Probability measure $P$

• The probability measure is a mapping $\mathcal{F} \to [0, 1]$ that satisfies the following axioms:

1. For any event $A$, $P(A) \geq 0$
2. If $A, B \in \mathcal{F}$ and if $A$ and $B$ are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$
3. The probability of the sample space satisfies $P(\Omega) = 1$
Properties of probability measures

4. \( P(A^c) = 1 - P(A) \)
5. For any event \( A \), \( P(A) \leq 1 \)
6. \( P(\emptyset) = 0 \)
7. If \( A \subset B \) then \( P(A) \leq P(B) \)
8. \( P(A \cup B) = P(A) + P(B) - P(AB) \)
Boole’s Inequality (Union Bound)

• For a countable set of events $A_1, A_2, ..., A_i$, we have:

$$P \left( \bigcup_{i} A_i \right) \leq \sum_{i} P(A_i)$$
Bonferroni’s Inequality

• Prove that for any two events $A$ and $B$, we have: $P(AB) \geq P(A) + P(B) - 1$
Bonferroni’s Inequality

• Generalize to the case of $n$ events $A_1, A_2, \ldots, A_n$ by showing that:

$$P(A_1 A_2 \cdots A_n) \geq P(A_1) + \cdots + P(A_n) - (n - 1)$$
Classical Probability and Counting

- An important class of probability spaces are those such that the set of outcomes, $\Omega$, is finite, and all outcomes have equal probability.

- The probability for any event $A$ is $P(A) = \frac{|A|}{|\Omega|}$, where $|A|$ is the number of elements in $A$ and $|\Omega|$ is the number of elements in $\Omega$. 
Permutations and Combinations

Poker — Just one set of cards
Magic — 5! possible sequences of cards
Binomial Coefficient

• There are $\binom{n}{k}$ ways to choose a subset of size $k$ elements, disregarding their order, from a set of $n$ elements

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
Pingala’s Meru Prastāra (~450 BC)
Pingala’s Meru Prastāra (~450 BC)

Ratio of successive terms of first Meru sequence (Fibonacci sequence) converge to the golden ratio

Multinomial Coefficient

- Principle of overcounting: cancel any possible ties or indistinguishable symbols

\[
\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! \cdot k_2! \cdots k_m!}
\]
Videos in Course Notes

https://uofi.app.box.com/s/qo8exgegq6aryw1yxip4
Enumerative Combinatorics

• Think of \( N \) as set of balls and \( X \) as set of boxes
• A function \( f : N \to X \) consists of placing each ball into some box
• If we can tell the balls apart, then the elements of \( N \) are called distinguishable, otherwise indistinguishable
• Similarly if we can tell the boxes apart, then elements of \( X \) are called distinguishable, otherwise indistinguishable
Distinguishability

Figure 1.22: Four functions with distinguishable balls and boxes

[R. P. Stanley, 2011]
Distinguishability

Figure 1.23: Balls indistinguishable

[R. P. Stanley, 2011]
Distinguishability

Figure 1.24: Boxes indistinguishable

[R. P. Stanley, 2011]
Distinguishability

Figure 1.25: Balls and boxes indistinguishable

[R. P. Stanley, 2011]
Counting

• 3 balls, 4 boxes
• Both distinguishable: $4^3 = 60$
• Balls indistinguishable, boxes distinguishable, use Feller’s “stars and bars” method of counting:
  \[
  \binom{n + k - 1}{n} = \binom{6}{3} = 20
  \]
### The Twelvefold Way

<table>
<thead>
<tr>
<th>Elements of $N$</th>
<th>Elements of $X$</th>
<th>Any $f$</th>
<th>Injective $f$</th>
<th>Surjective $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dist.</td>
<td>dist.</td>
<td>1. $x^n$</td>
<td>2. $(x)_n$</td>
<td>3. $x!S(n, x)$</td>
</tr>
<tr>
<td>indist.</td>
<td>dist.</td>
<td>4. $\binom{x}{n}$</td>
<td>5. $\binom{x}{n}$</td>
<td>6. $\binom{x}{n-x}$</td>
</tr>
<tr>
<td>dist.</td>
<td>indist.</td>
<td>7. $S(n, 0) + S(n, 1) + \cdots + S(n, x)$</td>
<td>8. 1 if $n \leq x$ 0 if $n &gt; x$</td>
<td>9. $S(n, x)$</td>
</tr>
<tr>
<td>indist.</td>
<td>indist.</td>
<td>10. $p_0(n) + p_1(n) + \cdots + p_x(n)$</td>
<td>11. 1 if $n \leq x$ 0 if $n &gt; x$</td>
<td>12. $p_x(n)$</td>
</tr>
</tbody>
</table>

[R. P. Stanley, 2011]
Problem to Consider

• If Alice tosses a coin until she sees a head followed by a tail, and Bob tosses a coin until he sees two heads in a row, then on average, Alice will require four tosses while Bob will require six tosses (try this at home!), even though head-tail and head-head have an equal chance of appearing after two coin tosses.

• Intuitively, first, both have to get a head.
• After that, if Alice fails by getting a head, then she still needs only one tail.
• Her first head does not get “reset” by failing her second try.
• But after getting a head, if Bob fails by getting a tail then he gets reset: he has to start all over.

[https://www.reddit.com/r/math/comments/4abm4k/expected_number_of_coin_flips_different_for/d0z0him/]
Alice (toss until HT)

HT
HHT, THT
HHHT, THHT
HHHHHT, THHHHT
HHHHHHT, THHHHHT
...


Bob (toss until HH)

HH
THH
HTHH
THTHH
HTHTHH
...

...
• Need a way to deal with sample spaces of infinite cardinality
Annual Conference on Cognitive Computational Neuroscience (CCN)
September 6-8, 2017

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