Proof of Poisson Approximation to Binomial

$$\frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{\lambda^k}{n^k} \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n}{n} \cdot \frac{(n-1)}{n} \cdots \frac{(n-k+1)}{n} \frac{\lambda^k}{k!} \left(1-\frac{\lambda}{n}\right)^{n-k}$$

Since k is fixed with $n \to \infty$, each of the ratios of n converges to 1.

Also, using formula $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = e^{-1}$ with $x = \frac{n}{\lambda}$, we get

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n/\lambda} = e^{-1} \quad \text{or} \quad \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}.$$

Thus we get the desired result by further noting

$$\left(1-\frac{\lambda}{\lambda}\right)^{-k} \to 1.$$

So get convergence to

$$e^{-\lambda} \frac{\lambda^k}{k!}.$$