The speed of a typical vehicle that drives past a police radar is modeled as an exponential random variable X with mean 50 mph. The police's radar measurement of speed, Y, has error that is normal with zero mean and std that is 1/10 vehicle's speed. What is joint pdf of X, Y?

$f_X(x) = \left(\frac{1}{50}\right) e^{-x/50}$, for $x \geq 0$. Also conditioned on $X=x$, the measurement Y has $N\left(x, \frac{x^2}{100}\right)$. So

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,(x/10)} e^{-(y-x)^2/(2x^2/100)}$$

so $$f_{XY}(x,y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{50} e^{-x/50} \frac{10}{\sqrt{2\pi}\, x} e^{-50(y-x)^2/x^2}$$

Let X, Y be independent standard normal random variables, and think about them in polar coordinates $R \geq 0$, $\Theta \in [0, 2\pi]$

$$X = R\cos\Theta, \quad Y = R\sin\Theta.$$

Show that Θ is uniform in $[0, 2\pi]$, R is Rayleigh with pdf:

$$f_R(r) = r e^{-r^2/2}, \quad r \geq 0$$

and $R \perp \Theta$.

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$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

Consider subset A of two-dimensional plane, let B be representation of same subset in polar coordinates:

$$(x,y) \in A \quad \text{iff} \quad (r,\theta) \in B, \quad \text{where } x = r\cos\theta, \; y = r\sin\theta.$$

so

$$\Pr[(R,\Theta) \in B] = \Pr[(X,Y) \in A] = \frac{1}{2\pi} \iint_{(x,y)\in A} e^{-(x^2+y^2)/2}\, dx\, dy = \frac{1}{2\pi} \iint_{(r,\theta)\in B} e^{-r^2/2}\, r\, dr\, d\theta.$$

Comparing to

$$P[(R,\Theta) \in B] = \iint_{(r,\theta)\in B} f_{R,\Theta}(r,\theta)\, dr\, d\theta,$$

we see $f_{R,\Theta}(r,\theta) = \frac{r}{2\pi} e^{-r^2/2}$, $r \geq 0$, $\theta \in [0, 2\pi]$

we get:

$$\left.\begin{aligned} f_R(r) &= \int_0^{2\pi} \frac{r}{2\pi} e^{-r^2/2}\, d\theta = r e^{-r^2/2}, \quad r \geq 0 \\ f_\Theta(\theta) &= \int_0^\infty \frac{r}{2\pi} e^{-r^2/2}\, dr = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi] \end{aligned}\right\} \text{as desired.}$$

Also $f_{R\Theta} = f_R f_\Theta$ so independent.