \geq 0. \tag{29}$$

(ii)

$$\langle [\nabla_A\Psi(F(X), X)]_\nu, [\nabla_B\Psi(F(X), X)]_\nu\rangle \geq \|(C - F(X) - X)_\nu G_\nu\|^2, \tag{30}$$

where $G_\nu = L_{C_\nu}^{-1}(C_\nu - F(X)_\nu - X_\nu)$.

We are now ready to extend Proposition 7 to SDCP(F) in which $\nabla F(X)$ has the Cartesian P-property or P_0-property.

Proposition 8. *Assume that $F : \mathcal{S} \mapsto \mathcal{S}$ is differentiable. Then the following holds:*

(i) f_γ is differentiable on $\mathcal{S}$ and any stationary point, X^*, of f_γ (i.e., $\nabla f_\gamma(X^*) = 0$) is a solution of SDCP(F) provided that $\nabla F(X^*)$ has the Cartesian P-property.

(ii) f_{FB} is differentiable on $\mathcal{S}$ and any stationary point, X^*, of f_{FB} (i.e., $\nabla f_{FB}(X^*) = 0$) is a solution of SDCP(F) provided that $\nabla F(X^*)$ has the Cartesian P_0-property.

Proof. (i) Assume that X^* is a stationary point of f_γ, i.e., $X^* \in \mathcal{S}$ and $\nabla f_\gamma(X^*) = 0$. It is easy to see that $R_\gamma(X), S_\gamma(X) \in \mathcal{S}$ for any $X \in \mathcal{S}$.

Assume that $R_\gamma(X^*) - \frac{1}{\gamma} S_\gamma(X^*) \neq 0$. From the assumption that $\nabla F(X)$ has the Cartesian P-property for any $X \in \mathcal{S}$, there must exist an $\nu \in \{1, \ldots, m\}$ such that

$$\langle [R_\gamma(X^*) - \frac{1}{\gamma} S_\gamma(X^*)]_\nu, [\nabla F(X^*)((R_\gamma(X^*) - \frac{1}{\gamma} S_\gamma(X^*))]_\nu \rangle > 0. \quad (31)$$

Furthermore, the inequality (29) implies that

$$\langle [R_\gamma(X^*) - \frac{1}{\gamma} S_\gamma(X^*)]_\nu, [S_\gamma(X^*) - \frac{1}{\gamma} R_\gamma(X^*)]_\nu \rangle \geq 0,$$

which, together with inequality (31), implies that $\nabla f_\gamma(X^*)$ cannot be zero. Thus, $R_\gamma(X^*) - \frac{1}{\gamma} S_\gamma(X^*) = 0$. Using the fact that $\nabla f_\gamma(X^*) = 0$, we see $S_\gamma(X^*) - \frac{1}{\gamma} R_\gamma(X^*) = 0$. Finally, since $\gamma > 1$, we easily see that $R_\gamma(X^*) = S_\gamma(X^*) = 0$, or equivalently $f_\gamma(X^*) = 0$. Hence X^* is a solution of SDLCP(F).

(ii) Assume that X^* is a stationary point of f_{FB}, i.e., $X^* \in \mathcal{S}$ and $\nabla f_{FB}(X^*) = 0$. Inequality (30) implies that, for any $\nu = 1, 2, \ldots, m$, either $\nu \in \mathcal{I}$ or $\nu \in \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ define a partition of the set $\{1, 2, \ldots, m\}$ and are defined as follows:

$$\mathcal{I} := \{ \nu \mid (F(X^*)_\nu + X^*_\nu)^{1/2} - F(X^*)_\nu - X^*_\nu = 0 \},$$

and

$$\mathcal{J} := \{ \nu \mid \langle [\nabla_A \Psi(F(X^*), X^*)]_\nu, [\nabla_B \Psi(F(X^*), X^*)]_\nu \rangle > 0 \}.$$

