Probability with Engineering Applications
ECE 313 – Section C – Lecture 6

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Counting for cellular communication

- In frequency division multiple access (FDMA), a service provider purchases a fixed number of calling frequencies f
- Assume at any given time, there are n callers distributed at random over r cells
- To avoid crosstalk, all callers within a cell and immediately adjacent cells must be assigned unique frequencies
- Frequencies can be reused otherwise

Counting for cellular communication

- If $\{n_i\}$ are the cell occupancy numbers, and i and j are adjacent cells, then $n_i + n_j \leq f$
- For purposes of frequency assignment, ordering of users within cell is irrelevant
- Thus we are partitioning n users into r subsets of sizes $n_1, n_2, \ldots, n_r$
- Hence the number of arrangements of n over r cells with specified occupancy numbers is given by the multinomial coefficient
- The total number of arrangements of n distinguishable users into r cells is r^n

Counting for cellular communication

- The probability of a particular set of occupancy numbers is therefore:

$$P(n_1, n_2, \dots, n_r) = \frac{\binom{n}{n_1\, n_2 \cdots n_r}}{n^r}$$

- This probability can then be used to determine the probability that f frequencies will suffice

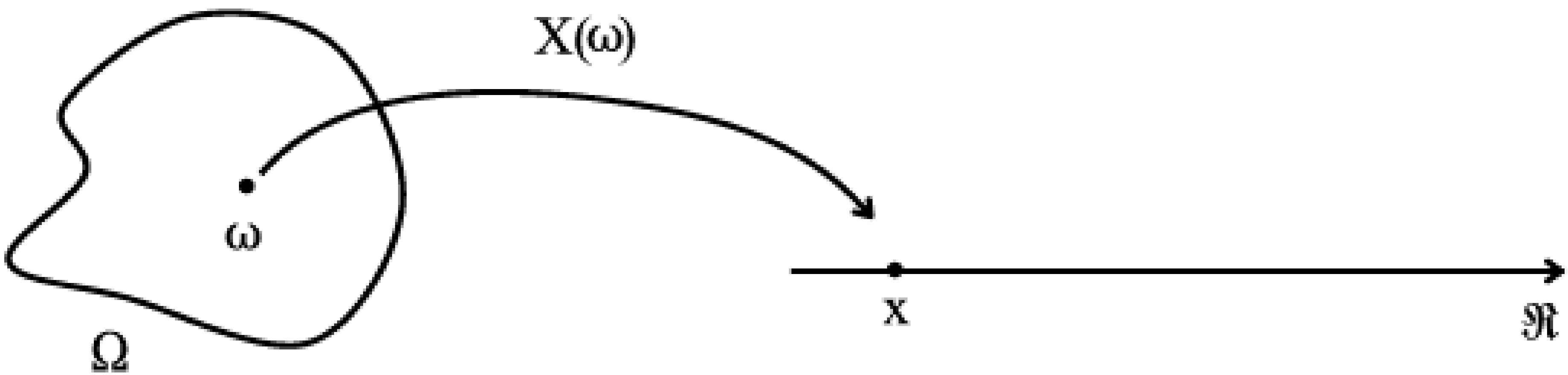
Random experiment

- A random experiment $\mathcal{E}$ is characterized by three components:
  - Possible outcomes (sample space), Ω
  - Collection of sets of outcomes of interest, $\mathcal{F}$
  - Numerical assessment of likelihood of occurrence of each outcome of interest, P

Random variable

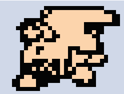
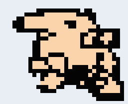
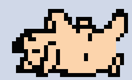
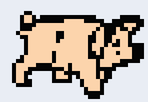
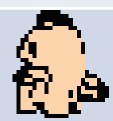
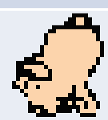
- Let X be a random variable for a probability space $(\Omega, \mathcal{F}, P)$
- If the probability experiment is performed, i.e. a value ω is selected from Ω, then the value of the random variable is $X(\omega)$
- The value $X(\omega)$ is called the *realized value* of X for outcome ω

Random variable is a mapping

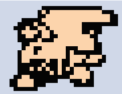
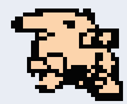
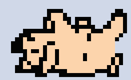
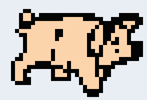
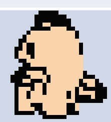
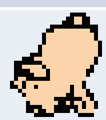


[cnx.org]

Pass the Pigs (like dice)

Position $\omega \in \Omega$		$X(\omega)$
Side (no dot)		1
Side (dot)		2
Razorback		3
Trotter		4
Snouter		5
Leaning Jowler		6

Pass the Pigs (with points)

Position $\omega \in \Omega$		$Y(\omega)$
Side (no dot)		0
Side (dot)		1
Razorback		5
Trotter		7
Snouter		15
Leaning Jowler		20

Discrete random variable

- A random variable is said to be *discrete* if there is a finite set $u_1, \ldots, u_n$ or a countably infinite set $u_1, u_2, \ldots$ such that
$$P\big\{X \in \{u_1, u_2, \ldots\}\big\} = 1$$

- The probability mass function (pmf) for a discrete random variable X, p_X, is defined by
$$p_X(u) = P\{X = u\}$$

