Probability with Engineering Applications
ECE 313 – Section C – Lecture 7

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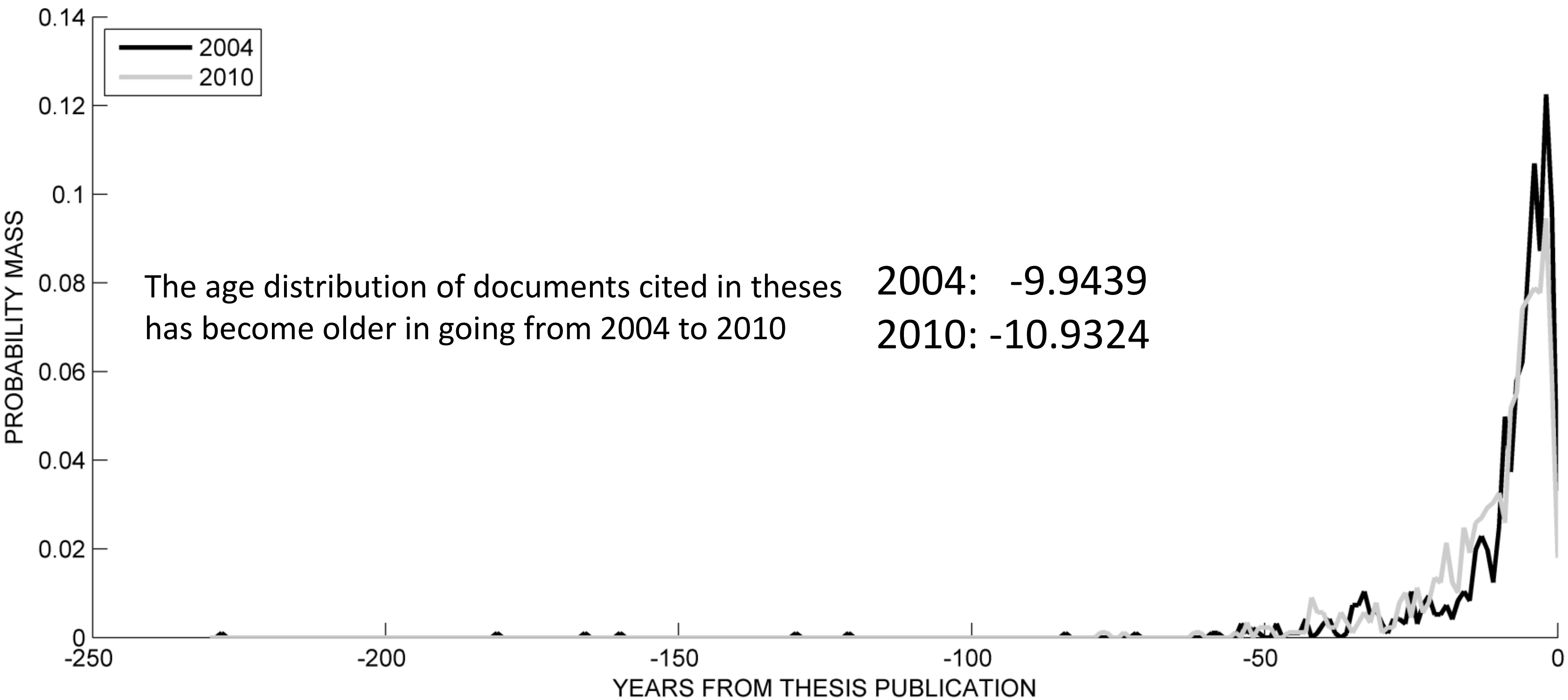
Mean of a random variable

- The mean of a random variable is a weighted average of the possible values of the random variable, such that the weights are given by the pmf:

$$E[X] = \sum_i u_i p_X(u_i)$$

Interpretation 1

- The expected value can often be thought of as a measure of location of a random variable



[Varshney, 2012]

Interpretation 2

- As per the laws of large numbers we will discuss later in the course, if we perform numerous unlinked repetitions of an experiment to produce a sequence of random variables $\{X_i\}$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \text{ converges to } E[X]$$

Interpretation 3

- How much would you pay to be offered a particular gamble?

