Gaussian noise

- thermal noise in resistors and all physical systems with dissipative capacitance

\[ V \sim N(0, \sigma^2) \quad \text{with} \quad \sigma^2 = 4kT R B \]

- \( B \) is bandwidth in Hz
- \( V \) is voltage across resistor \( R \) [ohms] at temperature \( T \) [K]
- \( k \) is Boltzmann's constant, \( 1.38 \times 10^{-23} \) [J/K]

- 3K universal background radiation
- 290K radiation from earth as seen from space
- shot noise from random arrivals of individual photons/electrons
- low-frequency noise in amplifiers, crystal oscillators, etc.
- many kinds of measured error
- variability in manufactured components
- variability in biological organisms (intelligence, etc.)
- large-scale systems composed of many loosely interlocking components

\[ \Rightarrow \text{central limit theorem.} \]
Let us introduce Gaussian distribution and come back to why next lecture.

\[ f_x(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \]

often denoted \( \mathcal{N}(\mu, \sigma^2) \) for normal distribution.

peak height is \( \frac{1}{\sqrt{2\pi}\sigma^2} \).

Standard normal has \( \mu = 0, \sigma = 1 \) so the cdf of standard normal often denoted

\[ \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy. \]

The survival function often denoted as

\[ \phi(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy = 1 - \Phi(x) = \Phi(-x). \]

Let us confirm standard normal integrates to 1 by considering \( I = \int_{-\infty}^{\infty} e^{x^2/2} dx \).

In polar coordinates:

\[
I^2 = \int_{0}^{\infty} \int_{0}^{\pi} e^{-r^2/2} r \, dr \, d\theta \cdot \int_{-\infty}^{\infty} e^{-x^2/2} dx
\]

\[ = \int_{0}^{\infty} \int_{0}^{\pi} e^{-(r^2/2) + (x^2/2)} r \, dr \, d\theta \]

\[ = \int_{0}^{\infty} \int_{0}^{\pi} r \, dr \, d\theta \cdot -2\pi e^{-r^2/2} \bigg|_{r=0}^{r=\infty} = -2\pi e^{-r^2/2} \bigg|_{r=0}^{r=\infty} = 2\pi. \]

so \( I = \sqrt{2\pi} \) and the claim is true.
Can also find the second moment

\[ E[X^2] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \, du \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot u \cdot u \exp\left(-\frac{u^2}{2}\right) \, du \]

\[ = -\frac{u}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \, du = 0 + 1 = 1. \]

Since mean zero, this is also \( \sigma^2 \), as we desired.

**Example** \( X \sim \mathcal{N}(10, 16) \). Find \( P(X \geq 15) \), \( P(X \leq 5) \), \( P(X^2 > 400) \)

Use fact \( \frac{X - 10}{4} \) is standard normal to use \( \Phi \) or \( Q \) function:

\[ P(X \geq 15) = P\left[ \frac{X - 10}{4} \geq \frac{15 - 10}{4} \right] = Q\left(\frac{15 - 10}{4}\right) = Q\left(\frac{5}{2}\right) \approx 1 - \Phi\left(\frac{5}{2}\right) \approx 0.1056 \]

\[ P(X \leq 5) = P\left[ \frac{X - 10}{4} \leq \frac{5 - 10}{4} \right] = \Phi\left(\frac{5 - 10}{4}\right) = \Phi\left(-\frac{5}{2}\right) = Q\left(\frac{5}{2}\right) = 0.1056 \]

\[ P(X^2 > 400) = P(X > 20) + P(X < -20) = P\left(\frac{X - 10}{4} > 2.5\right) + P\left(\frac{X - 10}{4} < -2.5\right) \]

\[ = Q(2.5) + Q(-2.5) = Q(2.5) = 1 - \Phi(2.5) = 0.0062. \]

Normality is preserved by linear transformation.

\( X \sim \mathcal{N}(\mu, \sigma^2) \), then \( Y = aX + b \) is also normal, \( E[Y] = a\mu + b \), \( \text{var}(Y) = a^2 \sigma^2 \).
Central Limit Theorem.

If many independent random variables add together, and each is small in magnitude compared to sum, sum is approximately Gaussian.

If $X$ is sum, $\bar{X}$ is Gaussian random variable with same mean/median,
then $X$ and $\bar{X}$ have approximately same CDF:

$$P(X \leq V) \approx P(\bar{X} \leq V).$$

An important special case is when $X$ is sum of $n$ Bernoulli r.v. each having same parameter $p$. That is $X \sim \text{Bin}(n, p)$.

[matlab]

Let $S_{n,p}$ be binomial r.v. with parameters $n, p$, so mean is $np$, variance is $np(1-p)$. So standardized sum is

$$\frac{S_{n,p} - np}{\sqrt{np(1-p)}}.$$

DeMoivre-Laplace Limit Theorem

Suppose $S_{n,p}$ with $p$ fixed, $0 < p < 1$, and any constant $c$.

$$\lim_{n \to \infty} P \left( \frac{S_{n,p} - np}{\sqrt{np(1-p)}} \leq c \right) = \Phi(c).$$

To actually do approximations using Gaussian for integer-valued r.v., we use a continuity correction:

$$P(X \leq k) \approx P(\bar{X} \leq k + 0.5)$$
$$P(X \geq k) \approx P(\bar{X} \geq k - 0.5)$$