Probability mass function

- The pmf is sufficient to determine the probability of any event determined by X, because for any set A,

$$P\{X \in A\} = \sum_{i:u_i \in A} p_X(u_i)$$

Probability mass function

- The pmf sums to unity:

$$\sum_i p_X(u_i) = 1$$

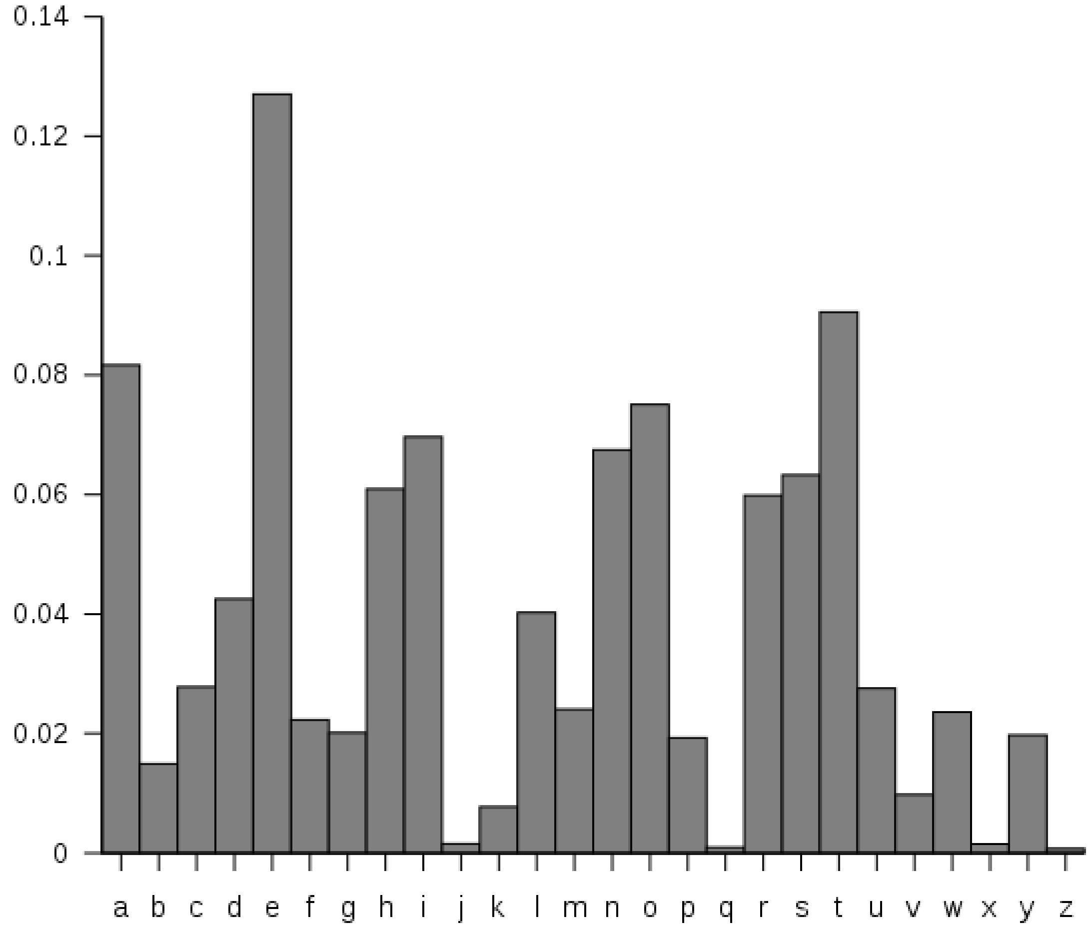
- The *support* of a pmf p_X is the set of u such that $p_X(u) > 0$

Example of pmf

- Consider a random variable Z corresponding to the flip of a fair coin, where heads maps to 0 and tails maps to 3

- What is the support of the pmf p_Z?
- What is the pmf p_Z?
- Does the pmf p_Z sum to unity?

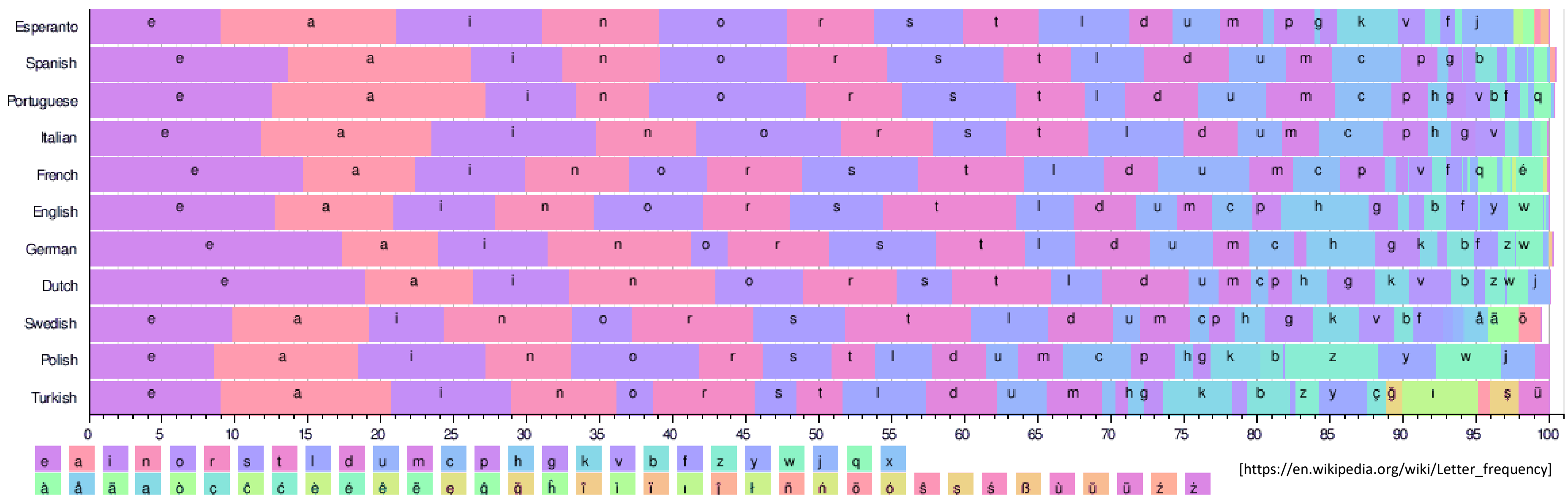
English letters



(map to number 1:26)

[By Nandhp - Own work; en:Letter frequency., Public Domain, https://commons.wikimedia.org/w/index.php?curid=9971073]

Convex combinations



[https://en.wikipedia.org/wiki/Letter_frequency]

- What if we first randomly select the language i of a text, and then look at the letter frequencies?

