Suppose a ball is thrown upward at time $t=0$ with initial height $X$ and initial upward velocity $Y$, such that $X \sim N(0, \frac{c^2}{2})$. The height at time $t$ is $H(t) = X + Yt - \frac{ct^2}{2}$, where $c$ is gravitational constant.

Let $T$ be r.v. for time when ball reaches maximum height, $M$ is maximum height.

Find joint pdf of $T$ and $M$, $f_{T,M}(t,m)$

The max height occurs when $H'(t) = 0$

$\Rightarrow$ Doing that yields $T = \frac{Y}{c}$.

$\Rightarrow$ Doing that yields $M = X + \frac{Y^2}{2c}$.

The support of $f_{xy}$ is positive quadrant of the plane, when $f_{xy}(xy) = e^{-xy}$.

The support of $f_{T,M}(t,m)$ is determined as follows. Must be that $t = \frac{y}{c}$ for some $y \geq 0$ or $y < 0$. Also $m = x + \frac{y^2}{2c} = x + \frac{t^2c}{2}$. So for $t \geq 0$ fixed, $(t,m)$ is in support if $m \geq \frac{t^2c}{2}$.

$f_{T,M}(t,m)$ support is $\mathcal{F}(t,m)$: $t \geq 0$, $m \geq \frac{t^2c}{2}$

The function $y$ is $y = m - \frac{t^2c}{2}$ and $y = tc$, so

$J = \begin{bmatrix} 0 & \frac{c}{t} \\ 1 & \frac{c}{t} \end{bmatrix}$

$\Rightarrow$ $\det(J) = \frac{c^2}{t^2}$

$f_{T,M}(t,m) = \frac{1}{\det(J)} f_{xy}(\phi^{-1}(\begin{bmatrix} t \\ m \end{bmatrix}))$, where $\phi(x,y) = \begin{bmatrix} \frac{t^2c}{2} + \frac{c}{t} \\ x \end{bmatrix}$

$\Rightarrow f_{T,M}(t,m) = \left\{ \begin{array}{ll}
\frac{c^2 e^{-\lambda\left(m - \frac{t^2c}{2} + tc\right)}}, & t \geq 0, m \geq \frac{t^2c}{2} \\
0, & \text{else}
\end{array} \right.$
Transformations of pdfs over many-to-one mappings

If there are multiple points in \((x,y)\) plane that map to a single point in the \((u,v)\) plane, we have to sum over all these points to get \(f_{uv}\) from \(f_{xy}\).

Example

Suppose \(W = \min \{x,y\}\) and \(Z = \max \{x,y\}\).

Express \(f_{wz}\) in terms of \(f_{xy}\).

Notice that

Let \(U = \{(w,z) : w < z\}\) be the region above the diagonal.

For any subset \(A\subseteq U\), \(\mathbb{P}(w,z) \in A) = \mathbb{P}(x,y) \in A) \cup \mathbb{P}(y,x) \in A)\) be the symmetric of the problem, and when the two sets in the union are disjoint,

\[
\mathbb{P}(w,z) \in A) = \mathbb{P}(x,y) \in A) + \mathbb{P}(y,x) \in A)
\]

So joint pdf is just

\[
f_{w,z}(w,z) = \begin{cases} f_{xy}(w,z) + f_{xy}(z,w), & w < z \\ 0, & \text{else} \end{cases}
\]

"Fold plane at diagonal and add."