Probability with Engineering Applications
ECE 313 – Section C – Lecture 8

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Kolmogorov (1933)

- The concept of mutual **independence** of two or more experiments holds, in a certain sense, a central position in the theory of probability… Historically, the independence of experiments and random variables represents the very mathematical concept that has given the theory of probability its peculiar stamp

Kolmogorov (1933)

- One of the most important problems in the philosophy of the natural sciences is—in addition to the well-known one regarding the essence of the concept of probability itself—to make precise the premises which would make precise the premises which would make it possible to regard any given real events as independent

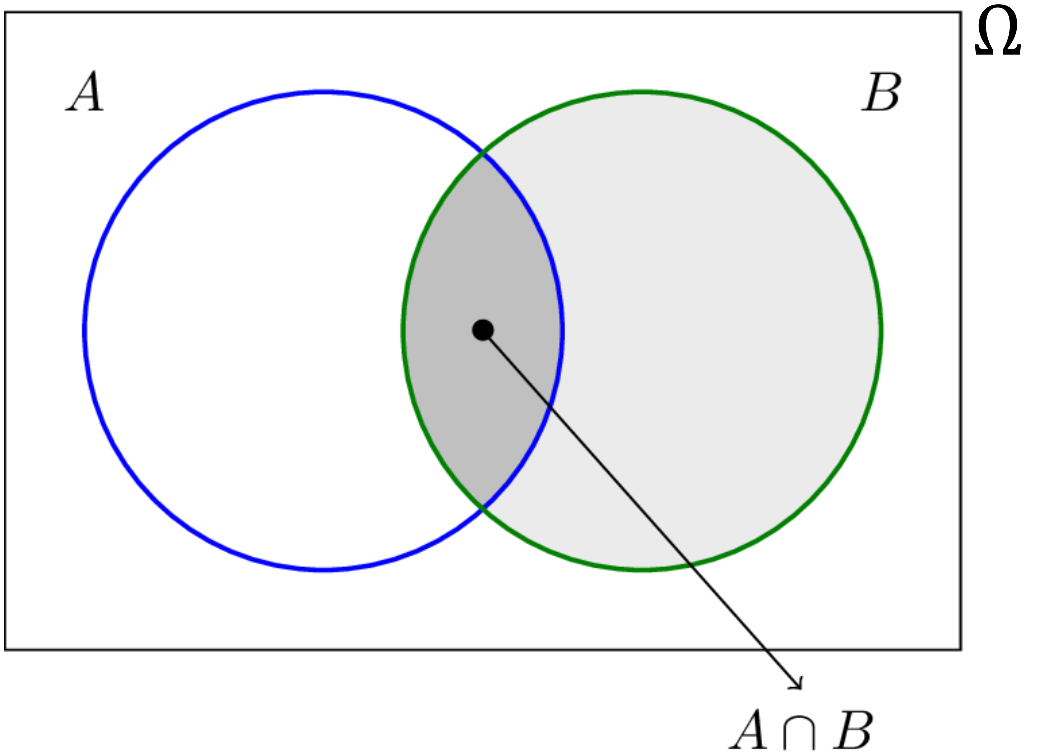
Fine (2002)

- The basic idea is that independence is an appropriate probability model when outcomes are **unlinked**. That is, without a causal connection in a physical setting or without one outcome being informative about another outcome in an information-theoretic or belief-based setting

Physically Unlinked

Conditional probability

- Probabilities based on information/knowledge
  - Revising the knowledge base should lead to revisions of probabilities



Formally defining independence?