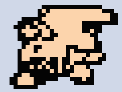
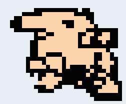
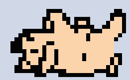
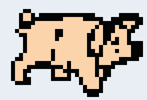
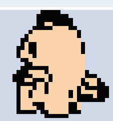
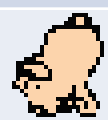
Convex combinations

- A convex combination p of pmfs $\{p^{(i)}\}$ is readily verified to be a pmf, in particular the pmf corresponding to the convex combination P of measures $\{P_i\}$:

$$P(A) = \sum_i \lambda_i P_i(A)\,, \qquad p(\omega) = \sum_i \lambda_i\, p^{(i)}(\omega)$$

- The $\{\lambda_i\}$ are the probabilities with which the ith language is chosen

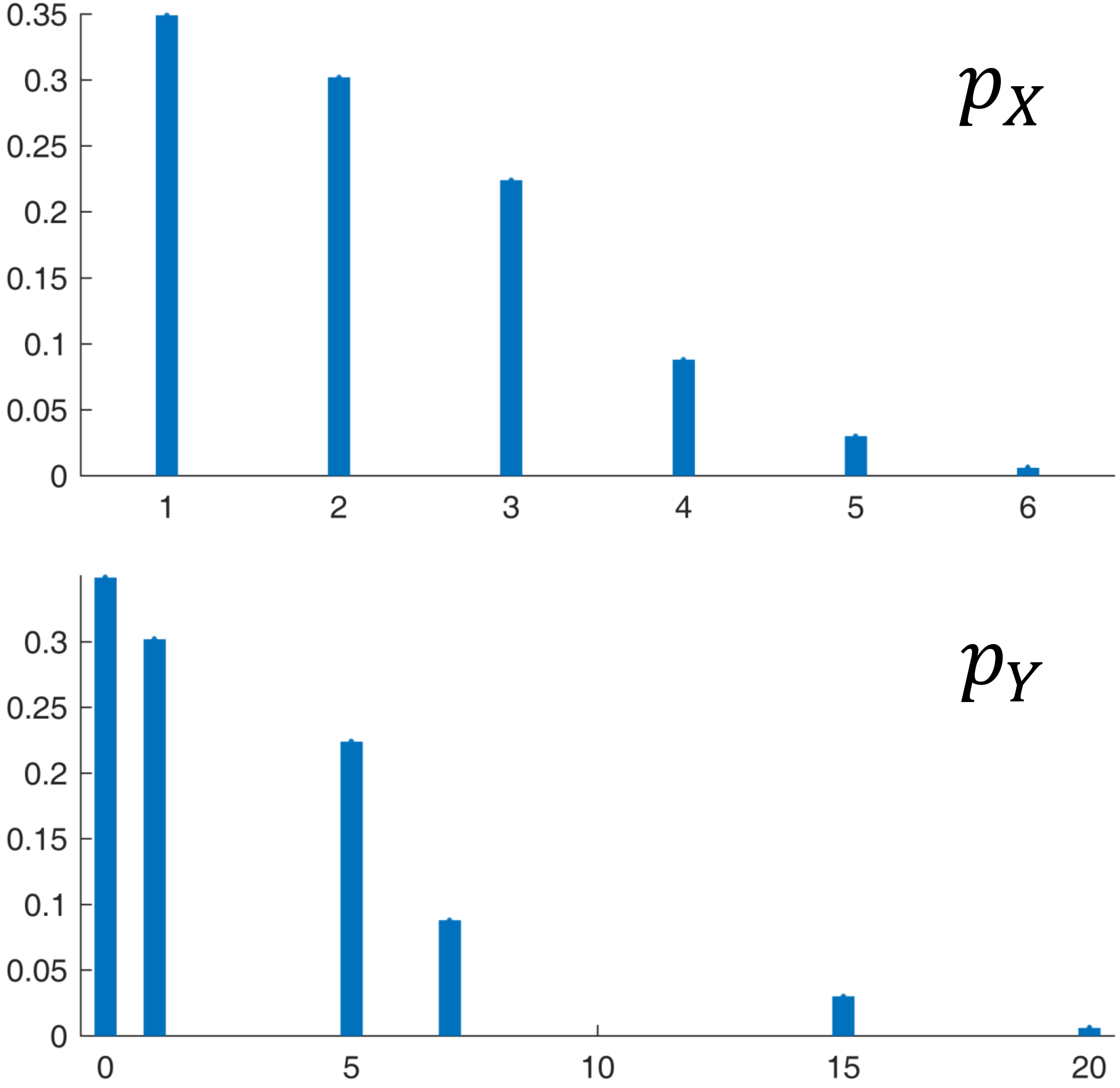
Pass the Pigs (empirical)

Position		Percentage
Side (no dot)		34.9%
Side (dot)		30.2%
Razorback		22.4%
Trotter		8.8%
Snouter		3.0%
Leaning Jowler		0.61%

