Gaussian noise

- thermal noise in resistors and all physical systems with dissipative component

$V \sim N(0, \sigma^2)$ where $\sigma^2 = 4kTRB$

B is bandwidth in Hz

V is voltage across resistor of R [ohms] at temperature T [K]

k is Boltzmann's constant, $1.38 \times 10^{-23} \left[\frac{\text{Watts}}{\text{Hz} \cdot \text{K}}\right]$.

- 3°K universal background radiation
- 290°K radiation from earth as seen from space
- shot noise from random arrivals of individual photons/electrons
- low-frequency noise in amplifiers, crystal oscillators, etc.
- many kinds of measurement error.
- variability in manufactured component parameters
- variability in biological organisms (intelligence, ...)
- large-scale systems composed of many loosely interacting components

↘ central limit theorem.

Let me introduce Gaussian distribution and come back to why next lecture.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

often denoted $\mathcal{N}(\mu,\sigma^2)$ for normal distribution.

peak height is $\frac{1}{\sqrt{2\pi\sigma^2}}$.

Standard normal has $\mu=0$, $\sigma^2=1$; the cdf of standard normal often denoted

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv.$$

The survival function often denoted as

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dx = 1-\Phi(x) = \Phi(-x).$$

Let us confirm standard normal integrates to 1, by considering $I = \int_{-\infty}^{\infty} e^{\{-x^2/2\}} dx$.

In polar coordinates:

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2} dx \int_{-\infty}^{\infty} e^{-v^2/2} dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+v^2)/2} \, dx \, dv$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r \, dr \, d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2/2} r \, dr = -2\pi e^{-r^2/2}\Big|_0^{\infty} = 2\pi.$$

so $I = \sqrt{2\pi}$ and the claim is true.

Can also find the second moment

$$E[X^2] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, u \cdot u \exp\left(\frac{-u^2}{2}\right) du$$

$$= -\frac{u}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right)\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du = 0 + 1 = 1.$$

Since mean zero, this is also σ^2, as we desired.

Example $X \sim N(10,16)$. Find $P[X \geq 15]$, $P[X \leq 5]$, $P[X^2 \geq 400]$

use fact $\frac{X-10}{4}$ is standard normal to use Φ or Q function

$$P[X \geq 15] = P\left[\frac{X-10}{4} \geq \frac{15-10}{4}\right] = Q\left(\frac{15-10}{4}\right) = Q\left(\frac{5}{4}\right) = 1 - \Phi\left(\frac{5}{4}\right) = 0.1056$$

$$P[X \leq 5] = P\left[\frac{X-10}{4} \leq \frac{5-10}{4}\right] = \Phi\left(\frac{5-10}{4}\right) = \Phi\left(\frac{-5}{4}\right) = Q\left(\frac{5}{4}\right) = 0.1056$$

$$Pr[X^2 \geq 400] = P[X \geq 20] + P[X \leq -20] = P\left[\frac{X-10}{4} \geq 2.5\right] + P\left[\frac{X-10}{4} \leq -7.5\right]$$

$$= Q(2.5) + \underbrace{Q(7.5)}_{\text{small}} \approx Q(2.5) = 1 - \Phi(2.5) = 0.0062.$$

Normality is preserved by linear transformation.

$X \sim N(\mu, \sigma^2)$, then $Y = aX + b$ is also normal, $E[Y] = a\mu + b$, $var[Y] = a^2\sigma^2$.

Central limit theorem.

If many independent random variables added together, and each is small in magnitude compared to sum, sum is approximately Gaussian.

If X is sum, $\tilde{X}$ is Gaussian random variable with same mean/variance, then X and $\tilde{X}$ have approximately same cdf:

$$P(X \le v) \approx P(\tilde{X} \le v).$$

An important special case is when X is sum of n Bernoulli r.v. each having same parameter p. That is $X \sim Bin(n, p)$.

[matlab].

Let $S_{n,p}$ be binomial r.v. with parameters n, p, so mean is np, variance is $np(1-p)$. So standardized version is

$$\frac{S_{n,p} - np}{\sqrt{np(1-p)}}.$$

<u>De Moivre - Laplace limit theorem</u> Suppose $S_{n,p}$ with p fixed, $0<p<1$, and any constant c.

$$\lim_{n \to \infty} P\left[\frac{S_{n,p} - np}{\sqrt{np(1-p)}} \le c \right] = \Phi(c).$$

To actually do approximations using Gaussian for integer-valued r.v., we use a continuity correction:

$$P(X \le k) \simeq P(\tilde{X} \le k + 0.5)$$
$$P(X \ge k) \simeq P(\tilde{X} \ge k - 0.5)$$