mean of continuous r.v.

$$E[g(X)] = \int_{-\infty}^{\infty} g(u)\, f_X(u)\, du$$

$$\mathrm{var}[g(X)] = \int_{-\infty}^{\infty} g(u)^2 f_X(u)\, du - \left(\int_{-\infty}^{\infty} g(u) f_X(u)\, du \right)^2$$

Example) Suppose X has the following pdf

$$f_X(u) = \begin{cases} A(1-u^2), & -1 \le u \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find A, $Pr[\{0.5 < X < 1.5\}]$, F_X, μ_X, $\mathrm{var}(X)$, σ_X.

To get A, use the normalization property

$$1 = \int_{-1}^{1} A(1-u^2)\, du = A\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1} = \frac{4A}{3}$$

so $A = \frac{3}{4}$.

Since X does not occur past 1, to get $Pr[\{0.5 < X < 1.5\}]$, only need to integrate to 1:

$$= \int_{\frac{1}{2}}^{1} \frac{3}{4}(1-u^2)\, du = \frac{3}{4}\left(u - \frac{u^3}{3}\right)\Big|_{1/2}^{1} = \frac{3}{4}\left(\frac{2}{3} - \frac{11}{24}\right) = \frac{5}{32}.$$

To get the cdf, from the pdf, we know it is 0 upto support and 1 after support.
To get middle part, we integrate:

$$F_X(c) = Pr[X \le c] = \int_{-1}^{c} \frac{3(1-u^2)}{4}\, du$$

$$= \frac{3}{4}\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{c} = \frac{2+3c-c^3}{4} \quad \text{for } -1 < c < 1.$$

In general, $F_X(c) = \begin{cases} 0, & c \le -1 \\ \frac{1}{4}(2+3c-c^3), & -1 < c \le 1 \\ 1, & c > 1. \end{cases}$

We can observe μ_X is zero since $u f_X(u)$ is an odd function mean of continuous r.v.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$\mathrm{var}[g(X)] = \int_{-\infty}^{\infty} g^2(x) f_X(x)\,dx - \left(\int_{-\infty}^{\infty} g(x) f_X(x)\,dx\right)^2$$

Example: Suppose X has the following pdf

$$f_X(x) = \begin{cases} A(1-x^2), & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find A, $\Pr[0.5 \le X \le 1.5]$, F_X, μ_X, $\mathrm{var}(X)$, σ_X.

To get A, use the normalization property

$$\int_{-1}^{1} A(1-x^2)\,dx = A\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{4A}{3} = 1$$

So $A = \frac{3}{4}$.

Since X does not occur past 1, to get $\Pr[0.5 \le X \le 1.5]$, we only need to integrate to 1:

$$\int_{0.5}^{1} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Big|_{0.5}^{1} = \frac{3}{4}\left(\frac{2}{3} - \frac{11}{24}\right) = \frac{5}{32}.$$

To get the cdf, from the pdf, we know it is 0 up to support and 1 after support. To get inside support, we integrate:

$$F_X(x) = \Pr[X \le x] = \int_{-1}^{x} \frac{3(1-t^2)}{4}\,dt$$

$$= \frac{3}{4}\left(t - \frac{t^3}{3}\right)\Big|_{-1}^{x} = \frac{2+3x-x^3}{4} \quad \text{for } -1 \le x \le 1.$$

In summary,

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}(2+3x-x^3), & -1 \le x \le 1 \\ 1, & x > 1. \end{cases}$$

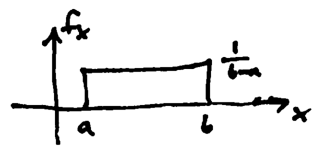
We can observe that μ_X is zero since $x f_X(x)$ is an odd function

$$\mathrm{var}(X) = E[X^2] = \int_{-\infty}^{\infty} u^2 f_X(u)\,du = \int_{-1}^{1} u^2 \frac{3}{4}(1-u^2)\,du = \frac{3}{4}\int_{-1}^{1}(u^2-u^4)\,du = \frac{1}{5}$$

so $\sqrt{\mathrm{var}(X)} = \sqrt{\frac{1}{5}} = \sigma_X$

Uniform distribution for r.v. X over $[a,b]$ if

$$f_X(u) = \begin{cases} \frac{1}{b-a} & , a \le u \le b \\ 0 & , \text{else} \end{cases}$$



$$E[X] = \frac{1}{b-a}\int_a^b u\,du = \frac{u^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$ midpoint.

$$E[X^2] = \frac{1}{b-a}\int_a^b u^2\,du = \frac{u^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

so $\mathrm{var}(X) = E[X^2] - E[X]^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(a-b)^2}{12}$ proportional to interval length squared.

