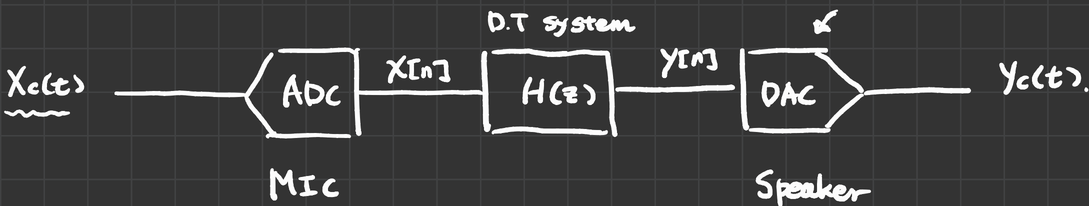
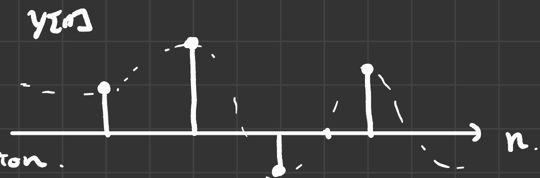
Reconstruction of CT signals from its samples.



→ Can we have $x_c(t) = y_c(t)$?

DAC: DT → CT

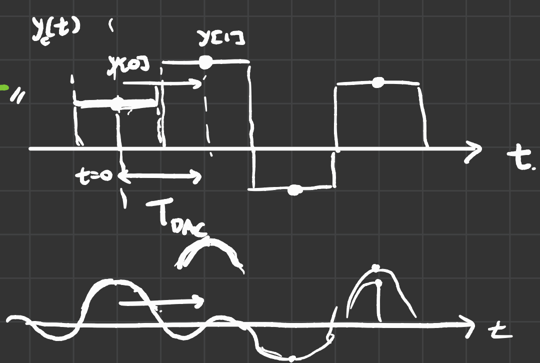
$y[n] \xrightarrow[\text{DAC.}]{} y_c(t)$



interpolation function.

"$$y_c(t) = \sum_{n=-\infty}^{\infty} g_c(t-nT)\cdot y[n]$$"

$$= \cdots + y[0]\cdot g_c(t)$$
$$+ y[1]\cdot g_c(t-T)$$
$$+ y[2]\cdot g_c(t-2T) + \cdots$$



$$``y_c(t) = \sum_{n=-\infty}^{\infty} g_c(t-nT)\cdot y[n]$$

CTFT $\updownarrow$

$$Y_c(\Omega) = \int_{-\infty}^{\infty} y_c(t)\cdot e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \sum_{n}^{\infty} g_c(t-nT)\cdot y[n]\cdot e^{-j\Omega t}\,dt$$

$$\hookrightarrow = \sum_{n}^{\infty} y[n] \int_{-\infty}^{\infty} g_c(t-nT)\, e^{-j\Omega t}\,dt.$$

$$\left\{\begin{array}{l} g_c(t) \xleftrightarrow{\text{CTFT}} G_c(\Omega) \\ \rightarrow g_c(t-nT) \longleftrightarrow e^{-j\Omega nT}\cdot G_c(\Omega). \end{array}\right.$$

$$= \sum_{n}^{\infty} y[n]\cdot e^{-j\Omega nT}\cdot G_c(\Omega)$$

$$= G_c(\Omega) \sum_{n}^{\infty} y[n]\cdot e^{-j\Omega Tn}$$

$$Y_d(\omega) = \sum_{n}^{\infty} y[n]\, e^{-j\omega n}$$

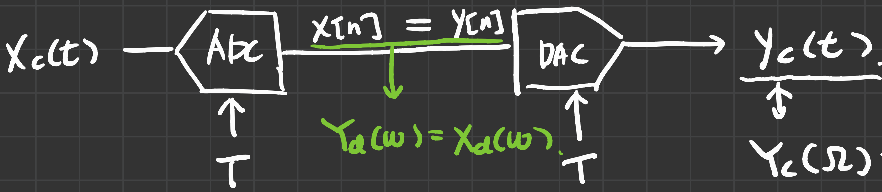
$$= G_c(\Omega)\cdot Y_d(\omega)\Big|_{\omega=\Omega T}$$

$$= G_c(\Omega)\cdot Y_d(\Omega T).''$$

$$``y_c(t) = \sum_{n=-\infty}^{\infty} g_c(t-nT)\cdot y[n]$$

$\updownarrow$ CTFT.

$$Y_c(\Omega) = G_c(\Omega)\cdot Y_d(\Omega T).$$



$$Y_c(\Omega) = G_c(\Omega)\, Y_d(\Omega T)$$

Assume Nyquist rate sampling at ADC, $T \cdot \Omega_{max} < \pi$

$$X_d(\omega) = \frac{1}{T} X_c\left(\frac{\omega}{T}\right) \qquad |\omega| \leq \pi$$

We want $x_c(t) = y_c(t)$.

