Joint Gaussianity Problems

1. Let X and Y be jG with mean zero, variance one, $\rho \neq 0$.
Find $E[Y^2|X]$.

Note that X and Y^2 are not jG.

Recall that $E[Z^2] = E[Z]^2 + var(Z)$ for a random variable Z.

Try to find the two terms that contribute to the second moment conditioned on $X=u$. We know given $X=u$, $Y \sim N(\rho u, 1-\rho^2)$ so

$$E[Y^2|X=u] = (\rho u)^2 + 1-\rho^2$$

so $E[Y^2|X] = (\rho X)^2 + 1-\rho^2$.

2. Suppose X and Y are standardized jG r.v. w/ $\rho = 0.5$
(a) find $var(3X-2Y)$
(b) find $E[Y|X=3]$.

(a) $var(3X-2Y) = 9var(X) - 12 \underbrace{cov(X,Y)}_{cov(X,Y) = \rho_{XY} \cdot \sigma_X \cdot \sigma_Y} + 4 var(Y) = 9 - 6 + 4 = 7.$
$= 9 var(X) - 12 \rho_{XY}\sigma_X\sigma_Y + 4 var(Y) =$

(b) Since X and Y jointly Gaussian, $E[Y|X=3]$ is LMMSE estimator

$$E[Y|X=3] = \mu_Y + \sigma_Y \rho_{X,Y}\left(\frac{3-\mu_X}{\sigma_X}\right) = 0 + 1\cdot\left(\tfrac{1}{2}\right)\left(\frac{3-0}{1}\right) = \frac{3}{2}.$$

3. Suppose X and Z are zero-mean jG with $\sigma_X^2 = 4$, $\sigma_Z^2 = \frac{17}{9}$, $E[XZ]=2$.
Let $Y = 2X - 3Z$. Find pdf of Y, conditional pdf of X given Y, joint pdf of X,Y.

Y is normal by jG, and clearly zero-mean. Just find variance

$$\sigma_Y^2 = E[(2X-3Z)^2] = 4E[X^2] + 9E[Z^2] - 12E[XZ]$$
$$= 4\cdot 4 + 9\cdot\tfrac{17}{9} - 12\cdot 2 = 9.$$

so $Y \sim N(0,9)$.

Next note that X and Y are jG.

The covariance of X and Y is
$$E[XY] = E[X(2X-3Z)]$$
$$= 2E[X^2] - 3E[XZ]$$
$$= 2\cdot 4 - 3\cdot 2$$
$$= 2.$$

The correlation coefficient is therefore $\rho_{XY} = \dfrac{E[XY]}{\sigma_X\sigma_Y} = \dfrac{2}{2\cdot 3} = \dfrac{1}{3}$.

Conditional expectation of X given Y, i.e. the LMMSE since jG is

$$E[X|Y] = \rho_{XY}\frac{\sigma_X}{\sigma_Y}Y = \frac{1}{3}\cdot\frac{2}{3}Y = \frac{2}{9}Y.$$

The conditional variance of X given Y is error of LMMSE:

$$(1-\rho_{XY}^2)\sigma_X^2 = \left(1-\tfrac{1}{9}\right)4 = \frac{32}{9}.$$

so $\tilde{X} = X|Y$ is $\cancel{N\left(\tfrac{2}{9}Y, \tfrac{32}{9}\right)}$.

$$f_{X|Y} \sim N\left(\tfrac{2}{9}y, \tfrac{32}{9}\right).$$

To finally find f_{XY}, just multiply $f_{XY} = f_X f_{X|Y}$, which yields:

$$f_{XY} = \frac{1}{2\pi\sqrt{32}}\exp\left\{-\frac{\frac{y^2}{9} + \frac{x^2}{4} - \frac{2}{3}\cdot\frac{xy}{2\cdot 3}}{2\left(1-\frac{1}{9}\right)}\right\}.$$