Probability with Engineering Applications
ECE 313 – Section C – Lecture 3

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Probability measure P

- The probability measure is a mapping $\mathcal{F} \rightarrow [0, 1]$ that satisfies the following axioms:

1. For any event A, $P(A) \geq 0$
2. If $A, B \in \mathcal{F}$ and if A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$
3. The probability of the sample space satisfies $P(\Omega) = 1$

Properties of probability measures

4. $P(A^c) = 1 - P(A)$
5. For any event A, $P(A) \leq 1$
6. $P(\emptyset) = 0$
7. If $A \subset B$ then $P(A) \leq P(B)$
8. $P(A \cup B) = P(A) + P(B) - P(AB)$

Boole's Inequality (Union Bound)

- For a countable set of events $A_1, A_2, \ldots$, we have:

$$P\left(\bigcup_i A_i\right) \leq \sum_i P(A_i)$$

Bonferroni's Inequality

- Prove that for any two events A and B, we have: $P(AB) \geq P(A) + P(B) - 1$

Bonferroni's Inequality

- Generalize to the case of n events $A_1, A_2, \dots, A_n$ by showing that:

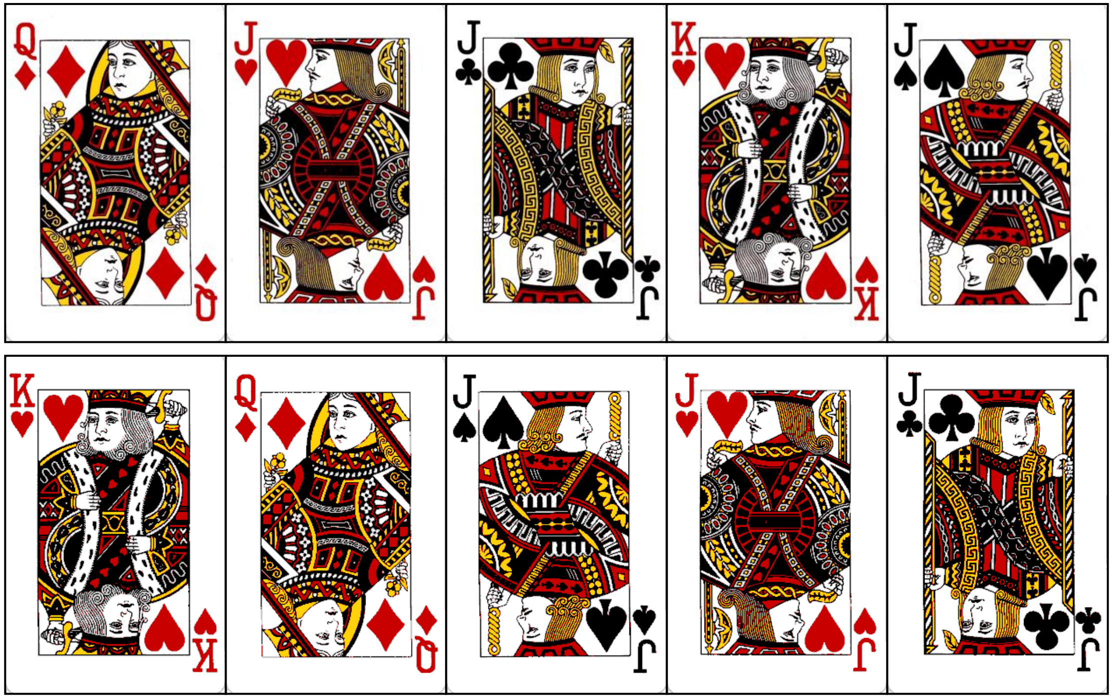
$$P(A_1 A_2 \cdots A_n) \geq P(A_1) + \cdots + P(A_n) - (n-1)$$

Classical Probability and Counting

- An important class of probability spaces are those such that the set of outcomes, Ω, is finite, and all outcomes have equal probability

- The probability for any event A is $P(A) = |A|/|\Omega|$, where $|A|$ is the number of elements in A and $|\Omega|$ is the number of elements in Ω