Find $g_c(t)$.

$\omega = \Omega \cdot T$

$$Y_c(\Omega) = G_c(\Omega) \cdot Y_d(\Omega T)$$

$$= G_c(\Omega) \cdot \frac{1}{T} \cdot X_c(\Omega)$$

$$|\Omega| \leq \pi/T$$

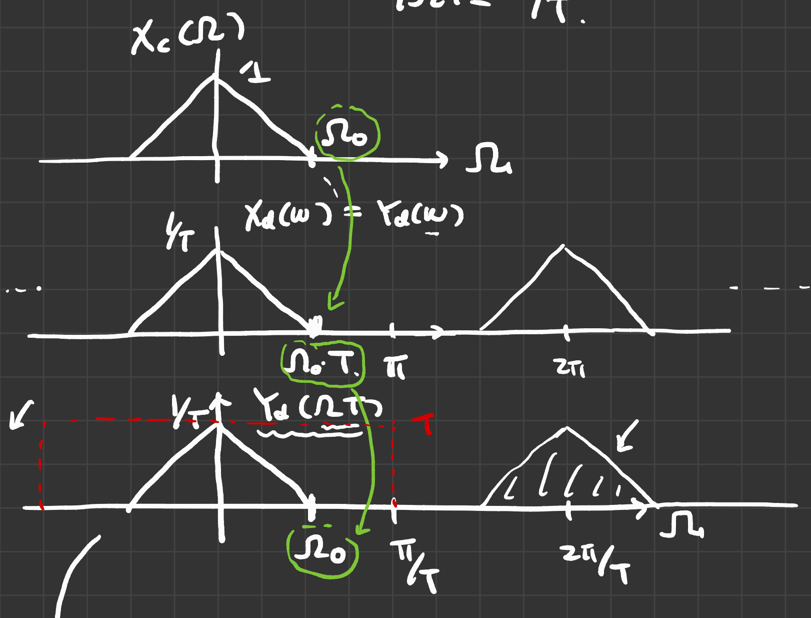
$Y_d(\omega) = X_d(\omega)$

$Y_d(\Omega T) = X_d(\Omega T)$

$\therefore X_d(\omega) = \frac{1}{T} X_c\left(\frac{\omega}{T}\right) \quad |\omega| \leq \pi$

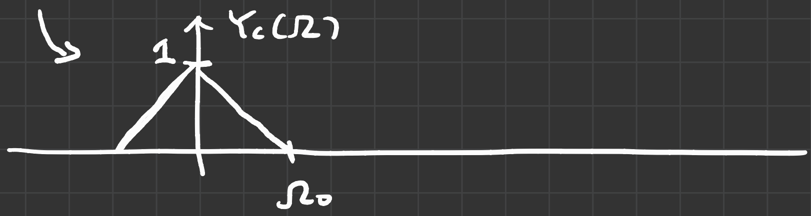
$Y_d(\Omega T) = \frac{1}{T} X_c(\Omega)$

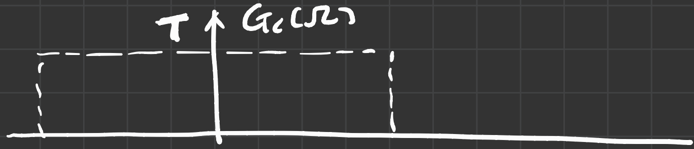
$|\Omega| \leq \pi/T$



$$X_d(\omega) = \frac{1}{T} X_c\left(\frac{\omega}{T}\right)$$

$$Y_d(\Omega T)$$





π/T inverse CTFT

$g_c(t)$? sinc function.

