Properties of geometric random variable

(mean, variance, memorylessness).

To find $E[L]$ is to condition on outcome of first trial. If outcome of first trial is one, then L is one. If outcome of first trial is zero, L is 1 + # additional trials until outcome one, denoted $\tilde{L}$. So

$$E[L] = p \cdot 1 + (1-p) E[1+\tilde{L}]$$

But L and $\tilde{L}$ have same distribution, since both count till get a one. (memoryless property).

So $E[L] = E[\tilde{L}]$, so

$$E[L] = 1 + (1-p)(E[L]+1) \Rightarrow E[L] = \frac{1}{p}.$$

similar reasoning for variance:

$$E[L^2] = p + (1-p) E[(1+\tilde{L})^2] \Rightarrow E[L^2] = p + (1-p) E[(1+L)^2]$$

expanding and simplifying:

$$E[L^2] = p + (1-p)(1 + 2E[L] + E[L^2])$$

using $E[L] = \frac{1}{p}$, solving for $E[L^2]$ yields $\frac{2-p}{p^2}$, so

$$\mathrm{var}[L] = E[L^2] - E[L]^2 = \frac{1-p}{p^2}.$$

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Problem (keys)

You just rented a large house and the owner gave you 5 keys, ...

1. when marking:

$$Pr[K=1] = \frac{1}{5}$$
$$Pr[K=2] = \frac{4}{5}\cdot\frac{1}{4} = \frac{1}{5}$$
$$Pr[K=3] = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3} = \frac{1}{5}$$
$$Pr[K=4] = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2} = \frac{1}{5}$$
$$Pr[K=5] = \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{5}$$

so $E[K] = \frac{1}{5}\sum_{i=1}^{5} i = 3.$

Note this equivalent to randomly lining up keys and trying them in order. The correct key is equally likely to be in any of the five spots.

2. This is a setting where a geometric rv models what happens, where success probability in any given trial is $p = \frac{1}{5}$.

Thus
$$Pr[K=k] = \left(\frac{4}{5}\right)^{k-1}\left(\frac{1}{5}\right), \quad k = 1, 2, \ldots$$

with duplicate keys:

2. Probability of getting correct key in any given trial is still $\frac{1}{5}$, so same.

1.
$$P_K(1) = \frac{2}{10}$$
$$P_K(2) = \frac{8}{10}\cdot\frac{2}{9} = \frac{8}{45}$$
$$P_K(3) = \frac{8}{10}\cdot\frac{7}{9}\cdot\frac{2}{8} = \frac{7}{45}$$
$$P_K(4) = \frac{8}{10}\cdot\frac{7}{9}\cdot\frac{6}{8}\cdot\frac{2}{7} = \frac{2}{15}$$
$$P_K(5) = \frac{8}{10}\cdot\frac{7}{9}\cdot\frac{6}{8}\cdot\frac{5}{7}\cdot\frac{2}{6} = \frac{1}{9}$$
$$P_K(6) = \frac{8}{10}\cdot\frac{7}{9}\cdot\frac{6}{8}\cdot\frac{5}{7}\cdot\frac{4}{6}\cdot\frac{2}{5} = \frac{4}{45}$$
$$P_K(7) = \frac{8}{10}\cdot\frac{7}{9}\cdot\frac{6}{8}\cdot\frac{5}{7}\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4} = \frac{1}{15}$$
$$P_K(8) = \frac{8}{10}\cdot\frac{7}{9}\cdot\frac{6}{8}\cdot\frac{5}{7}\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3} = \frac{2}{45}$$
$$P_K(9) = \frac{8}{10}\cdot\frac{7}{9}\cdot\frac{6}{8}\cdot\frac{5}{7}\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3} = \frac{1}{45}$$

analogy of lining up doesn't work in same way.

Banach Problem

Let X be the number of matches that remain when a match box is found empty. For $k = 0, 1, \ldots, n$, let L_k (or R_k) be the event that an empty box is first discovered in the left (respectively, right) pocket while the number of matches in the right (resp. left) pocket is k at that time.

Then pmf of X is:

$$p_X(k) = Pr[L_k] + Pr[R_k], \quad k = 0, 1, \ldots, n.$$

Viewing a left selection as "success" and right as "failure", $Pr[L_k]$ is probability there are n successes in $2n-k$ trials, and trial $2n-k+1$ is a success:

$$Pr[L_k] = \frac{1}{2} \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k}, \quad k = 0, \ldots, n$$

By symmetry, $Pr[L_k] = Pr[R_k]$, so

$$p_X(k) = Pr[L_k] + Pr[R_k] = \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k}, \quad k = 0, 1, \ldots, n.$$