Special case, when $a=0$, $b=1$.

Then

$$E[X^k] = \int_0^1 u^k\,du = \frac{u^{k+1}}{k+1}\Big|_0^1 = \frac{1}{k+1}$$ and variance is $\frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$.

Exponential distribution for r.v. T with parameter $\lambda > 0$ is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & , t \ge 0 \\ 0 & , \text{else} \end{cases}$$

Corresponding CDF is :

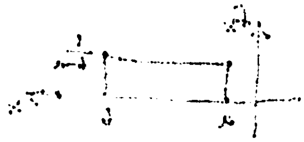
$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & , t \ge 0 \\ 0 & , t < 0. \end{cases}$$

$\mathrm{var}(X) = E[X^2] - \ldots = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \ldots$ [illegible]

so $\sigma_X = \sqrt{\mathrm{var}(X)} = \sqrt{\ldots}$

Uniform distribution: the r.v. X over $[a,b]$ is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{else} \end{cases}$$



midpoint: $E[X] = \int_a^b x \frac{1}{b-a}\,dx = \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2}$

$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{a^2+ab+b^2}{3}$

$\mathrm{var}(X) = E[X^2] - E[X]^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$ — increases as square of interval length

Special case, when $a=0$, $b=1$:

Then $E[X^2] = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$ and variance is $\frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$.

Exponential distribution: the r.v. T with parameter $\lambda > 0$ is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \ge 0 \\ 0, & \text{else} \end{cases}$$

cumulative CDF is:

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

Complementary cdf (survival function)

$$F_T^c(t) = Pr[T > t] = 1 - F_T(t) = \begin{cases} e^{-\lambda t} & t \geq 0 \\ 1 & t < 0. \end{cases}$$

To consider the moments:

$$E[T^n] = \int_0^\infty t^n \lambda e^{-\lambda t}\, dt \qquad \text{, use integration by parts}$$

$$= -n t^n e^{-\lambda t} \Big|_0^\infty + \int_0^\infty n t^{n-1} e^{-\lambda t}\, dt$$

$$= 0 + \frac{n}{\lambda} \int_0^\infty t^{n-1} \lambda e^{-\lambda t}\, dt = \frac{n}{\lambda} E[T^{n-1}]$$

from $E[T^n] = \frac{n}{\lambda} E[T^{n-1}]$ we can use induction.

to get the base cases, we can specifically get $E[T] = \frac{1}{\lambda}$ and $E[T^2] = \frac{2}{\lambda^2}$

Thus $E[T^n] = \frac{n!}{\lambda^n}$.

The $var(T)$ is $E[T^2] - E[T]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$ so std $= \frac{1}{\lambda}$ = mean.

The geometric in "discrete-time" and the exponential in "continuous-time" are quite strongly related.

→ e.g. both have a memoryless property.

$$E[T] = \int_0^\infty t \lambda e^{-\lambda t}\, dt = \left(-t e^{-\lambda t}\right)\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\, dt$$

$$= 0 - \frac{e^{-\lambda t}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

$$E[T^2] = \int_0^\infty t^2 \lambda e^{-\lambda t}\, dt = \left(-t^2 e^{-\lambda t}\right)\Big|_0^\infty + \int_0^\infty 2t e^{-\lambda t}\, dt$$

$$= 0 + \frac{2}{\lambda} E[T] = \frac{2}{\lambda^2}.$$

example

Time until small meteorite first lands in Sahara desert is modeled as exponential r.v. with mean 10 days. It is currently midnight. What is probability that a meteorite first lands some time between 6am and 6pm of first day?

Let X be time until event, measured in days.

Then X is $\sim \exp(\lambda = \frac{1}{10})$ since mean $= \frac{1}{\lambda} = 10$.

Want $\Pr[\frac{1}{4} \le X \le \frac{3}{4}] = \Pr(X \ge \frac{1}{4}] - \Pr(X > \frac{3}{4}) = e^{-1/40} - e^{-3/40} = 0.0476.$

where we used the survival function.