STAT 400 : DISCUSSION SECTION BD3 AND BD4 HANDOUT 4

SUBHADEEP PAUL

1. Theory and Discussions

1.1. Uniform and Exponential distributions.

- Pdf of uniform : $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$. Mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$. Pdf of exponential: $f(X) = \frac{1}{\theta} exp(-\frac{x}{\theta})$, $0 \leq x < \infty$. Mean θ, variance θ^2. CDF $1 - exp(-\frac{x}{\theta})$.
- Derivation of exponential: Waiting time for first event to happen in a poisson process with constant rate $\lambda = \frac{1}{\theta}$ events per interval. Recall Poisson distribution models number of such events in unit interval. Exponential distribution models the waiting time for the first event.

$$F(w) = P(W \leq w) = 1 - P(W > w) = 1 - P(\text{No change in time [0,w]}) = 1 - \exp(-\lambda w)$$

- Equivalence: So probability that the waiting time for first event is *more than* w units of time (follows exponential(λ) distribution) is equaivalent to probabilty that no events occur in w units of time (follows poisson (λw) distribution).

1.2. Gamma and Chi-Square distributions.

- Pdf of Gamma distribution : $f(X) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} exp(-\frac{x}{\theta})$, $0 \leq x < \infty$.
- Derivation : Now let W be the waiting time for α th event to occur. Gamma distribution models this waiting time

$$F(w) = P(W \leq w) = 1 - P(W > w) = 1 - P(\text{Less than } \alpha \text{events in time [0,w]})$$

$$= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k \exp(-\lambda w)}{k!}$$

- Equivalence: probability that the waiting time for αth event is *more than* w units of time ($P(T_\alpha > w)$, follows gamma distribution) is equaivalent to probabilty that less than α events occur in w units of time ($P(X_w < \alpha)$, follows poisson (λw) distribution).
- Since integrating the gamma pdf to find the probabilities is tough, we normally use the equivalent Poisson form.
- Chi square is a special case of gamma: mean$=r$, variance $2r$. Sum of squares of n standard normal RVs is χ^2_n.

1.3. Distributions from MGFs.

- You must know these 4 MGFs : Exponential : $\frac{1}{1-\theta t}$, Normal : $\exp(\mu t + \frac{\sigma^2 t^2}{2})$, Bernoulli : $(1 - p + pe^t)$, Binomial : $(1 - p + pe^t)^n$

2. Problems

Problem 1. In reliability study, stress and strength are random variables that generally follow Log Normal distribution. The strength of a material X is such that its distribution can be found by $X = e^Y$, or equivalently $Y = lnX$. If Y is N(10,1), find P (10000<X<20000). Find mean and variance of X.

Hint : Write the porbability in terms of Y first and then in terms of Z. For 2nd part use MGF of Y.

Problem 2. Suppose that the length of life of a human female, X, is modeled by the exponential p.d.f.,

$$f(X) = \frac{1}{80} exp(-\frac{x}{80}), 0 < x < \infty$$

(1) Compute the probability P(X > 10) and also compute the conditional probability P(X > 90|X> 80). What do you notice? (Memoryless property)

Problem 3. Every year on April 1, Anytown Tigers and Someville Lions play a soccer game. It is always a high-scoring game; the number of goals scored follows a Poisson process with the average rate of one goal per 5 minutes. (A soccer game consists of two halves, 45 minutes each.)

- What is the probability that the fourth goal is scored during the last 10 minutes of the first half?
- What is the probability that the fifth goal is scored during the last 15 minutes of the first half?
- What is the probability that it takes longer than 20 minutes to score 2 goals in the first half?

sol. Let X_t denote the number of goals scored in the interval [0,t]. X_t has a Poisson distribution with mean $\lambda t = 5t$. Break up the 45 minutes interval either in 5 minute intervals or 1 minute intervals. So both $P(35 < T_4 < 45) = P(T_4 > 35) - P(T_4 > 45) = P(X_{35} < 4) - P(X_{45} < 4)$ and $P(7 < T_4 < 9) = P(T_4 > 7) - P(T_4 > 9) = P(X_7 < 4) - P(X_9 < 4)$ are fine. For part 2, ($P(30 < T_5 < 45)$ or $P(6 < T_5 < 9)$. For part 3, $P(T_2 > 20) = P(X_{20} < 2)$ or $P(T_2 > 4) = P(X_4 < 2)$.