example 2 You get email according to a Poisson process at a rate of $\lambda = 0.2$ messages/hour.

You check email every hour. What is probability of finding 0 and 1 new messages?

Use Poisson pmf: $\dfrac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}$ with $\tau = 1$, $k = 0$ or $k = 1$.

$P(0,1) = e^{-0.2} = 0.819$ $\qquad$ $P(1,1) = 0.2\, e^{-0.2} = 0.164$

Suppose you have not checked email for a whole day: what is probability of no new messages?

$P(0,24) = e^{-0.2 \cdot 24} = 0.0083.$

example 1

Consider a Poisson process on $[0,T]$ with rate $\lambda > 0$, let $0 < \tau < T$.

Define X_1 to be number of counts during $[0,\tau]$, X_2 to be # counts during $[\tau,T]$,

X be total counts during $[0,T]$.

Let i, j, n be nonnegative integers s.t. $n = i + j$.

Find $Pr[X=n]$ in terms of $n, i, j, \tau, T, \lambda$:

$$\frac{e^{-\lambda T}(\lambda T)^n}{n!}$$

Find $Pr[X_1 = i] = \dfrac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}$

Find $Pr[X_2 = j] = \dfrac{e^{-\lambda(T-\tau)}(\lambda(T-\tau))^j}{j!}$

Erlang derivation

$$\Pr\{T_r > t\} = \sum_{k=0}^{r-1} \frac{\exp(-\lambda t)(\lambda t)^k}{k!}$$

$$f_{T_r}(t) = \frac{-d P\{T_r > t\}}{dt}$$

$$= \exp(-\lambda t)\left(\sum_{k=0}^{r-1} \lambda \frac{(\lambda t)^k}{k!} - \sum_{k=1}^{r-1} \frac{k \lambda^k t^{k-1}}{k!} \right)$$

$$= \exp(-\lambda t)\left(\sum_{k=0}^{r-1} \frac{\lambda^{k+1} t^k}{k!} - \sum_{k=1}^{r-1} \frac{\lambda^k t^{k-1}}{(k-1)!} \right)$$

$$= \exp(-\lambda t)\left(\sum_{k=0}^{r-1} \frac{\lambda^{k+1} t^k}{k!} - \sum_{k=0}^{r-2} \frac{\lambda^{k+1} t^k}{k!} \right)$$

$$= \frac{\exp(-\lambda t)\lambda^r t^{r-1}}{(r-1)!}$$

This is called the Erlang distribution with parameters r and λ.

- The mean of T_r is $\frac{r}{\lambda}$ since T_r is sum of r random variables of mean $\frac{1}{\lambda}$.
- Variance is $\frac{r}{\lambda^2}$.

Note that Erlang can be generalized to allow any $r > 0$, rather than just integer r by replacing $(r-1)!$ by $\Gamma(r)$, the gamma function, $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$.

Generalization is called gamma distribution.

You call a hotline and are told you are the 56th person in line, excluding person currently being served. Customers arrived according to Poisson process with $\lambda = 2$ / minute. What is average waiting time? What is probability of waiting more than an hour?

Y, Erlang of order 56, so

$$E[Y] = \frac{56}{\lambda} = 28.$$

$$\Pr[Y \geq 60] = \int_{60}^{\infty} \frac{\lambda^{56} y^{55} e^{-\lambda y}}{55!} dy$$

which is rather hard to compute; motivates use of central limit theorem and normal approximation

Linear scaling of pdfs

Let X be r.v. with pdf f_X, and let $Y = aX+b$, where $a>0$.

Then

$$Y = aX+b \quad \Rightarrow \quad f_Y(v) = f_X\left(\frac{v-b}{a}\right)\frac{1}{a}.$$

To obtain f_Y from f_X, stretch the graph of f_X horizontally by a factor a and shrink it vertically by a factor of a. This leaves the area still 1, yielding pdf of aX. Then shift horizontally by b.

Derivation:

Since $a>0$, event $\{aX+b \le v\}$ is same as $\{X \le \frac{v-b}{a}\}$ so cdf of Y expressed as

$$F_Y(v) = P\{aX+b \le v\} = P\{X \le \tfrac{v-b}{a}\} = F_X\left(\tfrac{v-b}{a}\right).$$

differentiate $F_Y(v)$ w.r.t. v, using chain rule and the fact $F_X' = f_X$:

$$f_Y(v) = F_Y'(v) = f_X\left(\frac{v-b}{a}\right)\frac{1}{a}.$$

This leads to "standardized" version of r.v. X as $\frac{X-\mu_X}{\sigma_X}$, having mean zero and variance one.

Let X denote pdf of historical temperature distribution for a given day in C°; let Y denote same temperature but in °F. Recall $Y = (1.8)X + 32$.

- $-40 = (1.8)(-40) + 32$ is when the scales meet.
- Fahrenheit often preferred for weather since degree size smaller and $(0,100)$ is common range: "triple digits" is quite hot and "below zero" quite cold.

Express f_Y in terms of f_X:

$$f_Y(c) = f_X\left(\frac{c-32}{1.8}\right)/1.8.$$