example

Time until small meteorite first lands in Sahara desert is modeled as exponential r.v. with mean 10 days. It is currently midnight. What is probability that a meteorite first lands some time between 6am and 6pm of first day?

Let X be time until event, measured in days.

Then X is $\sim exp(\lambda = \frac{1}{10})$ since mean $= \frac{1}{\lambda} = 10$.

Want $P_r[\frac{1}{4} \le X \le \frac{3}{4}] = P_r(X \ge \frac{1}{4}] - P_r(X > \frac{3}{4}) = e^{-1/40} - e^{-3/40} = 0.0476.$

where we used the survival function.

example Let T be an exponentially distributed r.v. with $\lambda = \ln 2$. Find an expression for $P(T \ge t]$ as a function of t for $t \ge 0$, and find $P(T \le 1 \mid T \le 2)$

$P(T \ge t) = F_T^c(t) = e^{-\lambda t} = e^{-(\ln 2)t} = 2^{-t}$

$P(T \le 1 \mid T \le 2) = \frac{P(T \le 1, T \le 2)}{P(T \le 2)} = \frac{P(T \le 1)}{P(T \le 2)} = \frac{1 - \frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$

Bayes

Memoryless property of exponential

Suppose T is exponentially distributed with parameter λ, then $P(T>t) = e^{-\lambda t}$.

Thus: $$P(T>s+t \mid T>s) = \frac{P(T>s+t, T>s)}{P(T>s)} = \frac{P(T>s+t)}{P(T>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(T>t)$$

so $P(T>s+t \mid T>s) = P(T>t)$. This is the memoryless property.

Interpretation in terms of lifetime. having not failed up to this time doesn't affect future time.

In fact exponential is one of the only nontrivial distributions to have this property.

$P(T>s+t \mid T>s) = P(T>t)$ can be rewritten as

$P(T>s+t) = P(T>s)\,P(T>t)$ by using fact $(T>s+t) \cap T(s) = (T>s+t)$.

Let $g(x) = \log P(T>s)$ as is legitimate for positive $P(T>s)$, we get

$g(s+t) = g(s) + g(t)$, which is the Cauchy equation.

One solution is $g(x) = \beta x$. (There are highly pathological other solutions that are unbounded in any finite interval and not measurable).

If $g(x) = \beta x$, then $P(L>s) = e^{\beta t}$

which yields $T \sim \mathcal{E}(\alpha)$.

Exponential distribution is continuous-time analogue of geometric distribution.

Similarly we can go from Bernoulli Process to Poisson Process.

Conversion is through a time-scaling operation.

$\Rightarrow$ see notes.

	Poisson	Bernoulli
Times of Arrival	Continuous	discrete
pmf of # arrivals	Poisson	Binomial
interarrival cdf	Exponential	Geometric
arrival rate	λ / unit time	p / per trial

example 2 You get email according to a Poisson process at a rate of $\lambda = 0.2$ messages/hour.

You check email every hour. What is probability of finding 0 and 1 new messages?

Use Poisson pmf: $\frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}$ with $\tau = 1$, $k=0$ or $k=1$.

$P(0,1) = e^{-0.2} = 0.819$ $\qquad$ $P(1,1) = 0.2\, e^{-0.2} = 0.164$

Suppose you have not checked email for a whole day: what is probability of no new messages?

$$P(0,24) = e^{-0.2 \cdot 24} = 0.0083.$$

example 1

Consider a Poisson process on $[0,T]$ with rate $\lambda > 0$, let $0 < \tau < T$.

Define X_1 to be number of counts during $[0,\tau]$, X_2 to be # counts during $[\tau, T]$,

X be total counts during $[0,T]$.

Let i, j, n be nonnegative integers s.t. $n = i + j$.

Find $Pr[X=n]$ in terms of $n, i, j, \tau, T, \lambda$:

$$\frac{e^{-\lambda T}(\lambda T)^n}{n!}$$

Find $Pr[X_1 = i] = \frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!}$

Find $Pr[X_2 = j] = \frac{e^{-\lambda(T-\tau)}(\lambda(T-\tau))^j}{j!}$