MATCHING COMBINATION	PRIZES	CURRENT ODDS (1 IN ...)	PREVIOUS ODDS (1 IN ...)
5 white balls and the Powerball	The grand prize	292,201,338	175,223,510
5 white balls	$1,000,000	11,688,054	5,153,633
4 white balls and the Powerball	$50,000 (formerly $10,000)	913,129	648,976
4 white balls	$100	36,525	19,088
3 white balls and the Powerball	$100	14,494	12,245
3 white balls	$7	580	360
2 white balls and the Powerball	$7	701	706
1 white balls and the Powerball	$4	92	111
The Powerball	$4	38	55

(At $500 million jackpot, expected payout is $2.03)

Interpretation 3

Variance

- The mean squared deviation is called the *variance*:

$$var[X] = E[(X - \mu_X)^2] = \sigma_X^2$$

- Also,

$$var[X] = E[X^2] - {\mu_X}^2$$

Why?

- Square root is called *standard deviation,* σ_X

Variance

- The mean squared deviation is called the *variance*, a measure of spread/dispersion:

$$var[X] = E[(X - \mu_X)^2]$$

- Also,

$$var[X] = E[X^2] - {\mu_X}^2$$

Why?

$$E[X^2 - 2X\mu_X + {\mu_X}^2] = E[X^2] - 2\mu_X E[X] + {\mu_X}^2$$
$$= E[X^2] - 2{\mu_X}^2 + {\mu_X}^2 = E[X^2] - {\mu_X}^2$$

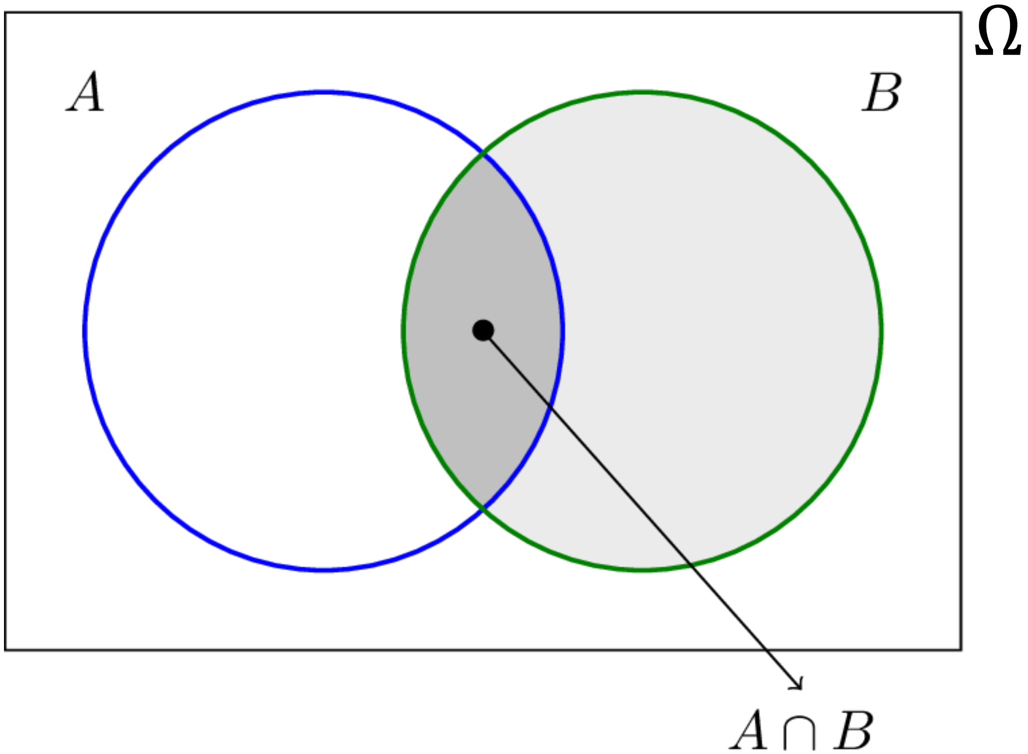
- Square root is called *standard deviation,* σ_X

Questions to Consider

1. In experiment involving two successive rolls of a die, you are told the sum of the two rolls is 8. How likely is it that the first roll was a 5?
2. In a word guessing game, the first letter of the word is a “t”. What is the likelihood the second letter is an “e”?
3. A spot shows up on a radar screen. How likely is it that there is an aircraft around?