If $\nu \in \mathcal{I}$, (25) and (26) imply that

$$[\nabla_A \Psi(F(X^*), X^*)]_\nu = [\nabla_B \Psi(F(X^*), X^*)]_\nu = 0.$$

On the other hand, since $\nabla F(X^*)$ has the Cartesian P_0-property, there exists a $\nu \in \mathcal{J}$ such that

$$\langle \nabla_A \Psi(F(X^*), X^*)_\nu, (\nabla F(X^*) \nabla_A \Psi(F(X^*), X^*))_\nu \rangle \geq 0,$$

which, together with the definition of set $\mathcal{J}$, contradicts the assumption that $\nabla f_{FB}(X^*) = 0$.

Therefore, $\mathcal{J} = \emptyset$ and $\mathcal{I} = \{1, 2 \ldots, m\}$, which implies that X^* is a solution of SDCP(F). $\square$

We remark that Proposition 8 provides an affirmative answer to the open question Q1 in [36], which asks whether there exists some appropriate property such that under this property, any stationary point of a merit function is a solution to SDCP(F). Indeed, as we demonstrate in Proposition 8, the Cartesian P-property proposed here provides us exactly such a property. It is also appropriate to point out that the analysis in this section may be extended to other merit functions studied in [36, 39].

5. Applications to a non-interior continuation method for SDCPs

In this section we apply the Cartesian P-property to the non-interior continuation method of Chen and Tseng [6] for solving SDCPs. We demonstrate that the Cartesian P-property ensures not only the nonsingularity of the Jacobian matrices but also the boundedness of neighborhoods used in the method, and thus extends these properties of the method from the monotone case to the nonmonotone case. Consequently, Chen-Tseng's algorithm is actually applicable to more general problems other than monotone problems.

Suppose, for SDCP(F) defined in (1), there exists a continuously differentiable function $\phi_\mu : \mathcal{S} \times \mathcal{S} \mapsto \mathcal{S}$, parameterized by a "smoothing parameter" $\mu > 0$, having the property that

$$\phi_\mu(A, B) \to 0 \text{ and } (A, B, \mu) \to (X, Y, 0) \Longrightarrow X \in S_+, Y \in S_+, \langle X, Y\rangle = 0.$$

Accordingly, (1) can be approximated by the smooth equation $H_\mu(X, Y) = 0$, where

$$H_\mu(X, Y) := (\phi_\mu(X, Y), F(X) - Y).$$

Typically, ϕ_μ can be constructed via any function from the $\mathcal{CM}$ class [38] which consists of convex continuously differentiable functions $g : \Re \mapsto \Re$ with the properties that

$$\lim_{\tau\to-\infty} g(\tau) = 0, \;\; \lim_{\tau\to\infty} g(\tau) - \tau = 0 \;\text{ and }\; 0 < g'(\tau) < 1 \text{ for all } \tau \in \Re.$$

Then, similar to Chen and Mangasarian's proposal in the LP/CP [3, 4], ϕ_μ can be chosen as follows

$$\phi_\mu(X, Y) := X - \mu g((X - Y)/\mu) \tag{32}$$

where, for any $A \in \mathcal{S}$ we have $g(A) = P^T\mathrm{Diag}[g(\lambda_1), \ldots, g(\lambda_n)]P$ with $P \in \mathcal{O}$ and $\lambda_1, \ldots, \lambda_n \in \Re$ satisfies $A = P^T\mathrm{Diag}[\lambda_1, \ldots, \lambda_n]P$. One important example of the $\mathcal{CM}$ class is as follows (see, Chen-Harker [1, 2], Kanzow [20, 21], and Smale [33])

$$g(\tau) := ((\tau^2 + 4)^{1/2} + \tau)/2.$$

With a particular choice of ϕ_μ, Chen-Tseng's algorithm starts with any $\mu > 0$ and $Z \in \mathcal{S} \times \mathcal{S}$. For a fixed μ, few Newton-type steps for $H_\mu(Z) = 0$ are applied to update, and then the parameter μ is decreased and re-iterate. Two important theoretical issues of this method are the solvability of Newton's equations and boundedness of neighborhoods, namely the nonsingularity of $\nabla H_\mu(Z)$ and the boundedness of the neighborhood

