What to do if Ω is uncountable, e.g. real numbers?

⇒ a nonempty interval of real numbers: $[a,b] = \{x : a \le x \le b\}$.

⇒ set of all real numbers $\mathbb{R} = (-\infty, \infty)$.

The usual event algebra in this case is called the Borel Algebra $\mathcal{B}$ and is the smallest algebra closed under formation of sets through countably many unions and intersections containing all finite intervals in $\mathbb{R}$.

→ voltage at a point in a circuit

→ weight of an indivual

→ waiting time until packet receipt or electron emission.

So we will be thinking of random experiment $\mathcal{E} = (\mathbb{R}, \mathcal{B}, P)$ where P is probability measure on $\mathcal{B}$ that satisfies the Kolmogorov axioms.

This is quite complicated and hard to specify.

Instead for applications in engineering, science, social science, etc., specify probability measures through alternate representation.

Cumulative distribution function (CDF).

→ restriction of probability measure P to just those events of the form $\{X : X \le x\} = (-\infty, x]$.

$$F_X(c) = P\{\omega : X(\omega) \le c\}$$
$$= P\{X \le c\} \text{ for short.}$$

While F_X only directly determines probability measure P on semi-infinite intervals, this suffices to determine P for all events. Proof is by Lebesgue-Stieltjes integral.

Though motivated by Ω that is uncountable, CDFs also defined for Ω that is discrete

Let $\mathbb{1}(x)$ be the Heaviside unit step function:

$$\mathbb{1}(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Consider An example of a r.v. X that is constant at value x_0, $P(X = x_0) = 1$.

Then, $$F_X(x) = P_r(X \leq x) = \begin{cases} 0, & x < x_0 \\ 1, & x \geq x_0 \end{cases} = \mathbb{1}(x - x_0).$$

Another example: binomial $(2, \frac{1}{2})$, so

$$p(X=0) = \tfrac{1}{4}, \quad p(X=1) = \tfrac{1}{2}, \quad p(X=2) = \tfrac{1}{4}.$$

Then $$F_X(x) = Pr(X \leq x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases} = \tfrac{1}{4}\mathbb{1}(x) + \tfrac{1}{2}\mathbb{1}(x-1) + \tfrac{1}{4}\mathbb{1}(x-2)$$

For a general pmf $p_X(u)$, the corresponding cdf is

$$F_X(c) = \sum_{u: u \leq c} p_X(u).$$

The jump height of the cdf at some value c is the probability with which c is taken on exactly by X.

To be precise about jumps, useful to formally define left and right limits.

LEFT: $F(x-) = \lim_{\substack{y \to x \\ y < x}} F(y)$ RIGHT: $F(x+) = \lim_{\substack{y \to x \\ y > x}} F(y)$.

The value of cdf at jump point is equal to right limit at that point.

Jump height: $$\Delta F_X(x) = F_X(x) - F_X(x-).$$

Properties of cdfs, whether continuous, discrete, etc. r.v.

Prop: A function F is the cdf of some random variable iff:

① F is nondecreasing

② $\lim_{c \to +\infty} F(c) = 1$, $\lim_{c \to -\infty} F(c) = 0$

③ F is right continuous, i.e. $F_X(c) = F_X(c+)$ for all c.

examples: a, c, f valid; others not.

---

for continuous-type random variables, cdf is integral of a function

$$F_X(c) = \int_{-\infty}^{c} f_X(u)\, du$$

where f_X is called the probability density function. (pdf)

The support of pdf f_X is set of u such that $f_X(u) > 0$.

By the fundamental thm of calculus, if f_X is continuous, then

$f_X = F_X'$.

In particular, this implies F_X is continuous.

But if F_X continuous, then no jumps, so for any constant value c,

$Pr[X = c] = 0$.

Instead interpret pdf value as probability that X is near c (in small interval around c), from limit definition of derivative.

Integral determines probability: $Pr[a \leq X < b] = \int_a^b f_X(u)\, du$.