Properties of Variance

\[ \text{Var}(X+b) = E[(X+b - E[X+b])^2] \]
\[ = E[(X+b - E[X] - b)^2] \]
\[ = E[(X - E[X])^2] \]
\[ = \text{Var}(X) \]

so shifting around doesn't change spread.

\[ \text{Var}(aX) = E[(aX - E[aX])^2] \]
\[ = E[(aX - aE[X])^2] \]
\[ = a^2 E[(X - E[X])^2] \]
\[ = a^2 \text{Var}(X) \]

so multiplying r.v. by a multiplies variance by \( a^2 \).

The random variable \( \frac{X - \mu_X}{\sigma_X} \) is called the standardized version of \( X \),

with mean 0 and variance 1.

\[ E\left[ \frac{X - \mu_X}{\sigma_X} \right] = \frac{1}{\sigma_X} \left( E[X] - \mu_X \right) = 0 \]

\[ \text{Var}\left[ \frac{X - \mu_X}{\sigma_X} \right] = \frac{1}{\sigma_X^2} E[(X - \mu_X)^2] = \frac{\sigma_X^2}{\sigma_X^2} = 1. \]
Exercise 1
\[
\frac{P(A \mid B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}
\]

Exercise 2

The sample space is
\[
\Omega = \{HHH, HHT, HTT, THT, THH, TTT\}
\]

Event \( B \) consists of four of these: \( B = \{HHH, HHT, HTT, TTT\} \) so \( P(B) = \frac{4}{8} = \frac{1}{2} \)

Event \( A \cap B \) consists of three elements: \( A \cap B = \{HHH, HHT, HTT\} \) so \( P(A \cap B) = \frac{3}{8} \)

Then
\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}
\]

Exercise 3

We can determine \( P(A \mid B) \) and \( P(B) \) by counting as before, or alternatively directly count size of sets \( A \cap B \) and \( B \) and divide.

\[\text{event includes the 5 shaded samples}\]

The set \( A = \{\max(x, y) = m\} \) shares two elements with \( B \) if \( m = 3, 4 \), shares one element if \( m = 2 \) and none if \( m = 1 \).

\[
P(A \mid B) = \begin{cases} 
\frac{1}{5} & \text{for } m = 3, 4 \\
\frac{1}{5} & \text{for } m = 2 \\
0 & \text{for } m = 1 
\end{cases}
\]