Joint Gaussian Distribution

Gaussians everywhere due to CLT. Lots of little effects add up.

Holds not just for single random variables but for two or more correlated random variables taken as an iid sequences.

$(U_1, V_1), (U_2, V_2), \ldots$ For example height/weight of elephants i seen on safari.

Suppose each is mean-zero for convenience.

Then as $n \to \infty$, $\left(\frac{U_1 + \cdots + U_n}{\sqrt{n}}, \frac{V_1 + \cdots + V_n}{\sqrt{n}} \right)$ has limiting bivariate distribution. By CLT this will be jointly Gaussian (jG).

Def: Random variables X and Y are jG if every linear combination $aX + bY$ is a Gaussian r.v.

$$f_{XY}(u,v) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left\{ - \frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2 + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2 - 2\rho \left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2(1-\rho^2)} \right\}.$$

Simplest setting is independent standard normal r.v.: W, Z

$$f_{W,Z}(\alpha, \beta) = \left(\frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \right) \left(\frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}} \right) = \frac{e^{-\frac{\alpha^2 + \beta^2}{2}}}{2\pi}.$$

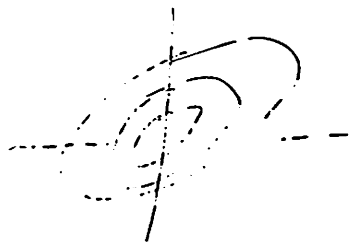
where $\mu_W = \mu_Z = 0$, $\sigma_x = \sigma_y = 1$, $\rho = 0$.

To get back general result:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = A \begin{bmatrix} W \\ Z \end{bmatrix} + \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

where $A = \begin{bmatrix} \sqrt{\frac{\sigma_X^2(1+\rho)}{2}} & -\sqrt{\frac{\sigma_X^2(1-\rho)}{2}} \\ \sqrt{\frac{\sigma_Y^2(1+\rho)}{2}} & \sqrt{\frac{\sigma_Y^2(1-\rho)}{2}} \end{bmatrix}$.

level sets are ellipses



Key properties

Suppose X, Y bivariate normal with $\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho$.

(a) $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$.

(b) any linear combination, $aX+bY$ is Gaussian r.v., i.e. X and Y are jG.

(c) ρ is correlation coeff between X and Y.

(d) $X \perp\!\!\!\perp Y$ iff $\rho = 0$.

(e) For estimation of Y from X, $L^*(X) = g^*(X)$, which is to say, linear estimators are good enough in Gaussian case

(f) Conditional distribution of Y given $X=u$ is $N(L^*(u), \sigma_e^2)$ where σ_e^2 is MSE for LMMSE estimator

To prove, restrict to $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$.

Then $f_{XY}(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-u^2+v^2-2\rho uv}{2(1-\rho^2)} \right\}$

$$= \underbrace{\left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \right]}_{\text{function of } u \text{ alone}} \underbrace{\left[\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(v-\rho u)^2}{2(1-\rho^2)}\right) \right]}_{\text{function of } v \text{ for } u \text{ fixed is } N(\rho u,\, 1-\rho^2).}$$

In particular first factor is $f_X(x)$ and second factor is $f_{Y|X}(y|x)$.

Since given $X=x$, conditional distribution of Y is $\mathcal{N}(\rho x, 1-\rho^2)$,

$g^*(x) = E[Y|X=x] = \rho x$

We can separately compute $L^*(x)$ and see it is also ρx, so ~~[illegible]~~

LMMSE coincides with MMSE estimator.

In higher dimensions

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} \exp\left(-\frac{(x-\mu)^T\Sigma^{-1}(x-\mu)}{2}\right).$$