Permutations and Combinations



Poker — Just one set of cards

Magic — 5! possible sequences of cards

Binomial Coefficient

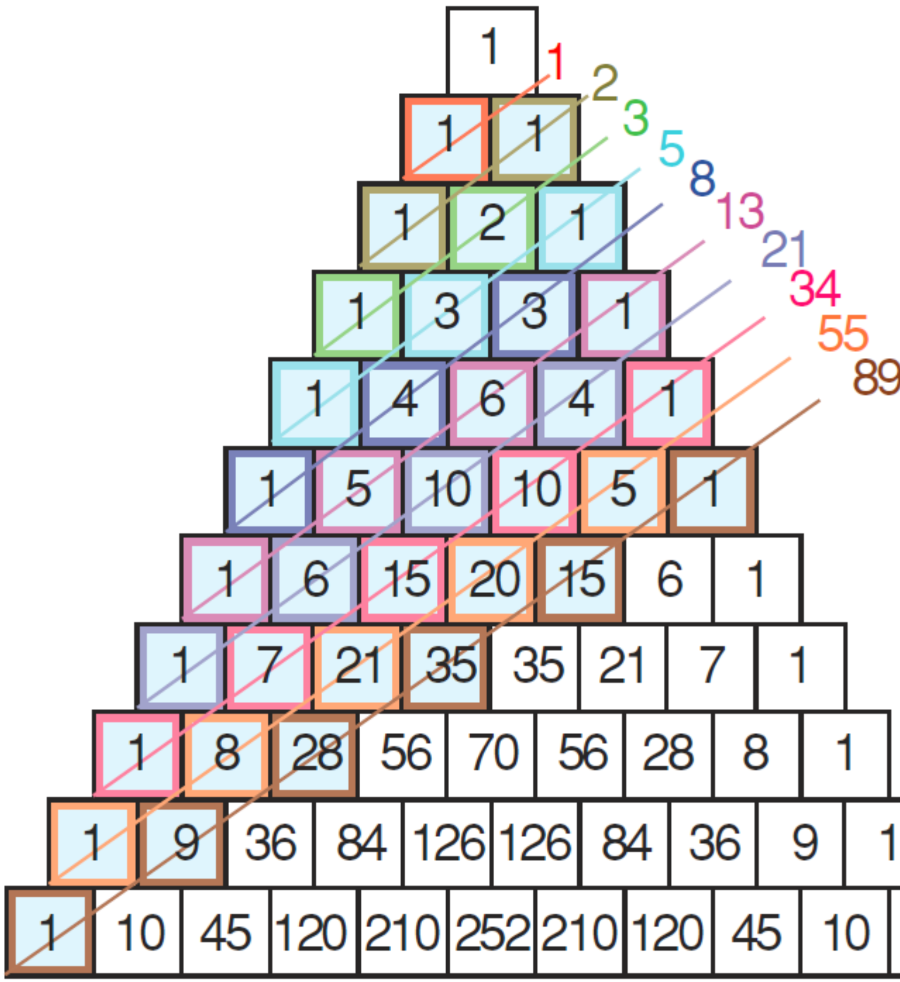
- There are $\binom{n}{k}$ ways to choose a subset of size k elements, disregarding their order, from a set of n elements

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Pingala’s Meru Prastāra (~450 BC)

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Pingala’s Meru Prastāra (~450 BC)



Ratio of successive terms of first Meru sequence (Fibonacci sequence) converge to the golden ratio

[L. R. Varshney, “Local Fidelity, Constrained Codes, and the Meru Prastara,” *IEEE Potentials*, vol. 27, no. 2, pp. 27-32, March/April 2008.]