Conditional Probability

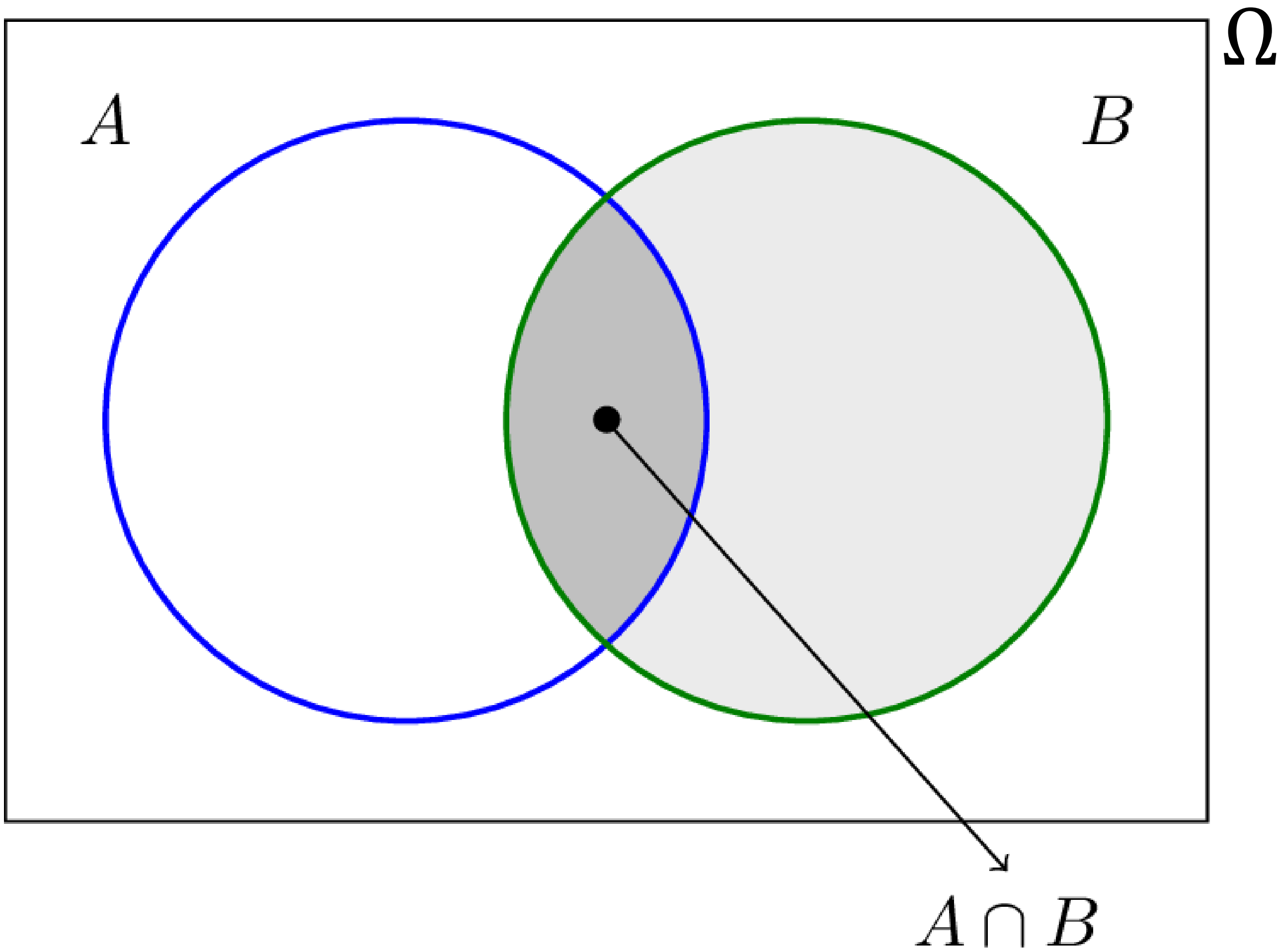
- Probabilities based on information/knowledge
  - Revising the knowledge base should lead to revisions of probabilities



