Consider a two-stage experiment, where a die is rolled to produce $X$. Then a fair coin is flipped $X$ times. Let $Y$ be the total number of heads.

Find $\Pr[Y = 3]$ and $\Pr[X = 3 | Y = 3]$.

By total probability:

$$\Pr[Y = 3] = \sum_{j=1}^{6} \Pr(Y = 3 | X = j) \Pr(X = j)$$

$$= \frac{1}{6} \left[ 0 + 0 + (\frac{3}{6})^3 + (\frac{1}{6})^3 + (\frac{5}{6})^3 + (\frac{1}{6})^3 \right]$$

$$= \frac{1}{6} \left[ 0 + 0 + \frac{1}{8} + \frac{1}{8} + \frac{10}{8} + \frac{10}{8} \right]$$

$$= \frac{1}{6}$$

Notice that $\Pr(Y = 3, X = 3)$ is $\frac{1}{6} \left( \frac{3}{6} \right) \left( \frac{1}{2} \right)^3 = \frac{3}{48}$.

So $\Pr[X = 3 | Y = 3] = \frac{\Pr[X = 3, Y = 3]}{\Pr[Y = 3]} = \frac{\frac{1}{48}}{\frac{1}{6}} = \frac{1}{8}$

(2) A test for a certain rare disease is correct 95% time: if a person has disease, test is positive with probability 0.95; if person does not have disease, test negative with probability 0.95. A random person in population has disease with probability 0.001. Given person test positive, what is probability of having disease?

Let $A$ be event that person has disease; $B$ event that test result positive.

We want $P(A | B)$.

$$P(A | B) = \frac{P(A) P(B | A)}{P(A) P(B | A) + P(A^c) P(B | A^c)}$$

$$= \frac{(0.001)(0.95)}{(0.001)(0.95) + (0.999)(0.05)}$$

$$= 0.0187$$

Even though test is fairly accurate, person who tested positive still unlikely (less than 2%) to have disease.
Consider a system of \( n \) urns \( U_1, U_2, \ldots, U_n \) with each urn \( u_i \) having different numbers of balls of different colors. In particular, urn \( u_i \) has \( r_i \) red balls and \( g_i \) green balls, so total \( u_i \) balls.

A random mechanism first selects an urn \( U = u_i \) with probability \( \frac{r_i}{n} \), then draws ball \( B \) at random from urn, so \( P(B = r \mid U = u_i) = \frac{r_i}{r_i + g_i} \).

What is total probability of getting \( r_i \):

\[
P(B = r) = \frac{\sum_{i=1}^{n} r_i}{r_i + g_i} = \frac{1}{n}.
\]

What is the probability \( P(U = u_i \mid B = r) \) that the selected urn is \( u_i \) given the observed draw was a red ball:

\[
P(U = u_i \mid B = r) = \frac{\frac{r_i}{n_i}}{\frac{1}{n}} = \frac{r_i}{n_i}.
\]

By applying Bayes' theorem, we get:

\[
P(U = u_i \mid B = r) = \frac{\frac{r_i}{n_i}}{\sum_{i=1}^{n} \frac{r_i}{n_i}}.
\]

On a multiple choice question having \( m \) choices, prior probability student knows answer is \( p \) if she has to guess (event \( G \)), all alternatives are equally probable. Find probability student knew answer to question (event \( K \)) given she answered correctly (event \( C \)).