$$\mathcal{N}_\beta := \left\{(Z, \mu) \in \mathcal{S} \times \mathcal{S} \times \Re_{++} : \; \|H_\mu(Z)\| \le \beta\mu\right\},$$

where $\beta \in \Re_{++}$ is a constant. It is appropriate to point out that other assumptions for the global and local superlinear convergence of the algorithm are less restrictive and can be easily satisfied when we replaced the monotone requirement by the Cartesian P-property .

The following result, which appears in [6, Lemma 2], gives the Jacobian of $\phi_\mu(X, Y)$. Notice that the Jacobian of a general matrix-valued function can be traced back to Löwner [25].

Lemma 4. *[6, Lemma 2] Fix any $\mu \in \Re_{++}$ and any $X, Y, U, V \in \mathcal{S}$. For ϕ_μ given by (32) with $g \in \mathcal{C}M$, we have that ϕ_μ is Fréchet-differentiable and*

$$\nabla\phi_\mu(X, Y)(U, V) = U - P^T((P(U - V)P^T) \circ C)P, \tag{33}$$

where $P \in \mathcal{O}$ and $\lambda_1, \ldots, \lambda_n \in \Re$ are such that $P^T\mathrm{Diag}[\lambda_1, \ldots, \lambda_n]P = (X - Y)/\mu$, $C \in \mathcal{S}$ and for i, j in the same diagonal block,

$$C_{ij} := \begin{cases} (g(\lambda_i) - g(\lambda_j))/(\lambda_i - \lambda_j) & \text{if } \lambda_i \neq \lambda_j \\ g'(\lambda_i) & \text{if } \lambda_i = \lambda_j. \end{cases} \tag{34}$$

At each step of Chen-Tseng's algorithm with iterate $(X, Y) \in \mathcal{S} \times \mathcal{S}$, we find a solution $(U, V) \in \mathcal{S} \times \mathcal{S}$ satisfying the Newton equation $\nabla H_\mu(X, Y)(U, V) = (R, S)$ for some given $(R, S) \in \mathcal{S} \times \mathcal{S}$. Equivalently, at each step, we solve the following equation systems:

$$\nabla\phi_\mu(X, Y)(U, V) = R, \qquad \nabla F(X)U - V = S.$$

It is shown in [6, Lemmas 6,7] that ∇H_μ is nonsingular if F is monotone and uniformly bounded if F is strongly monotone. We now extend these results to the SDLCP(L, Q) case when L has the Cartesian P-property.

Proposition 9. *Assume that $F(X) = L(X) + Q$ and L has the Cartesian P_0-property. Let ϕ_μ be given by (32) with $g \in \mathcal{C}M$. Then $\nabla H_\mu(Z)$ is nonsingular for all $Z \in \mathcal{S} \times \mathcal{S}$ and $\mu > 0$. If, in addition, L has the Cartesian P-property, we have that*

$$\sup_{\substack{0<\mu \\ Z\in\mathcal{S}\times\mathcal{S}}} |||\nabla H_\mu(Z)^{-1}||| < \infty.$$

Proof. Fix any $X, Y \in \mathcal{S}$ and $\mu \in \Re_{++}$. To show that $\nabla H_\mu(X, Y)$ is invertible, it suffices to prove that zero is the only solution of the linear system

$$\nabla\phi_\mu(X, Y)(U, V) = 0, \quad L(U) - V = 0. \tag{35}$$

From Lemma 4 we have that $\nabla\phi_\mu(X, Y)(U, V)$ is given by (33), where $P \in \mathcal{O}$ and $\lambda_1, \ldots, \lambda_n \in \Re$ are such that $P^T\mathrm{Diag}[\lambda_1, \ldots, \lambda_n]P = (X - Y)/\mu$, and C is given by (34). Define a linear transformation $\mathbf{B} : \mathcal{S} \mapsto \mathcal{S}$ by