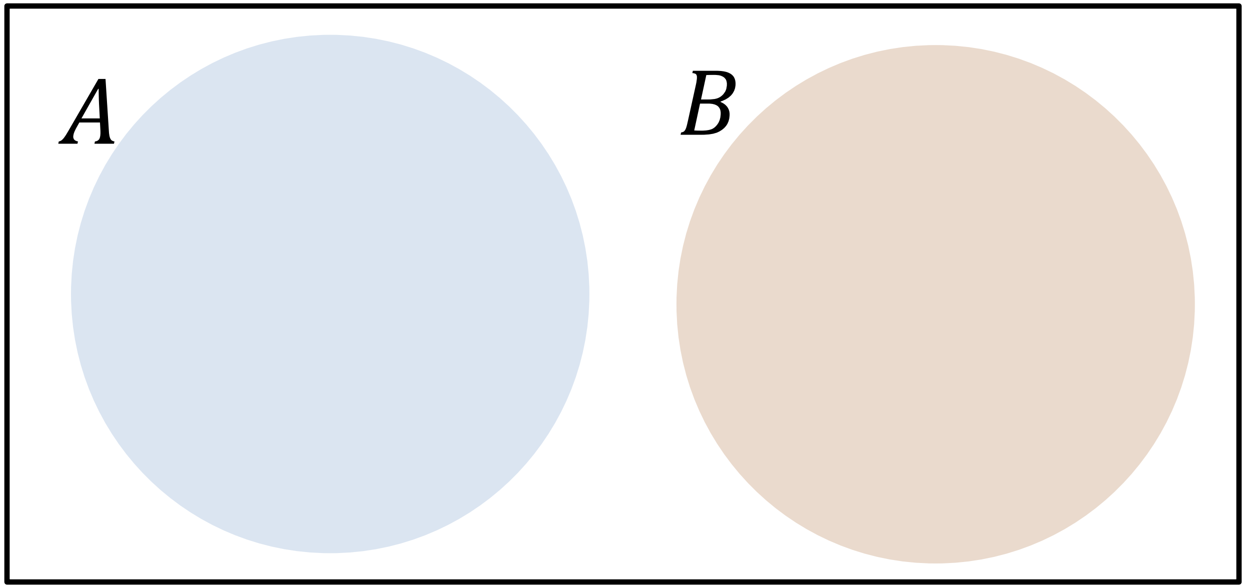
Formally defining independence?

- $P(B|A) = P(B)$ so knowledge that A happened does not impact the probability of B happening
- Since $P(B|A) = P(AB)/P(A)$, condition is equivalent to $P(AB) = P(A)P(B)$
- Note this definition is symmetric in A and B

Physically linked but still independent

- Roll a single die. Let A be event that outcome is even, $\{2, 4, 6\}$. Let B be event that outcome is a multiple of three, $\{3, 6\}$.
- Clearly $AB = \{6\}$, so $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(AB) = \frac{1}{6}$
- Thus $P(AB) = P(A)P(B)$, and therefore independent
- Also, knowing B still leaves A equiprobable

Very dependent



$$A \cap B = \emptyset$$

If A happens, it completely determines that B will not happen. The random variables are not at all independent.

A consequence of independence

1. A is independent of A
2. $P(A) = P(A)^2$
3. $P(A) = 0$ or 1
4. A is independent of B for all B

Pairwise independence

- Events A, B, C are pairwise independent if $P(AB) = P(A)P(B), P(AC) = P(A)P(C)$, and $P(BC) = P(B)P(C)$

- Note that pairwise independence of events does not imply that any one of the events is independent of the intersection of the other two events

(Mutual) independence

- Events A, B, C are (mutually) independent if they are pairwise independent and also $P(ABC) = P(A)P(B)P(C)$

- Likewise for n events (have to establish things for all inclusions/exclusions)

Independence of random variables

- Discrete random variables X and Y are independent if and only if

$$P\{X = i, Y = j\} = p_X(i)p_Y(j)$$

for all i, j

Exercise 1

- If $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 4\}$, $B = \{2, 3, 5\}$, then construct a pmf such that A and B are non-trivially independent

Exercise 2

- Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely.
- Are the events $A_i = \{1st\ \text{roll is}\ i\}$ and $B_j = \{\text{2nd roll is}\ j\}$ independent?

Exercise 3

- Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely.
- Are the events $\mathrm{A} = \{1\mathrm{st\ roll\ is\ }1\}$ and $\mathrm{B} = \{\mathrm{sum\ of\ two\ rolls\ is\ }5\}$ independent?

Exercise 4

- Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely.
- Are the events $\mathrm{A} = \{\mathrm{max\ of\ rolls\ is\ 2}\}$ and $\mathrm{B} = \{\mathrm{min\ of\ rolls\ is\ 2}\}$ independent?

Exercise 5

- Consider two independent rolls of a fair 6-sided die, and the following events:

$$A = \{\text{1st roll is } 1, 2, \text{or } 3\}$$
$$B = \{\text{1st roll is } 3, 4, \text{or } 5\}$$
$$C = \{\text{sum of two rolls is } 9\}$$

Compute the following quantities: $P(A)$, $P(B)$, $P(C)$, $P(AB)$, $P(AC)$, $P(BC)$, $P(ABC)$, and say what you can about independence relations