Sketch the pmfs p_X and p_Y

{1,2,3,4,5,6}
{0,1,5,7,15,20}

Probability mass function



Mean of a random variable

- The mean of a random variable is a weighted average of the possible values of the random variable, such that the weights are given by the pmf:

$$E[X] = \sum_i u_i p_X(u_i)$$

Mean of a random variable

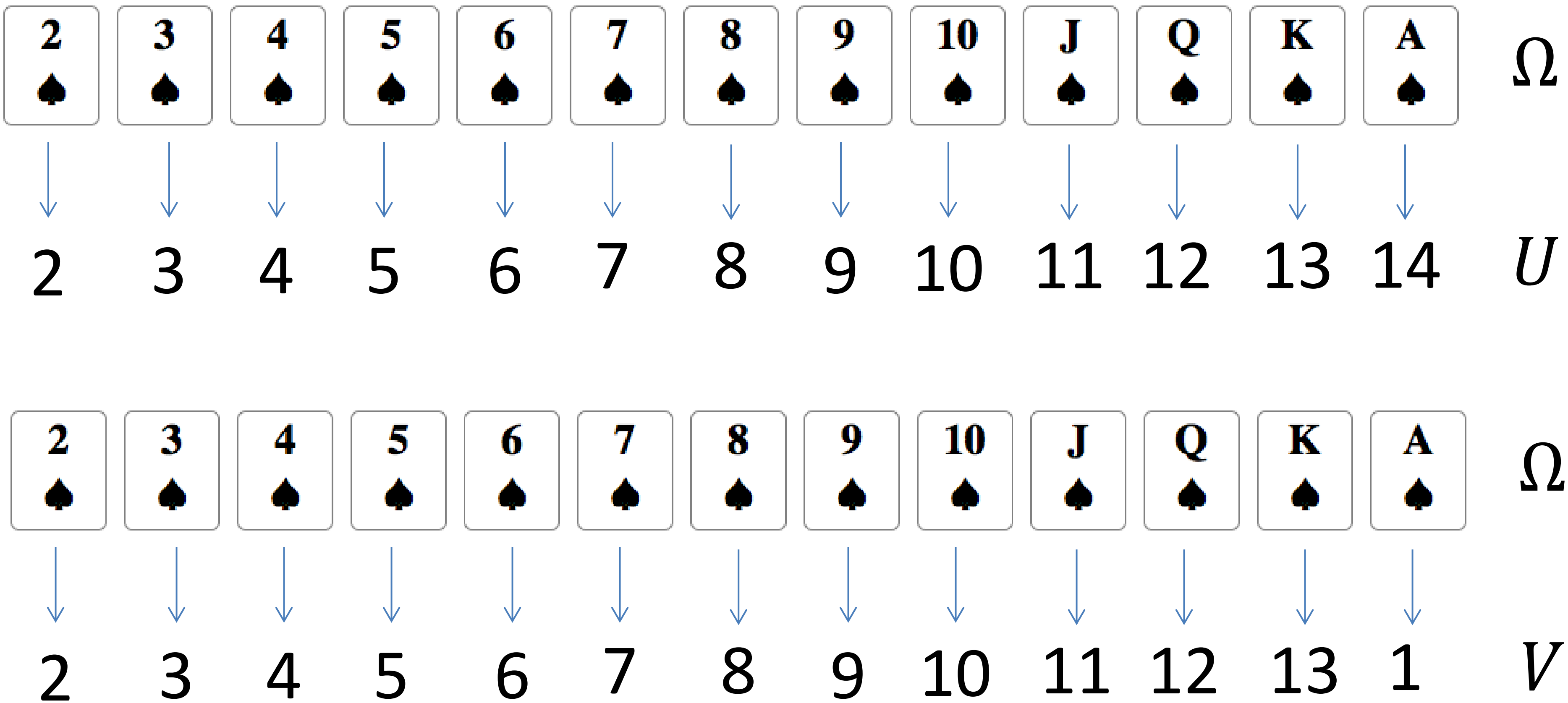
- What is the mean of the coin flip random variable Z?

Mean of a random variable

- What is the mean of the coin flip random variable Z?

$$E[Z] = \sum_i u_i p_Z(u_i) = 0 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = \frac{3}{2}$$

Mean of two random variables



What are means of random variables U and V?

Mean of two random variables

$$E[U] = \frac{1}{13}(2 + \cdots + 14) = \frac{104}{13} = 8$$

$$E[V] = \frac{1}{13}(1 + \cdots + 13) = \frac{91}{13} = 7$$

A new random variable

- Suppose we care about the “energy” of the card, so we consider $W = V^2$
- What is the support of the pmf of W?

- What is the mean of W?

A new random variable

- Suppose we care about the "energy" of the card, so we consider $W = V^2$
- What is the support of the pmf of W?