$$\mathbf{B}U := P^T((PUP^T) \circ C)P, \tag{36}$$

and let $\mathbf{A} := I - \mathbf{B}$. Then (35) becomes

$$\mathbf{A}U + \mathbf{B}V = 0, \quad L(U) - V = 0.$$

Eliminating V from the above equation, we have

$$\mathbf{A}U + \mathbf{B}L(U) = 0.$$

Observe that $0 < C_{ij} < 1$ for any i, j in the same diagonal block and hence $\mathbf{B}$ is invertible. Thus the above equation is equivalent to

$$\mathbf{B}^{-1}\mathbf{A}U + L(U) = 0. \tag{37}$$

We now prove that zero is the only solution of (37). Let $1/C$ be a matrix in $\mathcal{S}$ such that $(1/C)_{ij} := 1/C_{ij}$ for any i, j in the same diagonal block. Then

$$\mathbf{B}^{-1}U = P^T((PUP^T) \circ \frac{1}{C})P$$

and

$$\mathbf{A}U = P^T((PUP^T) \circ (I - C))P.$$

Consequently,

$$\mathbf{B}^{-1}\mathbf{A}U = P^T((PUP^T) \circ (\frac{1}{C} - E))P,$$

where $E \in \mathcal{S}$ with each of its diagonal blocks being the matrix of all ones. Hence we have

$$PU\mathbf{B}^{-1}\mathbf{A}UP^T = (PUP^T)((PUP^T) \circ (\frac{1}{C} - E)).$$

Noticing $0 < C_{ij} < 1$ for all i, j in the same diagonal block, we have that

$$(PU\mathbf{B}^{-1}\mathbf{A}UP^T)_{ii} = \sum_{j \in \mathcal{I}_i} (PUP^T)_{ij}^2 (1/C_{ij} - 1) \geq 0 \quad \text{for all } i = 1, \ldots, n.$$

where $\mathcal{I}_i$ is the set of indexes which belong to the same diagonal block as index i. Furthermore, $(PU\mathbf{B}^{-1}\mathbf{A}UP^T)_{ii} > 0$, unless the diagonal block, which contains index i, of U is zero. Let $\mathcal{I} = \{i | \text{the } i\text{th diagonal block of } U \text{ is zero}\}$ and $\mathcal{J}$ be the complement set of $\mathcal{I}$ in $\{1, 2, \ldots, m\}$. Then similar to the argument for Proposition 8(ii), we can prove that $U = 0$ is the only solution of (37). Therefore, $\nabla H_\mu(X, Y)$ is invertible.

To prove the uniform boundedness, we fix any $X, Y \in \mathcal{S}$ and $\mu > 0$. It follows from the first part that for any $(R, S) \in \mathcal{S} \times \mathcal{S}$, there is a unique $(U, V) \in \mathcal{S} \times \mathcal{S}$ satisfying the linear equation

$$\mathbf{A}U + \mathbf{B}V = R, \quad L(U) - V = S, \tag{38}$$

where $\mathbf{A} := I - \mathbf{B}$ and $\mathbf{B}$ is defined as in (36). The proof below is motivated by [6, Lemma 6] (note that we are working on diagonal elements of matrices, instead of inner product of matrices in [6]). Multiplying (38) on the left by P and on the right by P^T and letting $\tilde{U} := PUP^T$, $\tilde{V} := PVP^T$, $\tilde{R} := PRP^T$, $\tilde{S} := PSP^T$, and defining $\tilde{L}$ by $\tilde{L}(W) := PL(P^T WP)P^T$ for any $W \in \mathcal{S}$, we have

$$(I - C\circ)\tilde{U} + C \circ \tilde{V} = \tilde{R}, \quad \tilde{L}(\tilde{U}) - \tilde{V} = \tilde{S}.$$