$= \text{sinc}\left(\frac{\pi t}{T}\right).$

$$G_c(\Omega) = \begin{cases} T, & |\Omega| \le \pi/T \\ 0, & \text{otherwise} \end{cases}$$

(Ideal Low-pass Filter)

$$g_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_c(\Omega)\, e^{j\Omega t}\, d\Omega$$

(inverse CTFT)

$$= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} T \cdot e^{j\Omega t}\, d\Omega$$

$$= \frac{T}{2\pi}\left[\frac{1}{jt} e^{jt\Omega}\right]_{-\pi/T}^{\pi/T}$$

$$= \frac{T}{2\pi} \frac{1}{jt} \cdot \left[e^{jt\pi/T} - e^{-jt\pi/T} \right]$$

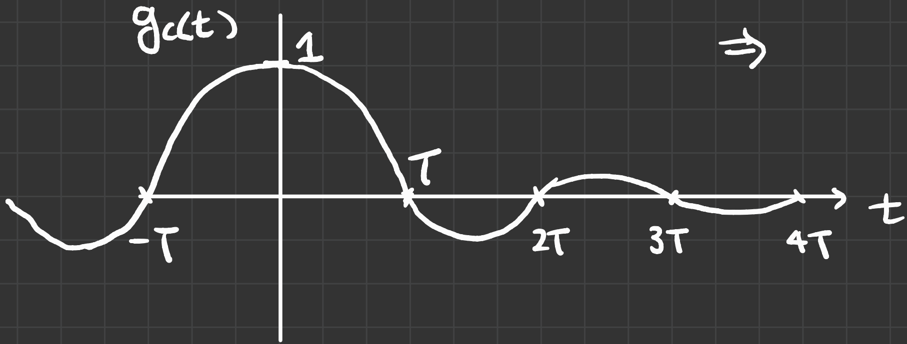
$$\cos t\pi/T + j\sin t\pi/T - (\cos t\pi/T - j\sin t\pi/T)$$

$$2j \sin t\pi/T$$

$$= \frac{T}{\cancel{2}\pi} \cdot \frac{1}{\cancel{j}t} \cdot \cancel{2}j \sin\left(\pi/T \cdot t\right)$$

$$= \frac{T}{\pi t} \cdot \sin\left(\pi/T \cdot t\right) \qquad \frac{\sin x}{x} = \operatorname{sinc} x$$

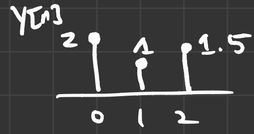
$$= \frac{\sin\left(\pi/T \cdot t\right)}{\pi/T \cdot t} = \operatorname{sinc}\left(\frac{\pi t}{T}\right)$$

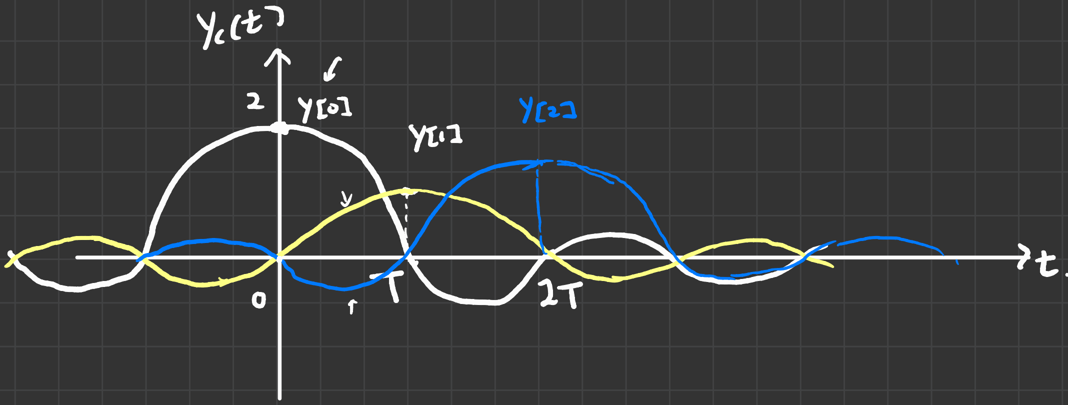


$\Rightarrow$

$g_c(t) = \operatorname{sinc}\left(\frac{\pi t}{T}\right)$: Ideal DAC interpolation kernel.