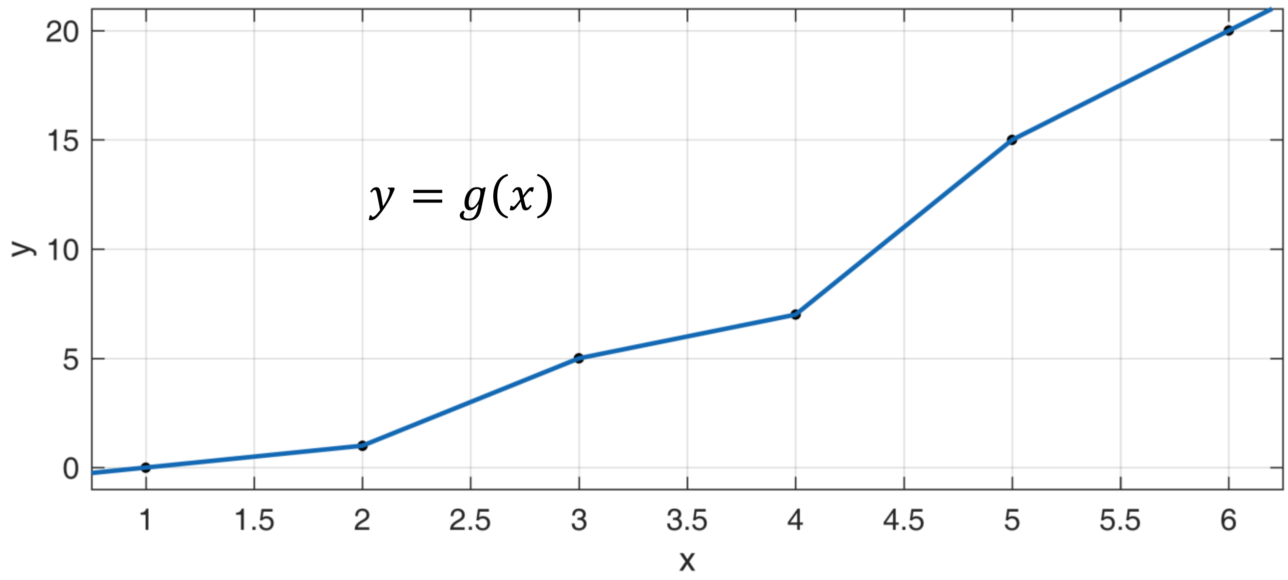
$$(1, 4, 9, 16, \ldots, 169)$$

- What is the mean of W?

$$E[W] = \frac{1}{13}(1 + \cdots + 169) = \frac{819}{13} = 63$$

Functions of random variables

$\boldsymbol{\omega \in \Omega}$	$X(\omega)$	$Y(\omega)$
Side (no dot)	1	0
Side (dot)	2	1
Razorback	3	5
Trotter	4	7
Snouter	5	15
Leaning Jowler	6	20



Functions of random variables

- In general, the law of the unconscious statistician (LOTUS) says that:

$$E[g(X)] = \sum_i g(u_i) p_X(u_i)$$

Linearity of expectation

- If the function has linear components, things become even easier, since the expectation operation is linear:

$$E[ag(X) + bh(X) + c]$$
$$= aE[g(X)] + bE[h(X)] + c$$

How does one prove linearity of expectation?

Notable functions

- Let $\mu_X = E[X]$
- The difference $X - \mu_X$ is called the *deviation* of X from its mean
- What is the mean of the deviation?

Notable functions

- Let $\mu_X = E[X]$
- The difference $X - \mu_X$ is called the *deviation* of X from its mean
- What is the mean of the deviation?

$$E[X - \mu_X] = E[X] - \mu_X = \mu_X - \mu_X = 0$$

Notable functions

- The mean squared deviation is called the *variance*:

$$var[X] = E[(X - \mu_X)^2]$$

- Also,

$$var[X] = E[X^2] - {\mu_X}^2$$

Why?

- Its square root is called the *standard deviation*

Notable functions

- The mean squared deviation is called the *variance*:

$$var[X] = E[(X - \mu_X)^2]$$

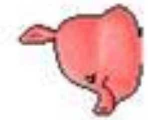
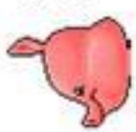
- Also,

$$var[X] = E[X^2] - \mu_X{}^2$$

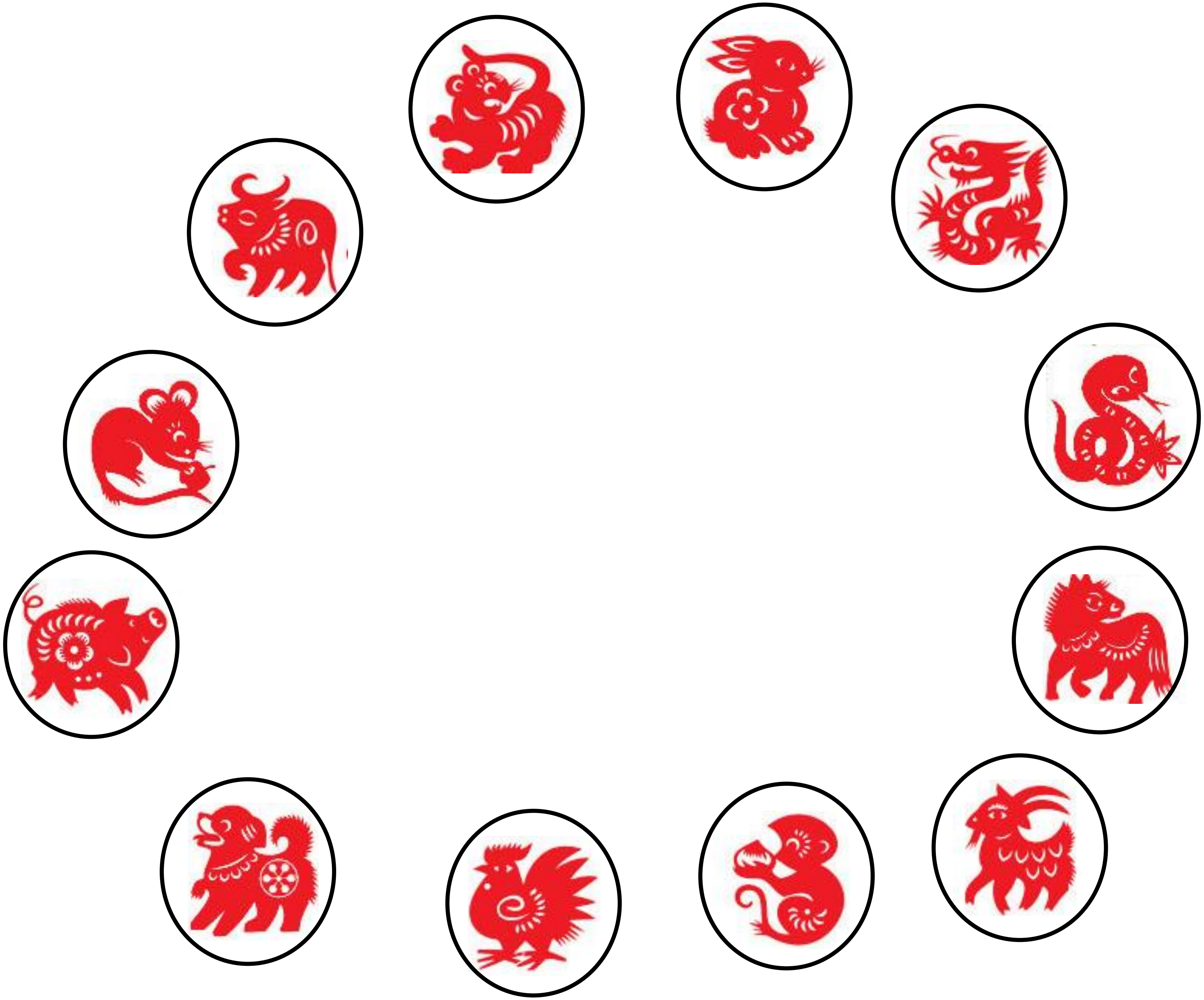
Why?

