$\triangle$

Sums of random variables

We know by linearity, $E[X+Y] = E[X] + E[Y]$ but what about the full distributional form of $X+Y$?

This problem often arises when considering additive measurement noise:

$$\begin{tikzcd} S \arrow[r] & \oplus \arrow[r] & Z \\ & N \arrow[u] & \end{tikzcd}$$

$Z = S + N$, signal plus noise.

$\boxed{\text{Sum of two dice}}$

Sums of integer-valued random variables:

Let $S = X+Y$, want p_S in terms of p_X and p_Y.

For fixed value k, possible ways to get $S=k$ can be indexed according to the value of X. For ~~S~~ $S=k$ to happen, have to have $X=j$ and $Y=k-j$, for some j.

By total probability,

$$\begin{aligned} p_S(k) &= P[X+Y=k] \\ &= \sum_j P[X=j, Y=k-j] \\ &= \sum_j p_{XY}(j, k-j). \end{aligned}$$

If $X \perp\!\!\!\perp Y$, then $p_{XY}(j,k-j) = p_X(j)\,p_Y(k-j)$ and so

$$p_S(k) = \sum_j p_X(j)\, p_Y(k-j)$$ which is the convolution operation.

$$p_S(k) = p_X * p_Y.$$

Example: Let $X \perp\!\!\!\perp Y$, with

$$p_X(x) = \begin{cases} 1/3, & x = 1,2,3 \\ 0, & \text{else} \end{cases} \qquad p_Y(y) = \begin{cases} 1/2, & x=0 \\ 1/3, & x=1 \\ 1/7, & x=2 \\ 0, & \text{else} \end{cases}$$

To calculate pmf of $S = X+Y$, note that range of possible values of S are integers $1, 2, \ldots, 5$, so $p_S(s) = 0$ for $s \neq 1, 2, \ldots, 5$.

For other values, let us calculate, one-by-one:

$$p_S(1) = \sum_x p_X(x) p_Y(1-x) = p_X(1) p_Y(0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.$$

$$p_S(2) = p_X(1) p_Y(1) + p_X(2) p_Y(0) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{18}$$

$$p_S(3) = p_X(1) p_Y(2) + p_X(2) p_Y(1) + p_X(3) p_Y(0) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

$$p_S(4) = p_X(2) p_Y(2) + p_X(3) p_Y(1) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{6}$$

$$p_S(5) = p_X(3) p_Y(2) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}.$$

2. Suppose X has binomial distribution with parameters m and p.
Y — n, p.

$X \perp\!\!\!\perp Y$, find distribution of $X+Y$.

Recall X is # heads in m tosses, Y is # heads in n tosses.

Thus $S = X+Y$ is # heads in $m+n$ tosses, i.e. binomial with $(m+n, p)$.

3. $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$, $X \perp\!\!\!\perp Y$, find distribution of $S = X+Y$

Since Poisson ≈ binomial, expect should be Poisson with $\lambda_1 + \lambda_2$.

Do formally using convolution:

$$p_S(k) = \sum_{j=0}^{k} \left(\frac{\lambda_1^j e^{-\lambda_1}}{j!} \right) \left(\frac{\lambda_2^{k-j} e^{-\lambda_2}}{(k-j)!} \right) = \left(\sum_{j=0}^{k} \frac{\lambda_1^j \lambda_2^{k-j}}{j!\,(k-j)!} \right) e^{-(\lambda_1+\lambda_2)}$$

$$= \left(\sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} \right) \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{where } \lambda = \lambda_1 + \lambda_2 \text{ and } p = \frac{\lambda_1}{\lambda}$$

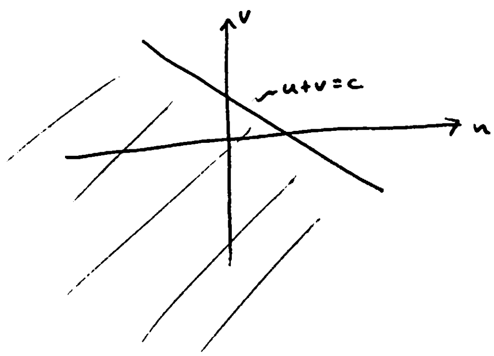
$$= \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{since } \sum \text{ is binomial sum.}$$

Continuous random variables.

$S = X+Y$, want to express f_S in terms of f_{XY}.

- First find cdf of S, then differentiate to get pdf.

- For any $c \in \mathbb{R}$, $\{S \le c\}$ is same as point (x,y) falls in lower diagonal:



Integrating over a region is by picking one dimension, then next:

$$F_S(c) = \Pr[S \le c] = \int_{-\infty}^{\infty} \int_{-\infty}^{c-u} f_{XY}(u,v)\,dv\,du$$

taking derivative:

$$f_S(c) = \frac{dF_S(c)}{dc} = \int_{-\infty}^{\infty} \frac{d}{dc}\left(\int_{-\infty}^{c-u} f_{XY}(u,v)\,dv \right) du$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(u, c-u)\,du.$$

Now since $X \perp\!\!\!\perp Y$, recover convolution:

$$= \int_{-\infty}^{\infty} f_X(u)\, f_Y(c-u)\,du$$

$$f_S = f_X * f_Y.$$

3 Consider $X \sim N(0,\sigma^2)$, $Y \sim N(0,\sigma^2)$, $X \perp\!\!\!\perp Y$

find pdf of $S = X+Y$.

$$f_S = \int_{-\infty}^{\infty} f_X(u)\, f_Y(c-u)\, du$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(c-u)^2}{2\sigma^2}} du$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2} - \frac{(c-u)^2}{2\sigma^2}} du$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left\{\frac{1}{2\sigma^2}\left(-u^2 - c^2 + 2cu - u^2\right)\right\} du$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left\{\frac{1}{2\sigma^2}\left(\quad\right)\right\} \exp\left\{\frac{1}{2\sigma^2}\left(\quad\right)\right\} du.$$

want pieces such that they go away, ie. integrate to 1, since known Gaussian pdfs.

to do so, use "completing the square: see notes.

eventually get

$$f_S(c) = \frac{1}{\gamma\sqrt{2\pi}} e^{-\frac{c^2}{2\gamma^2}} \qquad \text{where } \gamma^2 = 2\sigma^2.$$

Sum of two Gaussians is also Gaussian, with variance as sum of variances.

α-stable distributions are not so common.