(1) Suppose a coin is either: $H_1 \to$ biased with heads prob. $2/3$
$H_0 \to$ fair.

Measurement is five flips of coin. Let X be # times heads.

Describe ML and MAP decision rules, find P_F, P_M, and P_e for each using prior $(\pi_0, \pi_1) = (0.2, 0.8)$.

Likelihood matrix for each row is binomial:

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
H_1	$(\frac{1}{3})^5$	$5(\frac{2}{3})(\frac{1}{3})^4$	$10(\frac{2}{3})^2(\frac{1}{3})$	$10(\frac{2}{3})^3(\frac{1}{3})^2$	$5(\frac{2}{3})^4(\frac{1}{3})$	$(\frac{2}{3})^5$
H_0	$(\frac{1}{2})^5$	$5(\frac{1}{2})^5$	$10(\frac{1}{2})^5$	$10(\frac{1}{2})^5$	$5(\frac{1}{2})^5$	$(\frac{1}{2})^5$

in computing likelihood ratio, binomial coefficients cancel:

$$\Lambda(k) = \frac{(\frac{2}{3})^k (\frac{1}{3})^{5-k}}{(\frac{1}{2})^5} = 2^k \left(\frac{2}{3}\right)^5 \approx \frac{2^k}{7.6}.$$

ML rule is to declare H_1 whenever $\Lambda(x) \geq 1$, or equiv, $X \geq 3$.

For ML rule:

$$P_F = 10\left(\tfrac{1}{2}\right)^5 + 5\left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{2}$$

$$P_m = \left(\tfrac{1}{3}\right)^5 + 5\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)^4 + 10\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^3 = \tfrac{51}{243} \doteq 0.201.$$

$$P_e = \pi_0 P_F + \pi_1 P_m = \tfrac{1}{5}\cdot\tfrac{1}{2} + \tfrac{4}{5}(.201) \approx 0.26.$$

MAP rule is to decide H_1 when $\Lambda(x) \geq \frac{\pi_0}{\pi_1} = \frac{1}{4}$, or equiv. $X \geq 1$.

So MAP rule declares H_0 only if $X=0$.

$$P_F = 1 - \left(\tfrac{1}{2}\right)^5 \approx 0.97$$

$$P_m = \left(\tfrac{1}{3}\right)^5 = \tfrac{1}{243} \doteq 0.041$$

$$P_e = \tfrac{1}{5} P_F + \tfrac{4}{5} P_m \doteq 0.227$$

As expected P_e for MAP is smaller than for ML.

(2) <u>Detection with Poisson observations</u> : A deep space transmitter uses laser-based binary signalling. If zero, # photons X is Poisson ($\lambda_0 = 2$). If one, # photons X is Poisson ($\lambda_1 = 6$). Based on X, decide if 0 or 1 transmitted. Find ML, MAP, with $\pi_0/\pi_1 = 5$. Express in terms of X.

ML, MAP rules require $\Lambda(X)$:

$$\Lambda(k) = \frac{P(X=k \mid \text{one sent})}{P(X=k \mid \text{zero sent})} = \frac{e^{-\lambda_1}\lambda_1^k/k!}{e^{-\lambda_0}\lambda_0^k/k!} = \left(\frac{\lambda_1}{\lambda_0}\right)^k e^{-(\lambda_1-\lambda_0)} = 3^k e^{-4} = \frac{3^k}{54.6}.$$

So ML rule is to decide 'one' if $X \geq 4$, since want $\Lambda(X) \geq 1$.

MAP rule is $\Lambda(k) \geq \frac{\pi_0}{\pi_1} = 5$, i.e. $\frac{3^k}{54.6} \geq 5$ or $X \geq 6$.

(3) <u>Sensor Fusion</u> Two motion detectors are used to detect the presence of a person in a room, as part of an energy-saving device. First sensor outputs X and second Y. Both have possible outputs {0, 1, 2}, with larger numbers indicating person is present.

Let H_0 : absent
H_1 : present.

Likelihoods are:

	X=0	X=1	X=2
H_1	0.1	0.3	0.6
H_0	0.8	0.1	0.1

	Y=0	Y=1	Y=2
H_1	0.1	0.1	0.8
H_0	0.7	0.2	0.1

Assume sensors provide conditionally independent readings given hypothesis.

(a) Find likelihood matrix for joint observation (X, Y), and find ML rule. Break ties in favor of H_1.

(b) Find p_f, p_m, for ML rule

(c) Let $\pi_1 = 0.2$, $\pi_0 = 0.8$. Compute joint prob. matrix and find MAP rule.

(a) for MAP rule, compute P_F, P_M, $P_e = \pi_0 P_F + \pi_1 P_M$.

(e) compute P_e for ML rule.

(a) likelihood matrix (X,Y) is

(X,Y)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
H_1	0.01	0.01	$\underline{0.08}$	0.03	$\underline{0.03}$	$\underline{0.24}$	0.06	$\underline{0.06}$	$\underline{0.48}$
H_0	$\underline{0.56}$	$\underline{0.16}$	0.08	$\underline{0.07}$	0.02	0.01	$\underline{0.07}$	0.02	0.01

ML rule by underlining.

(b)

P_M is sum of entries in H_1 row not underlined:

$P_M = 0.01 + 0.01 + 0.03 + 0.06 = 0.11$

P_F is sum of entries in H_0 row not underlined:

$P_F = 0.08 + 0.02 + 0.01 + 0.02 + 0.01 = 0.14$

(c) joint prob matrix:

(X,Y)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
H_1	0.002	0.002	0.016	0.006	0.006	$\underline{0.048}$	0.012	0.012	$\underline{0.096}$
H_0	$\underline{0.448}$	$\underline{0.128}$	$\underline{0.064}$	$\underline{0.056}$	$\underline{0.016}$	0.008	$\underline{0.056}$	$\underline{0.016}$	0.008

MAP rule by underlining.

(d) $P_F = \Pr[(X,Y) \in \{(1,2),(2,2)\} / H_0] = 0.01 + 0.01 = 0.02$

$P_M = \Pr[(X,Y) \notin \{(1,2),(2,2)\} / H_1] = 1 - \Pr[(X,Y) \in \{(1,2),(2,2)\} / H_1]$

$= 1 - 0.24 - 0.48 = 0.28$

so $P_e = (0.8)(0.02) + (0.2)(0.28) = 0.072$

(also sum of not underlined entries in matrix).

(e) for ML,

$$P_e = (0.8)(0.14) + (0.2)(0.11) = 0.134.$$

of course more than error for ML.