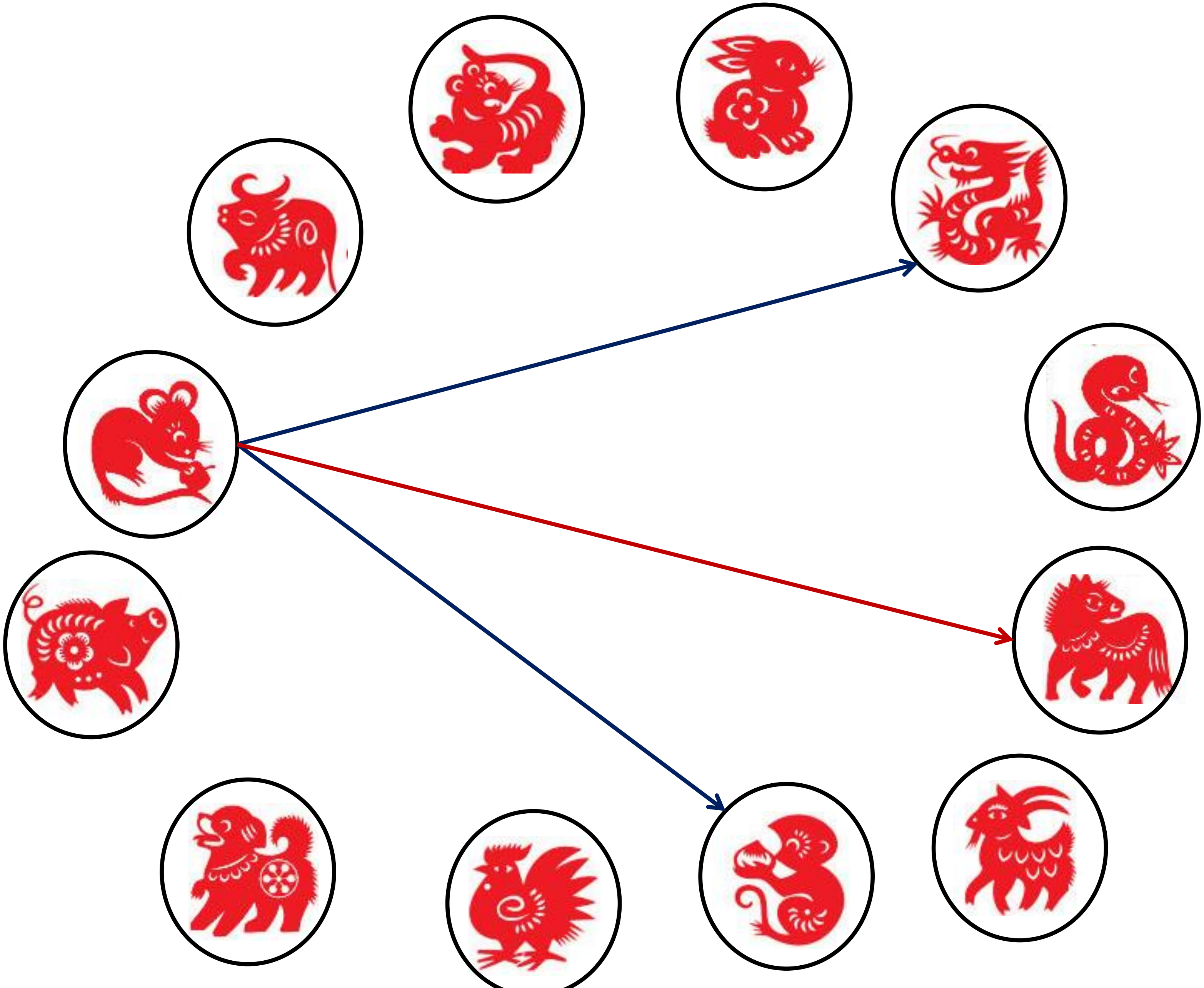
$$E[X^2 - 2X\mu_X + \mu_X{}^2] = E[X^2] - 2\mu_X E[X] + \mu_X{}^2$$
$$= E[X^2] - 2\mu_X{}^2 + \mu_X{}^2 = E[X^2] - \mu_X{}^2$$

