Probability Integral transformation

Use $X \sim \mathcal{U}(0,1)$ with realization $X = x$ to select a random value of desired cdf $\sim G(\cdot)$ $x = F_Y(y)$.

Note X only takes values that can be taken by cdf.

Then determine $Y = y$ from inverse cdf $y = F_Y^{-1}(x)$.

To verify this works

$$F_Y(y) = P(Y = G^{-1}(U) \leq y) = P(\{U : G^{-1}(U) \leq y\})$$

Let $A = \{U : G^{-1}(U) \leq y\}$ denote event that $F_Y(y) = P(A)$.

A can be re-represented as

$$B = \{U : U \leq G(y)\}.$$

So $F_Y(y) = P(A) = P(B) = P(\{U : U \leq G(y)\})$

recalling $U \sim \mathcal{U}(0,1)$ yields

$$F_Y(y) = P(\{U : U \leq G(y)\}) = G(y).$$

Multivariate distributions and densities

$\Omega \subset \mathbb{R}^n$ for $n > 1$.

A random vector $\vec{X} = (X_1, \ldots, X_n)$ taking values $\vec{x} \in \mathbb{R}^n$.

- for example various node voltages in a circuit
- state variables in a dynamical system

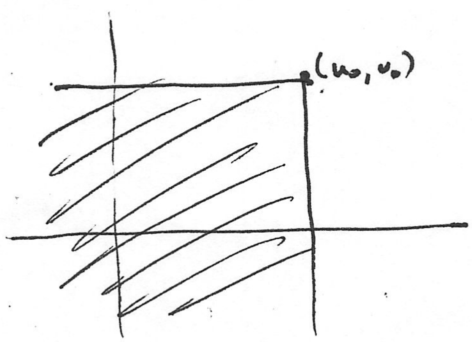
Joint distribution of multiple random variables

Let X, Y be random variables on single probability space $(\Omega, \mathcal{F}, P)$.

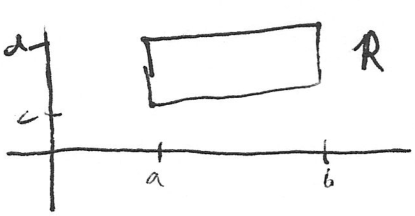
is the function of two variables defined by:

$$F_{X,Y}(u_0, v_0) = Pr[X \le u_0, Y \le v_0].$$

for any $(u_0, v_0) \in \mathbb{R}^2$



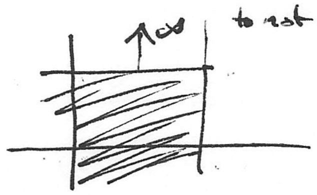
Discussion of why: Kolmogorov.



$P((X,Y) \in R) = F_{XY}(b,d) - F_{XY}(b,c) - F_{XY}(a,d) + F_{XY}(a,c)$

Marginalizing out from joint:

Prop: Suppose X, Y with F_{XY}. For each u fixed, $\lim_{v \to \infty} F_{XY}(u,v)$ exists. Call it $F_{XY}(u,\infty)$. Then $F_X(u) = F_{XY}(u,\infty)$. Also $F_Y(v) = F_{XY}(\infty, v)$.



to not care.

__Properties of cdfs.__

① $0 \le F(u,v) \le 1$ for all $(u,v) \in \mathbb{R}^2$

② $F(u,v)$ nondecreasing in u, nondecreasing in v

③ $F(u,v)$ right-continuous in u, right-continuous in v.

④ $\lim_{u \to -\infty} F(u,v) = 0$ for each v, $\lim_{v \to -\infty} F(u,v) = 0$ for each u.

⑤ $\lim_{u \to \infty} \lim_{v \to \infty} F(u,v) = 1$.

⑥ If $a < b$, $c < d$, then $F(b,d) - F(b,c) - F(a,d) + F(a,c) \ge 0$.

This is iff-type statement.

__Joint pmf:__

$p_{XY}(u,v)$:

	$Y=1$	$Y=2$	$Y=3$
$X=1$	0.1		0.1
$X=2$	0.2	0.2	
$X=3$	0.05	0.3	0.05

not symmetric about diagonal, so not iid.

Marginalizing by summing out: $p_Y(v) = \sum p_{XY}(u_i, v)$.

Properties of pmfs

① p is nonnegative

② $\sum \sum p = 1$

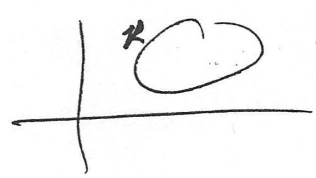
③ There are finite or countably infinite sets $\{u_1, \dots\}$ and $\{v_1 \dots\}$ such that $p(u,v) = 0$ if $u \notin \{u_1, u_2, \dots\}$ or if $v \notin \{v_1, v_2, \dots\}$.

Joint pdf:

There is function f_{XY} called joint pdf such that $F_{XY}(u_0, v_0)$ obtained for any $(u_0, v_0) \in \mathbb{R}^2$ by integration over corner:

$$F_{XY}(u_0, v_0) = \int_{-\infty}^{u_0} \int_{-\infty}^{v_0} f_{XY}(u,v)\, dv\, du$$

For any shape R [with piecewise differentiable boundary]



$$P((X,Y) \in R) = \iint_R f_{XY}(u,v)\, du\, dv.$$