Multinomial Coefficient

- Principle of overcounting: cancel any possible ties or indistinguishable symbols

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\ k_2! \cdots k_m!}$$

Videos in Course Notes

https://uofi.app.box.com/s/qo8exgegq6aryw1yxip4

Enumerative Combinatorics

- Think of N as set of balls and X as set of boxes
- A function $f: N \rightarrow X$ consists of placing each ball into some box
- If we can tell the balls apart, then the elements of N are called distinguishable, otherwise indistinguishable
- Similarly if we can tell the boxes apart, then then elements of X are called distinguishable, otherwise indistinguishable

Distinguishability

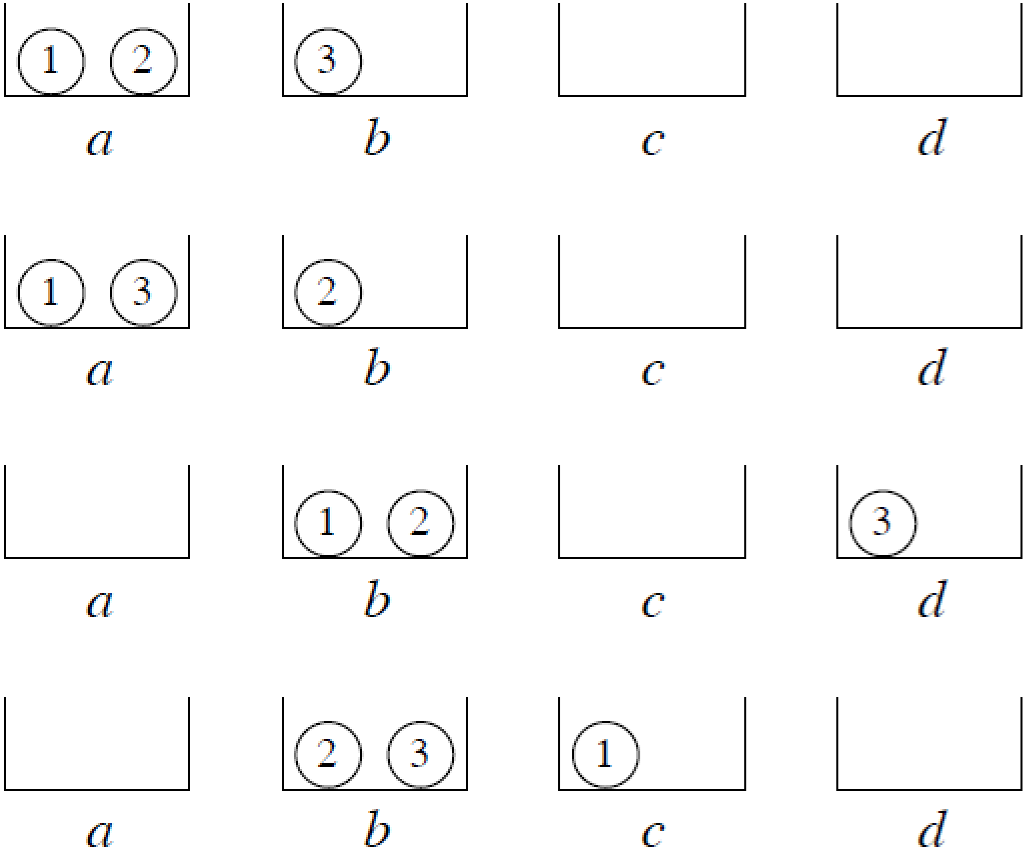
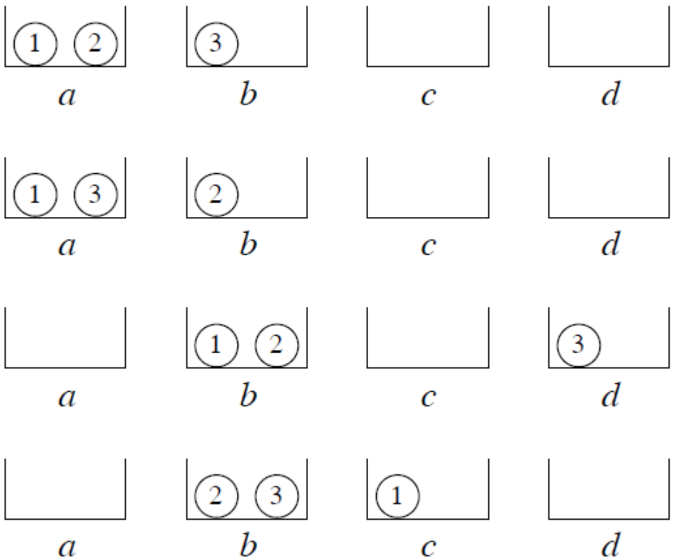