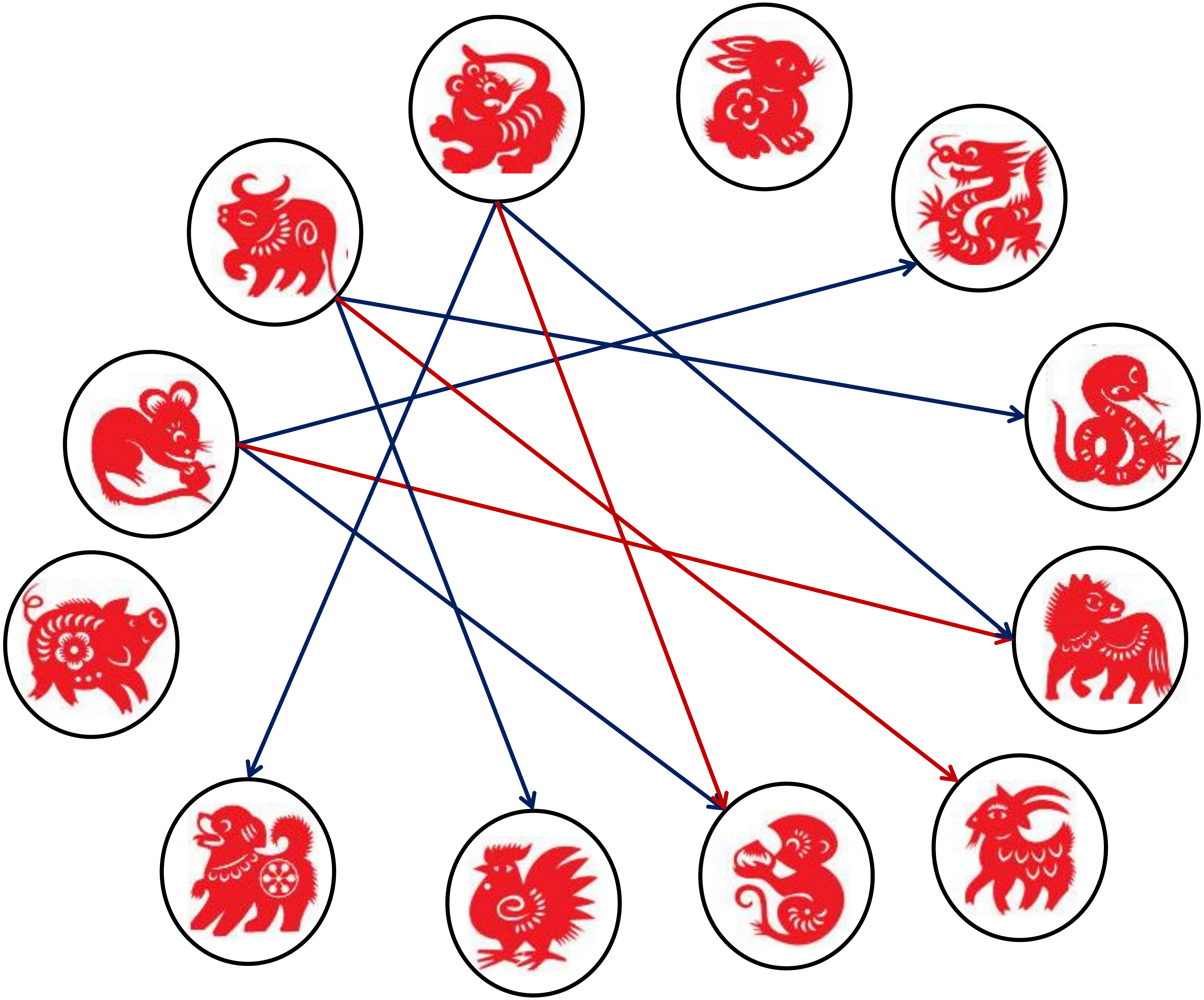
- Its square root is called the *standard deviation*

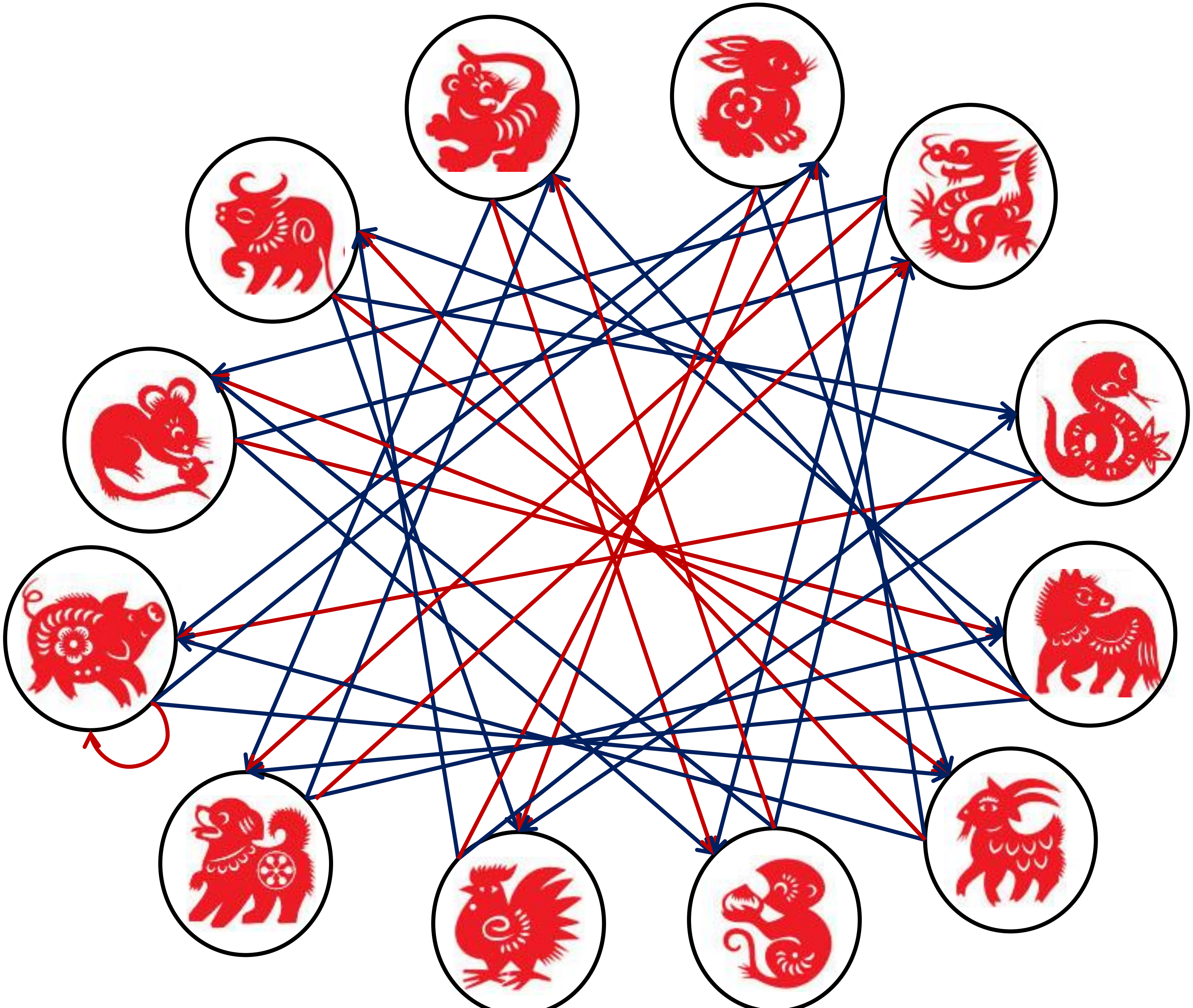
	Side 	Other Side 	Back 	Trotter 	Snout 	Leaning 
Side 	A SIDER Lose 1 Point	PIGOUT Lose ALL Points earned this round & your turn	+ 5 Points	+ 5 Points	+ 10 Points	+ 15 Points
Other Side 	PIGOUT Lose ALL Points earned this round & your turn	A SIDER Lose 1 Point	+ 5 Points	+ 5 Points	+ 10 Points	+ 15 Points
Back 	+ 5 Points	+ 5 Points	DOUBLE RAZORBACK + 20 Points	+ 10 Points	+ 15 Points	+ 20 Points
Trotter 	+ 5 Points	+ 5 Points	+ 10 Points	DOUBLE TROTTER + 20 Points	+ 15 Points	+ 20 Points
Snout 	+ 10 Points	+ 10 Points	+ 15 Points	+ 15 Points	DOUBLE SNOUTER + 40 Points	+ 25 Points
Leaning 	+ 15 Points	+ 15 Points	+ 20 Points	+ 20 Points	+ 25 Points	DOUBLE LEANER +60 Points