"$y_c(t) = \sum_{n=-\infty}^{\infty} g_c(t - nT) \cdot y[n]$",





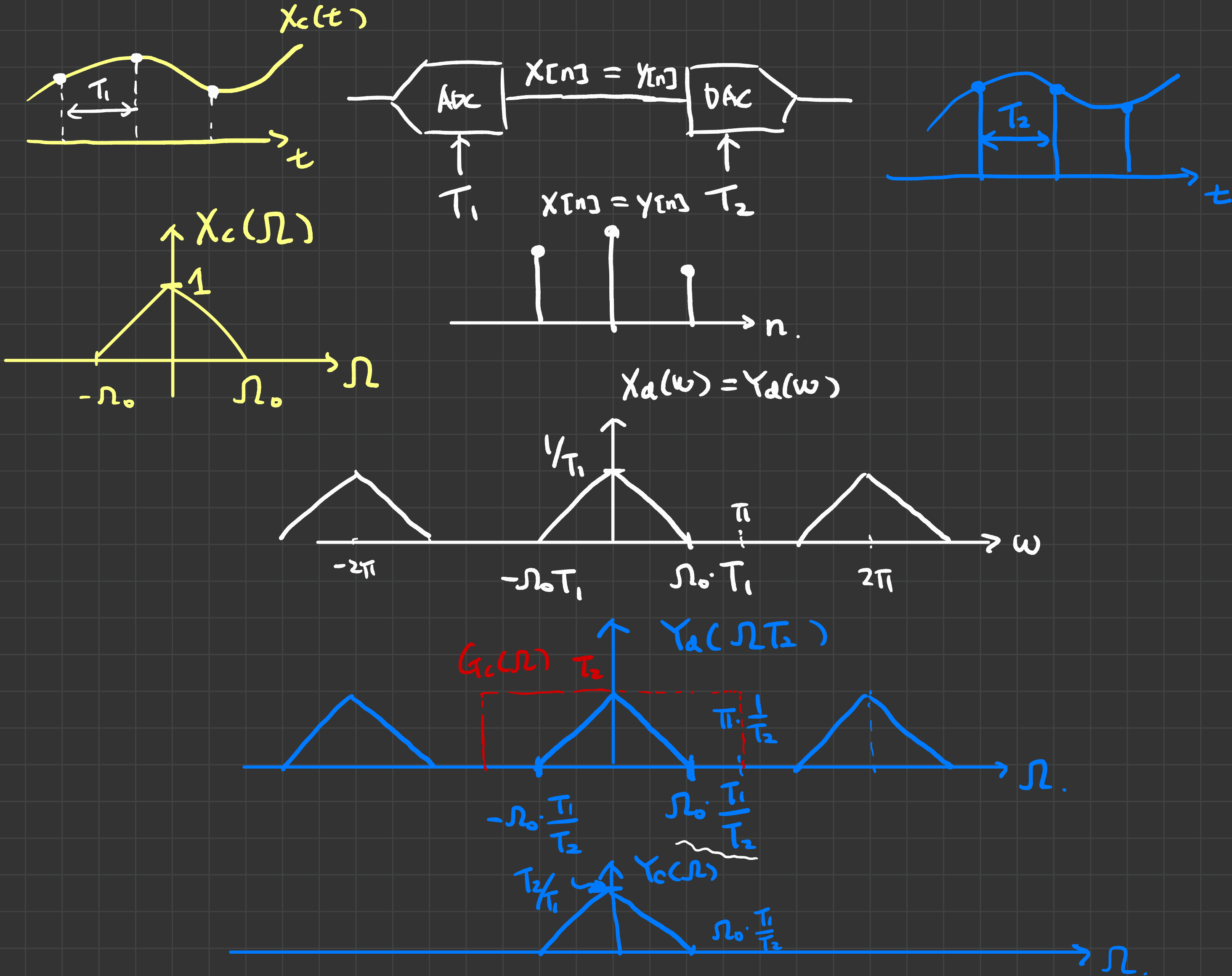
Shannon / Nyquist Sampling Thm

A bandlimitted signal $x_c(t)$
($X_c(\Omega) = 0$ for $|\Omega| \geq \Omega_{max}$)

can be reconstructed perfectly from its samples $x[n]$ taken with sampling rate

$$1/T \geq \frac{\Omega_{max}}{\pi} = 2 \cdot F_{max}$$

by the ideal interpolation kernel (sinc).



$x_c(t)$
T_1
t
ADC
$x[n] = y[n]$
DAC
T_1
$x[n] = y[n]$
T_2
T_2
t
$X_c(\Omega)$
1
$-\Omega_0$
Ω_0
Ω
n.
$X_d(\omega) = Y_d(\omega)$
$1/T_1$
π
ω
-2π
$-\Omega_0 T_1$
$\Omega_0 \cdot T_1$
2π
$Y_d(\Omega T_2)$
$G_c(\Omega)$
T_2
$\pi \cdot \frac{1}{T_2}$
Ω.
$-\Omega_0 \cdot \frac{T_1}{T_2}$
$\Omega_0 \cdot \frac{T_1}{T_2}$
T_2/T_1
$Y_c(\Omega)$
$\Omega_0 \cdot \frac{T_1}{T_2}$
Ω.