Eliminating $\tilde{V}$ yields

$$(I - C \circ +C \circ \tilde{L})\tilde{U} = \tilde{R} + C \circ \tilde{S}.$$

Recall the definition of $\alpha(L)$,

$$\alpha(L) := \min_{\|X\|=1} \max_{1\le \nu \le m} \langle X_\nu, L_\nu(X)\rangle.$$

Fix any $\rho \in (0, 1)$ satisfying $\rho < \alpha(L)$ so that $\tilde{L} \neq \rho I$. The above linear system can be written as

$$(I - (1-\rho)C\circ)\tilde{U} + C \circ (\tilde{L} - \rho I)\tilde{U} = \tilde{R} + C \circ S.$$

Since $0 < C_{ij} < 1$ for any i, j in the same diagonal block, it is readily seen that $(I - (1-\rho)C\circ)$ is an invertible linear mapping. Then the above system can be written as

$$\tilde{U} + (I - (1-\rho)C\circ)^{-1}C \circ (\tilde{L} - \rho I)\tilde{U} = (I - (1-\rho)C\circ)^{-1}(\tilde{R} + C \circ \tilde{S}).$$

Multiplying both sides of the above equation from righthand side by $(\tilde{L} - \rho I)\tilde{U}$ yields

$$\Delta_1 + \Delta_2 = \Delta_3,$$

where

$$\Delta_1 := \tilde{U}(\tilde{L} - \rho I)\tilde{U}, \quad \Delta_2 := (I - (1-\rho)C\circ)^{-1}C \circ (\tilde{L} - \rho I)\tilde{U}(\tilde{L} - \rho I)\tilde{U},$$

and

$$\Delta_3 := (I - (1-\rho)C\circ)^{-1}(\tilde{R} + C \circ \tilde{S})(\tilde{L} - \rho I)\tilde{U}.$$

It is known [6, P. 440] that the trace of Δ_2 is nonnegative. In fact, by following an argument similar to the proof of Proposition (8), we can show that for any $\upsilon \in \{1, \cdots, m\}$, $\mathrm{trace}([\Delta_2]_\upsilon) \ge 0$.

Using the bound $\|(I - (1-\rho)C\circ)^{-1}(\tilde{R} + C \circ \tilde{S})\| \le \sqrt{2}\|(R, S)\|/\rho$ [6, P. 440] and (4), we have that

$$\begin{aligned}
\max_{1\le \nu \le m} \mathrm{trace}[\Delta_3]_\nu &\le \max_{1\le \nu \le m} \sqrt{n_\upsilon}\|\Delta_3\| \\
&\le \max_{1\le \nu \le m} \sqrt{n_\upsilon}\|(I - (1-\rho)C\circ)^{-1}(\tilde{R} + C \circ \tilde{S})\|\|(\tilde{L} - \rho I)\tilde{U}\| \\
&\le \max_{1\le \nu \le m} \sqrt{n_\upsilon}\sqrt{2}\||\tilde{L} - \rho I\|/\rho\|(R, S)\|\|U\| \\
&\le \max_{1\le \nu \le m} \sqrt{2n_\upsilon}(\||L\|| + \rho)/\rho\|(R, S)\|\|U\|. \qquad (39)
\end{aligned}$$

Also, it follows from the Cartesian P-property of L that

$$\begin{aligned}
\max_{1\le \nu \le m} \mathrm{trace}[\Delta_1]_\nu &= \max_{1\le \nu \le m} \langle \tilde{U}_\nu, ((\tilde{L} - \rho I)\tilde{U})_\nu\rangle \\
&\ge \max_{1\le \nu \le m} \langle \tilde{U}_\nu, \tilde{L}_\nu(\tilde{U})\rangle - \rho\|U\|^2 \\
&\ge \alpha(L)\|U\|^2 - \rho\|U\|^2 \\
&= (\alpha(L) - \rho)\|U\|^2. \qquad (40)
\end{aligned}$$