CHINESE ZODIAC
RABBIT
1939, 1951, 1963, 1975, 1987, 1999 Luckiest of all signs, you are also talented and articulate. Affectionate, yet shy, you seek peace throughout your life. Marry a Sheep or Boar. Your opposite is the Cock.
DRAGON
1940, 1952, 1964, 1976, 1988, 2000 You are eccentric and your life complex. You have a very passionate nature and abundant health. Marry a Monkey or Rat late in life. Avoid the Dog.
SNAKE
1941, 1953, 1965, 1977, 1989, 2001 Wise and intense with a tendency towards physical beauty. Vain and high tempered. The Boar is your enemy. The Cock or Ox are your best signs.
HORSE
1942, 1954, 1966, 1978, 1990, 2002 Popular and attractive to the opposite sex. You are often ostentatious and impatient. You need people. Marry a Tiger or a Dog early, but never a Rat.
SHEEP
1943, 1955, 1967, 1979, 1991, 2003 Elegant and creative, you are timid and prefer anonymity. You are most compatible with Boars and Rabbits but never the Ox.
MONKEY
1944, 1956, 1968, 1980, 1992, 2004 You are very intelligent and are able to influence people. An enthusiastic achiever, you are easily discouraged and confused. Avoid Tigers. Seek a Dragon or a Rat.
TIGER
1938, 1950, 1962, 1974, 1986, 1998 Tiger people are aggressive, courageous, candid and sensitive. Look to the Horse and Dog for happiness. Beware of the Monkey.
OX
1937, 1949, 1961, 1973, 1985, 1997 Bright, patient and inspiring to others. You can be happy by yourself, yet make an outstanding parent. Marry a Snake or Cock. The Sheep will bring trouble.
RAT
1936, 1948, 1960, 1972, 1984, 1996 You are ambitious yet honest. Prone to spend freely. Seldom make lasting friendships. Most compatible with Dragons and Monkeys. Least compatible with Horses.
The Chinese Zodiac consists of a 12 year cycle, each year of which is named after a different animal that imparts distinct characteristics to its year. Many Chinese believe that the year of a person's birth is the primary factor in determining that person's personality traits, physical and mental attributes and degree of success and happiness throughout his lifetime. To learn about your Animal Sign, find the year of your birth among the 12 signs running around the border. If born before 1936, add 12 to the year you were born to find your year.
COCK
1945, 1957, 1969, 1981, 1993, 2005 A pioneer in spirit, you are devoted to work and quest after knowledge. You are selfish and eccentric. Rabbits are trouble. Snakes and Oxen are fine.
DOG
1946, 1958, 1970, 1982, 1994, 2006 Loyal and honest you work well with others. Generous yet stubborn and often selfish. Look to the Horse or Tiger. Watch out for Dragons.
BOAR
1947, 1959, 1971, 1983, 1995, 2007 Noble and chivalrous. Your friends will be lifelong, yet you are prone to marital strife. Avoid other Boars. Marry a Rabbit or a Sheep.









	Rat	Ox	Tiger	Rabbit	Dragon	Snake	Horse	Sheep	Monkey	Cock	Dog	Boar
Rat					1		-1		1			
Ox						1		-1		1		
Tiger							1		-1		1	
Rabbit								1		-1		1
Dragon	1								1		-1	
Snake		1								1		-1
Horse	-1		1								1	
Sheep		-1		1								1
Monkey	1		-1		1							
Cock		1		-1		1						
Dog			1		-1		1					
Boar				1				1				-1