Classical Conditional Probability

- Consider the probability of an event A, $P(A)$
- If we are now informed that event B has occurred, how should we revise $P(A)$ so that it is the *conditional probability* $P(A|B)$?

- $P(A) = \frac{|A|}{|\Omega|}$ gets revised to $P(A|B) = \frac{|AB|}{|B|}$



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

[https://www.probabilitycourse.com/chapter1/1_4_0_conditional_probability.php]

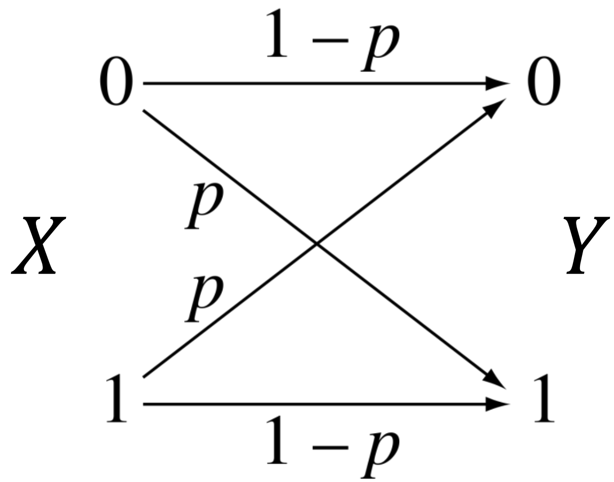
Conditional Probability

- Conditional probabilities can be viewed as a probability law on a new universe B, since all of the conditional probability is concentrated on B

Conditional Probability Properties

1. $P(B|A) \geq 0$
2. $P(B|A) + P(B^c|A) = 1$
3. $P(\Omega|B) = 1$
4. $P(AB) = P(A)P(B|A)$
5. $P(ABC) = P(C)P(B|C)P(A|BC)$

Example in Digital Communication



- $P(Y = 1|X = 0) = p = P(Y = 0|X = 1)$
- $P(Y = 0|X = 0) = 1 - p = P(Y = 1|X = 1)$

Example in Photonics

- Suppose X represents actual count of emitted photons in a given time period, and Y is the measured value
- The sensor is imperfect and occasionally drops/adds a single count, but designed so no negative counts
- Model correct count as not depending on true value of X so $P(Y = y|X = x) = p_c$
- For $x = 0$, errors defined by $P(Y = 1|X = 0) = 1 - p_c$
- For $x > 0$, errors defined by $P(Y = x + 1|X = x) = \frac{1}{2}(1 - p_c) = P(Y = x - 1|X = x)$

Exercise 1

- Let $\Omega = \{0,1,\ldots,9\}$, $p(0) = .35$, $p(1) = .25$, and $p(2) = \cdots = p(9) = .05$
- Let $A = \{0, 4, 5\}$ and $B = \{2, 3, 5\}$
- Evaluate $P(A|B)$

Exercise 2

- We toss a fair coin three successive times. We wish to find the conditional probability $P(A|B)$ when the events A and B are as follows:

$$A = \{\text{more heads than tails}\}$$
$$B = \{\text{1st toss is a head}\}$$

Exercise 3

- A fair 4-sided die is rolled twice (so all sixteen possibilities are equally likely)
- Let X and Y be the result of the first and second rolls.
- Find $P(A|B)$ when $A = \{\max(X, Y) = m\}$ and $B = \{\min(X, Y) = 2\}$ for each $m = 1, 2, 3, 4$