Suppose a ball is thrown upward at time $t=0$ with initial height X and initial upward velocity Y, such that $X \perp\!\!\!\perp Y$, $\exp(\lambda)$.

The height at time t is $H(t) = X + Yt - \frac{ct^2}{2}$ where c is gravitational constant

Let T be r.v. for time when ball reaches maximum height, M is maximum height.

Find joint pdf of T and M, $f_{T,M}(t,m)$

The max height occurs when $H'(t) = 0$

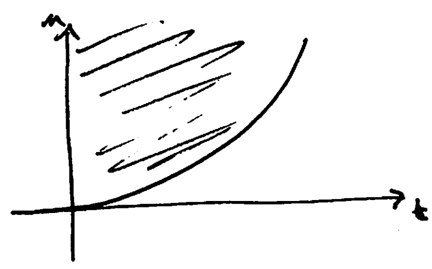
→ Doing that yields $T = Y/c$.

→ Doing that yields $M = X + \frac{Y^2}{2c}$.

The support of f_{XY} is positive quadrant of the plane, where $f_{XY}(x,y) = \lambda^2 e^{-\lambda(x+y)}$.

The support of $f_{T,M}(t,m)$ is determined as follows. Must be that $t = \frac{y}{c}$ for some $y \geq 0$ or $y = tc$. Also $m = x + \frac{y^2}{2c} = x + \frac{t^2 c}{2}$. So for $t \geq 0$ fixed, (t,m) is in support if $m \geq \frac{t^2 c}{2}$.

$f_{T,M}(t,m)$ support is $\{(t,m) : t \geq 0,\ m \geq \frac{t^2 c}{2}\}$



function g is $x = m - \frac{t^2 c}{2}$ and $y = tc$, so

$$J = \begin{bmatrix} 0 & \frac{1}{c} \\ 1 & \frac{y}{c} \end{bmatrix} \quad \text{so} \quad |\det(J)| = \frac{1}{c}$$

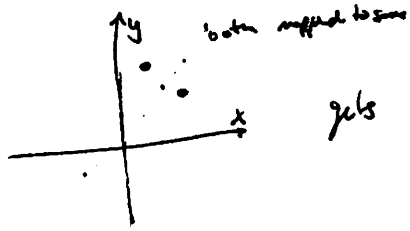
$$f_{T,M}(t,m) = \frac{1}{|\det J|} f_{XY}\left(g^{-1}\left(\begin{bmatrix} t \\ m \end{bmatrix}\right)\right) = \begin{cases} \lambda^2 c \exp\left\{-\lambda\left(m - \frac{t^2 c}{2} + tc\right)\right\}, & t \geq 0,\ m \geq \frac{t^2 c}{2} \\ 0, & \text{else.} \end{cases}$$

Transformations of pdfs under many-to-one mapping

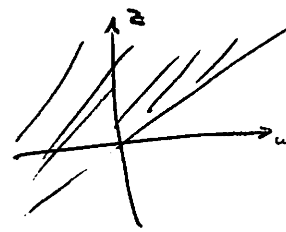
If there are multiple points in (x,y) plane that map to a single point in the (w,z) plane, we have to sum over all those points to get f_{ZW} from f_{XY}.

Example Suppose $W=\min\{X,Y\}$ and $Z=\max\{X,Y\}$.
Express $f_{W,Z}$ in terms of f_{XY}.

Notice that



gets mapped to



Let $U=\{(w,z): w<z\}$, the region above the diagonal.

For any subset $A\subset U$, $\{(W,Z)\in A\} = \{(X,Y)\in A\}\cup\{(Y,X)\in A\}$ by the symmetry of the problem, and where the two sets in the union are disjoint.

$$P\{(W,Z)\in A\} = P\{(X,Y)\in A\} + P\{(Y,X)\in A\}$$
$$= \iint_A f_{XY}(u,v) + f_{Y,X}(u,v)\,du\,dv$$
$$= \iint_A f_{XY}(u,v) + f_{X,Y}(v,u)\,du\,dv$$

So joint pdf is just

$$f_{W,Z}(w,z) = \begin{cases} f_{XY}(w,z) + f_{X,Y}(z,w), & w<z \\ 0, & \text{else.} \end{cases}$$

"fold plane at diagonal and add"