Independence of Random Variables

Recall two events are independent if $P(AB) = P(A)P(B)$. [unlinked either physically or by belief update].

events A, B, C are independent if pairwise independent and also $P(ABC) = P(A)P(B)P(C)$

Def: random variables X and Y are independent if any pair of events of the form $\{X \in A\}$ and $\{Y \in B\}$ are independent. That is:

$$P\{X \in A,\ Y \in B\} = P\{X \in A\} P(Y \in B).$$

Taking A and B as semi-infinite intervals, $A = \{u : u \le u_0\}$ and $B = \{v : v \le v_0\}$ shows that if $A \perp\!\!\!\perp B$, the cdf factors:

$$F_{XY}(u_0, v_0) = F_X(u_0) F_Y(v_0).$$

This is actually an iff statement on cdf factorization $\Leftrightarrow$ independence.

Same factorization result holds for pmfs, pdfs:

$$p_{XY}(u,v) = p_X(u)\, p_Y(v) \quad \text{for pmf}$$
$$f_{XY}(u,v) = f_X(u)\, f_Y(v) \quad \text{for pdf.}$$

For several r.v.s:

$$F_{XYZ}(u,v,w) = F_X(u) F_Y(v) F_Z(w)$$
$$p_{XYZ}(u,v,w) = p_X(u)\, p_Y(v)\, p_Z(w)$$
$$f_{XYZ}(u,v,w) = f_X(u) f_Y(v) f_Z(w).$$

(all inclusions / exclusions are already taken care of).

The most commonly encountered joint experiment of independent variates is independent and identically distributed (iid): mutually independent and having common cdf.

For independent X, Y, show that

$E[XY] = E[X]E[Y]$:

$$E[XY] = \sum_x \sum_y xy\, p_{XY}(x,y)$$

$$= \sum_x \sum_y xy\, p_X(x) p_Y(y) \quad \text{by independence}$$

$$= \sum_x x p_X(x) \sum_y y\, p_Y(y)$$

$$= E[X]\, E[Y].$$

similarly, $E[g(X)h(Y)] = E[g(X)]\,E[h(Y)]$ for any functions g and h.

Consider zero-mean independent random variables, $X = \tilde{X} - E[\tilde{X}]$, $Y = \tilde{Y} - E[\tilde{Y}]$.

Then $\operatorname{var}(X+Y) = E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$

$$= E[X^2] + 2E[XY] + E[Y^2]$$

$$= E[X^2] + 2\cancel{E[X]E[Y]}^{\,0} + E[Y^2]$$

$$= \operatorname{var}(X) + \operatorname{var}(Y).$$

<u>How to determine whether independence holds for a given joint pdf</u>.

If $X \perp\!\!\!\perp Y$, then $f_{XY}(x,y) = f_X(x) f_Y(y)$.

Prop: $X \perp\!\!\!\perp Y$ iff: for all $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ for all $v \in \mathbb{R}$.

<u>proof</u> ($\Rightarrow$). If condition holds, then for any $(u,v) \in \mathbb{R}^2$ there are two possible cases

(i) $f_X(u) = 0$ so $f_{XY}(u,v) = 0$, so $f_{XY}(u,v) = f_X(u) f_Y(v)$.

(ii) $f_X(u) > 0$ so $f_{XY}(u,v) = f_X(u) f_{Y|X}(v|u) = f_X(u) f_Y(v)$.

Since factorization true for all $(u,v) \in \mathbb{R}^2$, X and Y are independent.

(Notice the conditional probability being same as unconditional probability step)

(*) if $X \perp\!\!\!\perp Y$, then $f_{XY}(u,v) = f_X(u) f_Y(v)$ for all $(u,v) \in \mathbb{R}^2$, so if $f_X(u) > 0$, then $f_{Y|X}(v|u) = \frac{f_{XY}(u,v)}{f_X(u)} = f_Y(v)$ for all $v \in \mathbb{R}$.

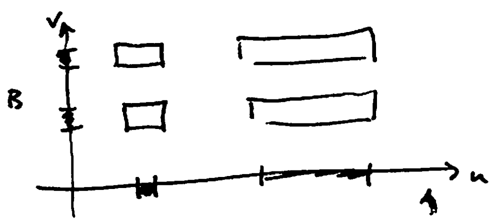
Define <u>product sets</u>.

Suppose A, B are each sets of disjoint line segments.

Let $|A|$ be total length of all line segments in A, likewise for $|B|$.

The product set $A \times B = \{(u,v) : u \in A, v \in B\}$.

The total area of $A \times B$, $|A \times B|$ is $|A| \times |B|$.



A subset S in $\mathbb{R}^2$ has <u>swap property</u> if for any two points $(a,b) \in S$ and $(c,d) \in S$, the points (a,d) and (c,b) also in S.

<u>Prop</u>: Let $S \in \mathbb{R}^2$, Then S is a product set iff it has the swap property.

Necessary condition for independence:

<u>Prop</u>: If $X \perp\!\!\!\perp Y$, jointly continuous r.v., then support of f_{XY} is a product set.

<u>proof</u>: by assumption $f_{XY}(u,v) = f_X(u) f_Y(v)$. So if $f_{XY}(a,b) > 0$ and $f_{XY}(c,d) > 0$ it must be that $f_X(a)$, $f_Y(b)$, $f_X(c)$, and $f_Y(d)$ are all strictly positive.

So $f_{XY}(a,d) > 0$ and $f_{XY}(c,b) > 0$. That is support of f_{XY} has swap property and is therefore a product set.

Corollary: (necessary and sufficient in special case).

Suppose (X,Y) uniformly distributed over a set S in the plane.

Then X and Y are independent iff S is a product set.

Example: Decide whether X and Y are independent for

$$f_{XY}(u,v) = \begin{cases} 9u^2v^2, & u,v \in [0,1]. \\ 0, & \text{else}. \end{cases}$$

Yes: $f_{X,Y}(u,v) = f_X(u) f_Y(v)$, where $f_X(u) = f_Y(u) = \begin{cases} 3u^2, & u \in [0,1] \\ 0, & \text{else}. \end{cases}$

Before next class, review convolution and Fourier transforms.