Exercise 1

for independence, want $P(AB) = P(\{2\})$ to equal $P(A)P(B) = P(\{1,2,4\})P(\{2,3,5\})$.

Let us choose $P(\{2\}) = \frac{1}{4}$ to satisfy the independence condition.

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

This implies $P(\{1,4\}) = \frac{1}{4}$, and $P(\{3,5\}) = \frac{1}{4}$, and let us split these evenly, so

$P(\{1\}) = \frac{1}{8}$

$P(\{2\}) = \frac{1}{4}$

$P(\{3\}) = \frac{1}{8}$

$P(\{4\}) = \frac{1}{8}$

$P(\{5\}) = \frac{1}{8}$

$P(\{6\}) = \frac{1}{4}$ $\leftarrow$ forced by normalization.

Exercise 2

$P(A_i B_j) = P(\text{result of two rolls is } (i,j)) = \frac{1}{16}$

$P(A_i) = \dfrac{\#\text{ pairs with first roll } i}{\text{all pairs}} = \frac{4}{16} = \frac{1}{4}$

$P(B_j) = \dfrac{\#\text{ pairs with second roll } j}{\text{all pairs}} = \frac{4}{16} = \frac{1}{4}$

$P(A_i)P(B_j) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} = P(A_i B_j)$ so independent.

Exercise 3

$P(AB) = P(\text{result of two rolls is } (1,4)) = \frac{1}{16}$

$P(A) = \frac{4}{16} = \frac{1}{4}$ from before

$P(B) = \dfrac{\#\text{ elements of } B}{\text{total pairs}} = \dfrac{|\{(1,4),(2,3),(3,2),(4,1)\}|}{16} = \frac{4}{16} = \frac{1}{4}$

$P(A)P(B) = \frac{1}{16} = P(AB)$ so independent.

Exercise 4

$P(AB) = P(\text{result of rolls is } (2,2)) = \frac{1}{16}$

$P(A) = \frac{3}{16}$

$P(B) = \frac{5}{16}$

so $P(A)P(B) = \frac{15}{16^2} \neq \frac{1}{16}$, so not independent.

Exercise 5

$$P(AB) = \frac{1}{6} \neq \frac{1}{2}\cdot\frac{1}{2} = P(A)P(B)$$

$$P(AC) = \frac{1}{36} \neq \frac{1}{2}\cdot\frac{4}{36} = P(A)P(C)$$

$$P(BC) = \frac{1}{12} \neq \frac{1}{2}\cdot\frac{4}{36} = P(B)P(C)$$

$P(ABC) = \frac{1}{36} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{36} = P(A)P(B)P(C)$, so this equality not enough for independence.