① Consider a two-stage experiment, where a die is rolled to produce X. Then a fair coin is flipped X times. Let Y be total number of heads.

Find $\Pr[Y=3]$ and $\Pr[X=3 \mid Y=3]$.

By total probability:

$$\Pr[Y=3] = \sum_{j=1}^{6} P(Y=3 \mid X=j)\, p_X(j)$$

$$= \frac{1}{6}\left[0 + 0 + \binom{3}{3}\left(\tfrac{1}{2}\right)^3 + \binom{4}{3}\left(\tfrac{1}{2}\right)^4 + \binom{5}{3}\left(\tfrac{1}{2}\right)^5 + \binom{6}{3}\left(\tfrac{1}{2}\right)^6\right]$$

$$= \frac{1}{6}\left[0 + 0 + \frac{1}{8} + \frac{1}{4} + \frac{10}{32} + \frac{20}{64}\right]$$

$$= \frac{1}{6}.$$

Notice that $\Pr(Y=3, X=3]$ is $\frac{1}{6}\left[\binom{3}{3}\left(\frac{1}{2}\right)^3\right] = \frac{1}{6}\left(\frac{1}{2}\right)^3 = \frac{1}{48}$

So $\Pr[X=3 \mid Y=3] = \dfrac{\Pr[X=3, Y=3]}{\Pr[Y=3]} = \dfrac{(1/48)}{(1/6)} = \dfrac{1}{8}$

② A test for a certain rare disease is correct 95% time: if a person has disease, test is positive with probability 0.95; if ~~person~~ does not have disease, test negative with probability 0.95. A random person in population has disease with probability 0.001. Given person test positive, what is probability of having disease?

Let A be event that person has disease; B event that test result positive. We want $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(A^c)\,P(B \mid A^c)}$$

$$= \frac{(0.001)(0.95)}{(0.001)(0.95) + (0.999)(0.05)}$$

$$= 0.0187.$$

Even though test is fairly accurate, person who tested positive still unlikely (less than 2%) to have disease.

③ Consider a system of n urns $u_1, u_2, \ldots, u_n$ with each urn having different numbers of balls of different colors. In particular urn u_i has r_i red balls and g_i green balls, so total n_i balls.

A random mechanism first selects an urn $U = u_i$ with probability $\frac{1}{n}$, then draws ball B at random from urn, so $P(B=r \mid U=u_i) = \frac{r_i}{r_i+g_i}$.

What is total probability of getting r:

$$P(B=r) = \sum_{i=1}^{n} \frac{r_i}{r_i+g_i} \cdot \frac{1}{n}.$$

What is the probability $Pr[U=u_i \mid B=r]$ that the selected urn is u_i given the observed draw was a red ball.

$$Pr[U=u_i] = \frac{1}{n}, \quad Pr(B=r \mid U=u_i) = \frac{r_i}{n_i}, \quad Pr[B=r] = \sum_{i=1}^{n} \frac{r_i}{n_i} \cdot \frac{1}{n}.$$

By applying Bayes thm, we get:

$$Pr[U=u_i \mid B=r] = \frac{r_i/n_i}{\sum_{i=1}^{n} r_i/n_i}.$$

④ On a multiple choice question having m choices, prior probability student knows answer is p. If s/he has to guess (event G), all alternatives are equally probable. Find probability student knew answer to question (event K) given s/he answered correctly (event C).