Figure 1.22: Four functions with distinguishable balls and boxes

[R. P. Stanley, 2011]

Distinguishability



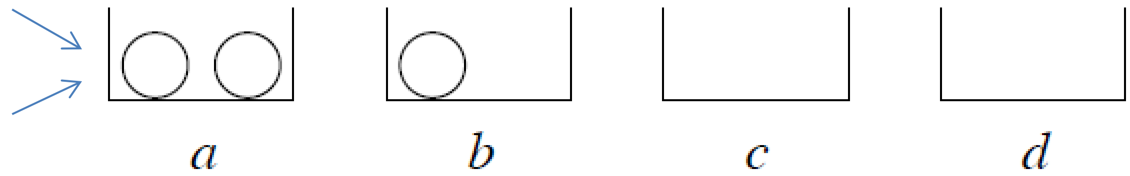
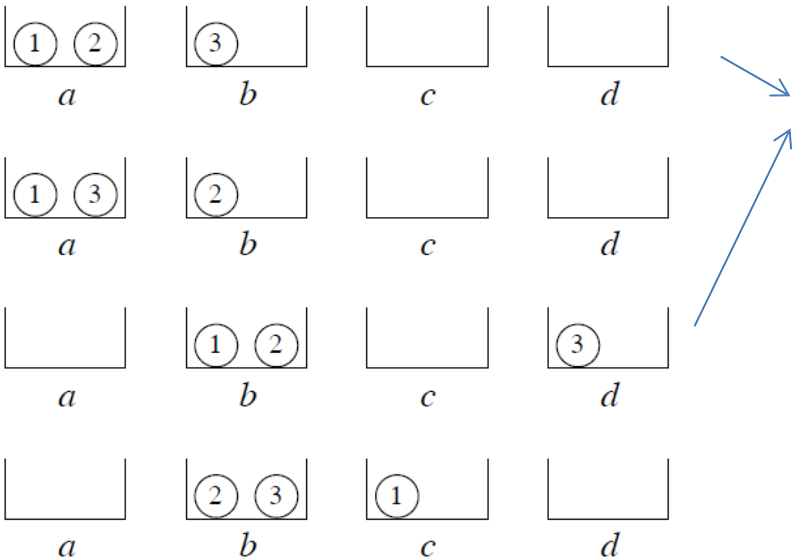


Figure 1.23: Balls indistinguishable

[R. P. Stanley, 2011]

Distinguishability



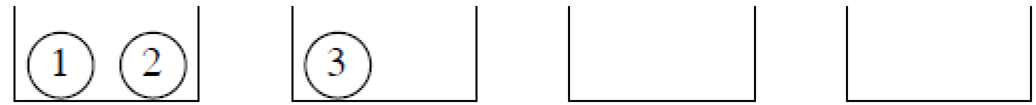


Figure 1.24: Boxes indistinguishable

[R. P. Stanley, 2011]

