Time until small mussels first lands in Sahara desert is modeled as exponential r.v. with mean 10 days. It is currently midnight. What is probability that mussels first lands some time between 6am on 6pm of first day?

Let $X$ be time until event, measured in days.

Then $X$ is $\text{exp} \left( \frac{1}{10} \right)$ since mean $= \frac{1}{\lambda} = 10$.

Thus $\Pr(X \leq x) = 1 - e^{-\frac{x}{10}}$.

Want $\Pr(\frac{9}{4} \leq x \leq \frac{3}{4}) = \Pr(x \leq \frac{3}{4}) - \Pr(x < \frac{9}{4}) = e^{-\frac{3}{4}} - e^{-\frac{9}{4}} = 0.0476$.

which we used in survival form.

Example: Let $T$ be an exponentially distributed r.v. with $\lambda = \ln 2$. Find an expression for $\Pr(T \leq t)$ as a function of $t > 0$, $t < 0$, and find $\Pr(T < 1 | T > 2)$

$\Pr(T \leq t) = e^{-\lambda t} = e^{-(\ln 2) t}$

$\Pr(T < 1 | T > 2) = \frac{\Pr(T < 1, T > 2)}{\Pr(T > 2)} = \frac{\Pr(T < 1)}{\Pr(T > 2)} = \frac{1 - \frac{1}{2}}{1 - \frac{1}{2}} = \frac{2}{3}$

Examples
Meaningless property of Exponential

Suppose $T$ is exponentially distributed with parameter $\lambda$, then $\mathbb{P}(T > t) = e^{-\lambda t}$.

Thus: $\frac{P(T > s + t \mid T > s)}{P(T > s)} = \frac{P(T > s + t)}{P(T > s)} \cdot \frac{e^{-\lambda(s + t)}}{e^{-\lambda s}} = e^{-\lambda t} \cdot P(T > t)$.

So $P(T > s + t \mid T > s) = P(T > t)$. This is the meaningless property.

Interpretation: long of lifetime, having not failed up to time $s$ from doesn't affect fate here.

In fact exponential is one of the only nontrivial distributions to have this property.

$P(T > s + t \mid T > s) = P(T > t)$ can be rewritten as:

$P(T > s + t) = P(T > s) \cdot P(T > t)$ by using $f_T(T > s + t) = f_T(T > s)f_T(T > t)$.

Let $g(x) = \log P(T > s)$ as is legitimate & positive $P(T > s)$, we get

$g(s + t) = g(s) + g(t)$ which is the Cauchy equation.

One solution is $g(x) = \beta x$. (There are highly pathological other solutions that are unbounded in any finite interval and not measurable).

If $g(x) = \beta x$, then $P(T > s) = e^{-\beta s}$

which yields $T \sim \mathcal{E}(\lambda)$. 

Exponential distribution is continuous-time analogue of geometric distribution.

Similarly we can go from Bernoulli Process to Poisson Process.

Conversion is through a time-scaling operation.

$\Rightarrow$ see notes.
<table>
<thead>
<tr>
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<th>Poisson</th>
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<td>arrival rate</td>
<td>( \lambda )/unit time</td>
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You get email according to a Poisson process at a rate of \( \lambda = 0.2 \) messages/hour. You check email every hour. What is probability of finding 0 and 1 new messages?

Use Poisson pdf: 
\[
\frac{e^{-\lambda T} (\lambda T)^k}{k!}
\]

with \( T=1 \), \( k=0 \) or \( k=1 \).

\[
P(0,1) = e^{-0.2} \cdot 0.819
\]

\[
P(1,1) = 0.2 \cdot e^{-0.2}, \quad 0.164
\]

Suppose you have not checked email for a whole day; what is probability of no new messages?

\[
P(0,24) = e^{-0.2 \cdot 24} = 0.00037
\]

Example 1

Consider a Poisson process on \([0,T]\) with rate \( \lambda > 0 \), let \( 0 < t < T \).

Let \( X_t \) be number of counts during \([0,t]\), \( X_T \) be # counts during \([t,T]\), \( X \) be total counts during \([0,T]\).

Let \( c_{ij} \) be nonnegative integers s.t. \( n = i+j \).

Find \( \Pr[X = n] \) in lim of \( n,i,j,T,T, \lambda \):

\[
\frac{e^{-\lambda T} (\lambda T)^n}{n!}
\]

Find \( \Pr[X_i = i] \):

\[
\frac{e^{-\lambda T} (\lambda T)^i}{i!}
\]

Find \( \Pr[X_j = j] \):

\[
\frac{e^{-\lambda (T-t)} (\lambda (T-t))^j}{j!}
\]