Noticing the nonnegativeness of all traces of $[\Delta_2]_\nu$, and using the inequalities (39) and (40), we obtain that

$$(\alpha(L)-\rho)\|U\|^2 \le \max_{1\le\nu\le m} \sqrt{2n_\upsilon}(|||L|||+\rho)/\rho\|U\|\|(R,S)\|,$$

and hence

$$\|U\| \le \kappa\|(R,S)\| \qquad \text{where } \kappa := \max_{1\le\nu\le m}\sqrt{2n_\upsilon}\frac{|||L|||+\rho}{\alpha(L)-\rho}.$$

Since $V = L(U) - S$, this yields

$$\begin{aligned}\|(U,V)\|^2 &\le \|U\|^2 + (|||L|||\|U\| + \|S\|)^2\\ &\le \kappa^2\|(R,S)\|^2 + (\kappa|||L|||+1)^2\|(R,S)\|^2.\end{aligned}$$

Therefore

$$|||\nabla H_\mu(Z)^{-1}||| \le \sqrt{\kappa^2 + (\kappa|||L|||+1)^2}.$$

Thus $|||\nabla H_\mu(Z)^{-1}|||$ is uniformly bounded. □

The next issue we want to address is the boundedness of the neighborhood $\mathcal{N}_\beta$ under the condition of the Cartesian P-property. This issue is important because the boundedness of neighborhood ensures the iterates produced by the Chen-Tseng's algorithm remain bounded and hence there exists an accumulative point which is a solution to SDCP(L, Q). One sufficient condition for the boundedness is given below.

A. [6, condition A2] F is a Lipschitz continuous and uniformly R_0-function in the sense that, for any sequence $X^k \in \mathcal{S}$, $k = 1, 2, \ldots$, satisfying

$$\|X^k\| \to \infty, \ \lim_{k\to\infty}\frac{X^k}{\|X^k\|} \in \mathcal{S}_+, \ \text{and} \ \lim_{k\to\infty}\frac{F(X^k)}{\|X^k\|} \in \mathcal{S}_+,$$

we have

$$\liminf_{k\to\infty}\frac{\langle X^k, F(X^k)\rangle}{\|X^k\|} > 0.$$

Proposition 10. *Consider the SDLCP (2). Suppose L has the Cartesian P-property. Then condition (**A**) holds with $F(X) = L(X) + Q$.*

Proof. First we establish the following fact. Suppose $X \in \mathcal{S}_+$ and $L(X) \in \mathcal{S}_+$ with $\|X\| = 1$. Then for any sequence $\{X^k \in \mathcal{S}\}$ converging to X, we have

$$\liminf_{k\to\infty}\langle X^k, L(X^k)\rangle \ge \liminf \max_{1\le\nu\le m}\langle X^k_\nu, L_\nu(X^k)\rangle \ge \alpha(L) > 0. \tag{41}$$

We will see Proposition 10 follows from this fact.

Now apply (41) to the sequence X^k in (**A**), the result follows by noticing that for any $\|X^k\| \to \infty$,

$$\liminf\frac{\langle X^k, F(X^k)\rangle}{\|X^k\|^2} = \liminf\frac{\langle X^k, L(X^k)\rangle}{\|X^k\|^2} \ge \alpha(L).$$

This completes our proof. □

Combining Proposition 10 and [6, Lemma 8] we have the following corollary.

Corollary 6. *Assume $F(X) = L(X) + Q$ and L has the Cartesian P-property. Then for any $\beta > 0$ and $\mu_0 > 0$, the set $\{(Z, \mu) \in \mathcal{N}_\beta | \, 0 < \mu \leq \mu_0\}$ is bounded.*

We conclude this section by remarking that, for the Cartesian P-SDLCP (2), Chen-Tseng's algorithm is globally convergent and the linear convergence rate is achieved (because of Proposition 9 and Corollary 6). Under further standard nondegeneracy condition at the solution, the algorithm also achieves local superlinear convergence (see [6, Propositions 1 and 2].

