One last problem

Observations $X_1, \ldots, X_T$ from altimeter are of the form $X_t = bt + W_t$ where b is rate of ascent of the drone. ($b<0$ is descent), and $W_1, \ldots, W_T$ are independent observation noise $\sim N(0,1)$.

Obtain the ML estimator for b given observations $u_1, \ldots, u_T$. Estimator is unbiased if mean of estimator equals parameter being estimated; determine if MLE is unbiased.

For each t, $X_t \sim N(bt, 1)$, so $f_{X_t}(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(u_t - bt)^2}{2}}$.

Joint is just product, so

$$f_{X_1, X_2, \ldots, X_T}(u_1, \ldots, u_T) = \prod_{t=1}^{T} f_{X_t}(u_t) = \frac{1}{(2\pi)^{T/2}} \exp\left\{ -\sum_{t=1}^{T} \frac{(u_t - bt)^2}{2} \right\}.$$

The joint pdf is the likelihood of observations for given parameter value b.

The estimator $\hat{b}_{ML}$ is value that maximizes likelihood or equivalently minimizes $\sum_{t=1}^{T} \frac{(u_t - bt)^2}{2}$:

$$\frac{d}{db} \sum_{t=1}^{T} \frac{(u_t - bt)^2}{2} = \sum_{t=1}^{T} (u_t - bt)(-t) = b\sum_{t=1}^{T} t^2 - \sum_{t=1}^{T} u_t t.$$

Setting equal to zero:

$$\hat{b}_{ML} = \sum_{t=1}^{T} u_t t \Big/ \sum_{t=1}^{T} b^2.$$

Since $E[X_t] = bt$ for all t, we find $E[\hat{b}_{ML}] = b$ so estimator is unbiased.

Joint pdfs of functions of random variables

Last time we saw some examples of this, through problems. Now let us consider the general case.

Linear mapping

W, Z are new random variables that are both linear functions of X and Y, with f_{XY}.

In particular $W = aX + bY$ and $Z = cX + dY$, so

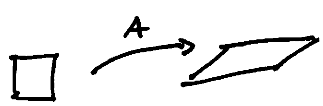
$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix}.$$

The determinant of A is $\det(A) = ad - bc$. If $\det A \neq 0$, then there is an inverse mapping

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

An important property of linear mappings is that if $\mathcal{R}$ is a set in the space corresponding to (x,y) and $\mathcal{S}$ is its image, then

$\text{area}(\mathcal{S}) = |\det A| \, \text{area}(\mathcal{R})$.



Prop: Suppose $\begin{bmatrix} W \\ Z \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix}$ where $(X,Y) \sim f_{XY}$, and A is matrix with $\det A \neq 0$.

Then
$$f_{W,Z}(w,z) = \frac{1}{|\det A|} f_{XY}\left(A^{-1} \begin{bmatrix} w \\ z \end{bmatrix}\right).$$

Example Suppose $W = X - Y$ and $Z = X + Y$. Express $f_{W,Z}$ in terms of f_{XY}.

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, so $\det A = 2$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

So $f_{WZ}(w,z) = \frac{1}{2} f_{XY}\left(\frac{w+z}{2}, -\frac{w+z}{2}\right)$.

$W = X+Y$, $Z = Y$; find joint pdf ~~f_{WZ}~~ and marginal f_W.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \det A = 1 \qquad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

So $f_{WZ}(w,z) = f_{XY}(w-z, z) = f_X(w-z) f_Y(z)$.

to get marginal, integrate out z:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(w-z)\, f_Y(z)\, dz.$$

The convolution.

For one-to-one nonlinear mappings.

Prop: Suppose $\begin{bmatrix} W \\ Z \end{bmatrix} = g\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right)$ where $\begin{bmatrix} X \\ Y \end{bmatrix} \sim f_{XY}$ and g is a one-to-one mapping.

Suppose Jacobian J of g exists, is continuous, and has nonzero determinant everywhere.

Then $\begin{bmatrix} W \\ Z \end{bmatrix}$ has joint pdf:

$$f_{WZ}(w,z) = \frac{1}{|\det J|} f_{XY}\left(g^{-1}\left(\begin{bmatrix} w \\ z \end{bmatrix}\right)\right).$$

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ex. Let X, Y be independent exponential random variables with common parameter λ. Define $U = X+Y$ and $V = X-Y$. Find joint/marginal pdfs of U and V.

Given that $f_{XY}(x,y) = \frac{1}{\lambda^2} e^{-(x+y)/\lambda}$, $x \geq 0$, $y \geq 0$.

Since $u = x+y$ and $v = x-y$, it is always true that $|v| < u$, there is only one solution given by

$$x = \frac{u+v}{2} \qquad y = \frac{u-v}{2}.$$

So Jacobian of transformation is

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so $f_{UV}(u,v) = \frac{1}{2\lambda^2} e^{-u/\lambda}$, $0 < |v| < u < \infty$.

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example

Let $X \sim \mathcal{U}(0,1)$, $Y \sim \mathcal{U}(0,1)$, $X \perp\!\!\!\perp Y$.

Let $Z = (-2 \ln X)^{1/2} \cos(2\pi Y)$.

Find $f_Z(z)$.

Use auxilliary variable $W = Y$:

$x_1 = e^{-[z \sec(2\pi w)]^2/2}$

$y_1 = w$.

$$J(z,w) = \begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} -z \sec^2(2\pi w) e^{-[z \sec(2\pi w)]^2/2} & \frac{\partial x_1}{\partial w} \\ 0 & 1 \end{vmatrix}$$

$$= -z \sec^2(2\pi w) e^{-[z \sec(2\pi w)]^2/2}.$$

so applying formula:

$$f_{ZW}(z,w) = z \sec^2(2\pi w) e^{-[z \sec(2\pi w)]^2/2}, \quad -\infty < z < +\infty, \quad 0 < w < 1.$$

and $f_Z(z) = \int_0^1 f_{ZW}(z,w)\, dw = e^{-z^2/2} \int_0^1 z \sec^2(2\pi w) e^{-[z \tan(2\pi w)]^2/2}\, dw$

Let $u = z \tan(2\pi w)$ so $du = 2\pi z \sec^2(2\pi w)\, dw$ and note that as w varies from 0 to 1, u varies from $-\infty$ to $+\infty$.

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \underline{\int_{-\infty}^{\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

$Z \sim N(0,1)$.