Proof of Relation (64), Spectral norm convergence rate for precision matrix

Proof. We follow the argument in Rothman et al (2008). Let \( \hat{\Delta} = \hat{K}_\lambda - K \), \( \Xi = \hat{R} - R \), \( S_u = \{(j,k) : |\omega_{jk}| \geq u, j \neq k \} \), \( S_u^c = \{(j,k) : |\omega_{jk}| < u, j \neq k \} \) and \( W_u = \{(j,k) : |\xi_{jk}| \geq u, j \neq k \} \). Clearly, \( \xi_{jj} = 0 \). Since \( \varepsilon_0 \leq \rho(\Sigma) = \rho(\Omega^{-1}) \leq \varepsilon_0^{-1} \), then for all \( j \), \( \varepsilon_0^{1/2} \leq v_{jj} \leq \varepsilon_0^{-1/2} \). Note that \( K = V\Omega V \) and \( K_{jk} = \omega_{jk}v_{jj}v_{kk} \), we have

\[
|K_{\Sigma_u^c}|_1 = \sum_{j \neq k} |K_{jk}| I(|\omega_{jk}| < u)
\leq \varepsilon_0^{-1} \sum_{j \neq k} |\omega_{jk}| I(|\omega_{jk}| < u)
\leq \varepsilon_0^{-1} p^2 u^{-1} D^-(u).
\]

By the argument of proving Theorem 3.1, we have that

\[
|\hat{\Delta}|^2_F \lesssim |\Xi W_u|^2_F + u^2 S_u + u|K_{\Sigma_u^c}|_1, \quad S_u = |S_u|.
\]

Hence we obtain

\[
\rho(\hat{\Delta})^2 \lesssim |\Xi W_u|^2_F + p^2 D^-(u).
\]

Now, by the argument of proving [RBLZ08, Theorem 2],

\[
\rho(\hat{\Omega}_\lambda - \Omega) \leq \rho(\hat{\Delta})\rho(\hat{V}^{-1})\rho(V^{-1}) + \rho^2(\hat{V}^{-1} - V^{-1})\rho(\hat{\Delta})
+ \rho(\hat{V}^{-1} - V^{-1})(|\rho(\hat{K}_\lambda)|\rho(V^{-1}) + \rho(\hat{V}^{-1})\rho(K)).
\]

(1)

Under \( \max[p^{1/q}n^{-1+1/q} \log(p/n)^{1/2}] \lesssim \lambda \), we have \( \rho(\hat{V}^2 - V^2) = O_p(\lambda) \). Since \( \varepsilon_0 \leq v_{jj} \leq \varepsilon_0^{-1} \) holds for all \( j \), we have \( \rho(\hat{V}^{-1} - V^{-1}) = O_p(\lambda) \). Then the first term on the RHS of (1) is the dominating term for the spectral norm rate of convergence and (64) [numbered in the paper] follows from (56) [numbered in the paper]. \( \Box \)

References