6. Conclusion

In this paper, we introduce a Cartesian P-property for linear transformations from $\mathcal{S}$ to $\mathcal{S}$ and show that several popular numerical approaches for solving monotone SDLCPs can be extended to solve SDLCPs with this Cartesian P-property. Relations with other P-type properties are also investigated. In particular, we have the following one-way implication of properties for general linear transformations L:

$$\begin{aligned} \text{strong monotonicity} &\Longrightarrow \text{Cartesian } P\text{-property} \Longrightarrow P_2\text{-property} \\ &\Longrightarrow \begin{cases} P\text{-property} \\ \text{GUS-property.} \end{cases} \end{aligned}$$

The reverse implications are not true in general. However, for some special linear transformations, these definitions might be equivalent. For instance, we have the following equivalent relations for the Lyapunov transformation L_A:

$$\begin{aligned} A \text{ is positive definite} &\Longleftrightarrow \text{strong monotonicity} \Longleftrightarrow \text{Cartesian } P\text{-property} \\ &\Longleftrightarrow P_2\text{-property.} \end{aligned}$$

If $(A + A^T)$ is nonsingular, we have that for L_A

$$\text{GUS-property} \Longleftrightarrow P_2\text{-property} \Longleftrightarrow \text{Cartesian } P\text{-property.}$$

Moreover, we show that the Cartesian P-property implies the Lipschitz continuity of the solution map for SDLCP. We demonstrate, by studying the issue of stationary points, that the merit function approach can be extended to solve the Cartesian P-SDLCPs. We also show that a non-interior continuation method can be used to solve this class of problems.

So far, the concept of Cartesian P-property is only defined for linear transformations between the space of symmetric matrices. Its extension to nonlinear transformations between the space of symmetric matrices is straightforward and is presented as follows.

Definition 3. *Given a transformation $F : \mathcal{S} \mapsto \mathcal{S}$, we say that*

(i) F has the Cartesian P-property if for any $X, Y \in \mathcal{S}$ ($X \neq Y$), there exists an index $\nu \in \{1, \ldots, m\}$ such that

$$\langle (X - Y)_\nu, (F(X) - F(Y))_\nu \rangle > 0.$$

(ii) *F has the uniform Cartesian P-property if for any* $X, Y \in \mathcal{S}$ $(X \neq Y)$ *and any* $P \in \mathcal{O}$*, there exists an index* $\nu \in \{1, \ldots, m\}$ *and a positive scalar* ρ *such that*

$$\langle (X - Y)_\nu, (F(X) - F(Y))_\nu \rangle \geq \rho \|X - Y\|^2.$$

(iii) *F has the Cartesian* P_0*-property if for any* $X, Y \in \mathcal{S}$ $(X \neq Y)$*, there exists an index* $\nu \in \{1, \ldots, m\}$ *such that*

$$X_\nu \neq Y_\nu \text{ and } \langle (X - Y)_\nu, (F(X) - F(Y))_\nu \rangle \geq 0.$$

It is easy to show that if F has the uniform Cartesian P-property with a constant ρ, then for any $X, Y \in \mathcal{S}$, there exists an index $\nu \in \{1, \ldots, m\}$ such that $\langle Y_\nu, [\nabla F(X) Y]_\nu \rangle \geq \rho \|Y\|^2$. Furthermore, F has the Cartesian P_0-property, then so does $\nabla F(X)$ for any $X \in \mathcal{S}$. Most importantly, it can be easily verified that the first part of Proposition 9 holds true if F has the Cartesian P_0-property, and the second part of Proposition 9 holds true if F has the uniform Cartesian P-property.

Finally it is appropriate to point out that the concept of Cartesian P-property lies between the concept of strong monotonicity and the concept of P_2-property, and provides a characterization of an important class of tractable nonmonotone SDCPs.

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