Distinguishability

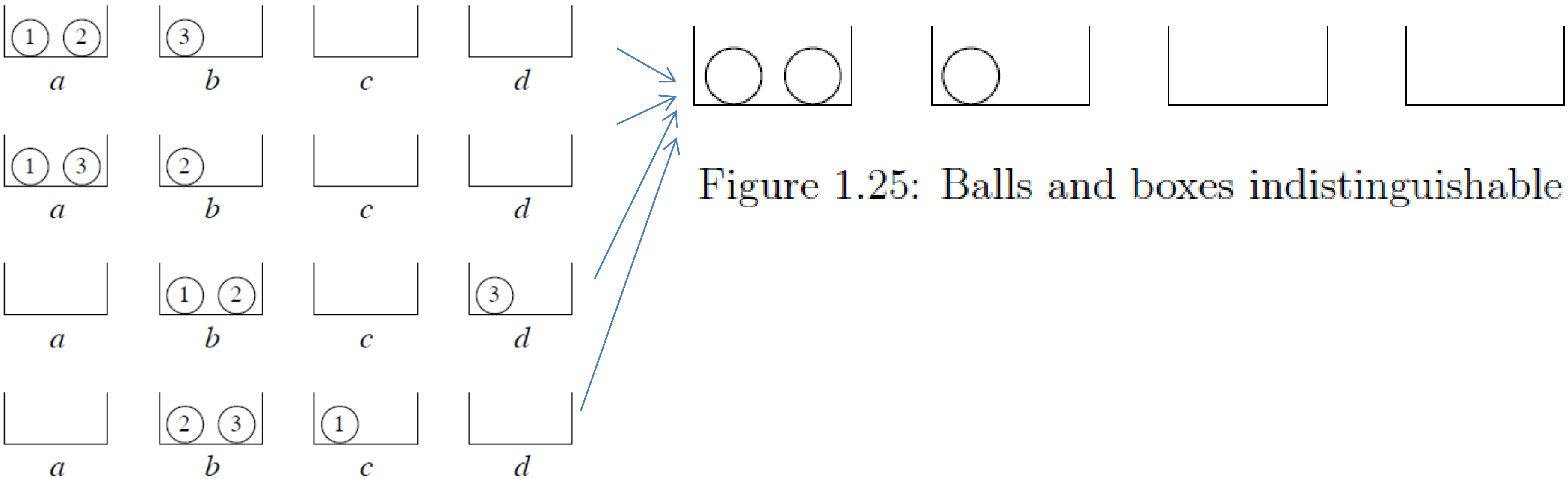


Figure 1.25: Balls and boxes indistinguishable

[R. P. Stanley, 2011]

Counting

- 3 balls, 4 boxes
- Both distinguishable: $4^3 = 60$
- Balls indistinguishable, boxes distinguishable, use Feller's "stars and bars" method of counting: $\binom{n+k-1}{n} = \binom{6}{3} = 20$

The Twelvefold Way

Elements of N	Elements of X	Any f	Injective f	Surjective f
dist.	dist.	1. x^n	2. $(x)_n$	3. $x!S(n,x)$
indist.	dist.	4. $\left(\!\binom{x}{n}\!\right)$	5. $\binom{x}{n}$	6. $\left(\!\binom{x}{n-x}\!\right)$
dist.	indist.	7. $S(n,0)+S(n,1)$ $+\cdots+S(n,x)$	8. 1 if $n \leq x$ 0 if $n > x$	9. $S(n,x)$
indist.	indist.	10. $p_0(n)+p_1(n)$ $+\cdots+p_x(n)$	11. 1 if $n \leq x$ 0 if $n > x$	12. $p_x(n)$

[R. P. Stanley, 2011]

Problem to Consider

- If Alice tosses a coin until she sees a head followed by a tail, and Bob tosses a coin until he sees two heads in a row, then on average, Alice will require four tosses while Bob will require six tosses (try this at home!), even though head-tail and head-head have an equal chance of appearing after two coin tosses.

[E. Klarreich, "Mathematicians Discover Prime Conspiracy," *Quanta Magazine*, 13 March 2016.]

- Intuitively, first, both have to get a head.
- After that, if Alice fails by getting a head, then she still needs only one tail.
- Her first head does not get “reset” by failing her second try.
- But after getting a head, if Bob fails by getting a tail then he gets reset: he has to start all over.

[https://www.reddit.com/r/math/comments/4abm4k/expected_number_of_coin_flips_different_for/d0z0him/]

Alice (toss until HT)

HT

HHT, THT

HHHT, THHT

HHHHT, THHHT

HHHHHT, THHHHT

…

Bob (toss until HH)

HH

THH

HTHH

THTHH

HTHTHH

…

- Need a way to deal with sample spaces of infinite cardinality

Annual Conference on Cognitive Computational Neuroscience (CCN)
September 6-8, 2017

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