TRANSPORTATION COST AND FUNCTIONAL INEQUALITIES FOR SUB-RIEMANNIAN HEAT SEMIGROUPS

NATHANIEL ELDRIDGE
UNIVERSITY OF NORTHERN COLORADO
JOINT WORK WITH FABRICE BAUDOIN

• $(M,g)$ sub-Riemannian manifold. Sub-gradient $\nabla$, sub-Laplacian $L$, heat semigroup $P_t$.
• $\mathcal{P}(M)$ is the space of probability measures on $M$. $W_2$ is 2-Kantorovich–Wasserstein distance and $\text{He}_2$ is 2-Hellinger distance (equivalent to total variation).
• Reverse Poincaré inequality (rPI):
  $$|\nabla P_tf|^2 \leq C(t)(P_tf^2) - (P_t f)^2$$
• Hellinger–Kantorovich distance:
  $$W_{a,b}(\mu_0, \mu_1) := \sup \left\{ \int_M \phi_1 d\mu_1 - \int_M \phi_0 d\mu_0 : \partial_s \phi + a |\nabla \phi|^2 + b \phi^2 \leq 0 \right\}$$
• $W_{1,0}$ is Wasserstein distance, $W_{0,1}$ is Hellinger
• Reverse log Sobolev (rLSI):
  $$P_t f |\nabla \ln P_t f|^2 \leq C(t) (P_t (f \ln f) - (P_t f) \ln P_t f)$$
• Rényi-type divergence:
  $$T_{a,b}(\mu_0, \mu_1) := \sup \left\{ \int_M \phi_1 d\mu_1 - \int_M \phi_0 d\mu_0 : \partial_s \phi + a \nabla \ln \phi_s + b \phi_s \ln \phi_s \leq 0 \right\}$$

Preprint will be posted at https://sites.google.com/site/fabricebaudoinwebpage/articles_1 and on arXiv shortly.
HYPOELLIPTIC DIFFUSIONS IN INFINITE DIMENSIONS

MASHA GORDINA
UNIVERSITY OF CONNECTICUT

The classical definition of hypoellipticity in $\mathbb{R}^n$ is difficult to use in infinite dimensions directly. One of the consequences of hypoellipticity is smoothness of transition probabilities for the corresponding parabolic equation which has an interpretation in infinite dimensions.

We will describe two infinite-dimensional settings. One is Wiener or heat kernel measures on infinite-dimensional Heisenberg groups. The smoothness of the heat kernel measure in this case has been shown by Baudoin-G-Melcher using a generalized curvature-dimension condition, and by Driver-Eldridge-Melcher using stochastic calculus techniques. Another setting is path spaces over sub-Riemannian manifolds. Baudoin-G-Q.Feng recently proved a number of results on smoothness of horizontal Wiener measure on foliated manifolds.

**Theorem 1** (Smoothness of heat kernel measures). For a vector field $X$ on an infinite-dimensional space $W$ there a density $z^X$ on $W$ such that

$$
\int_W (Xf) d\mu = \int_W fz^X d\mu, \quad f : W \to \mathbb{C}.
$$

Notation:

**Hypoellipticity** in $\mathbb{R}^n$: $Lu \in C^\infty$ implies $u \in C^\infty$

**Heat kernel** (transition probability) is the fundamental solution to $\partial_t - \mathcal{L} = 0$

**Smoothness of transition probabilities** for any multi-index $\alpha$ there is a function $z_\alpha$ such that

$$
\int_{\mathbb{R}^n} (D^\alpha f)(p_t dx) = -\int_{\mathbb{R}^n} f z_\alpha (p_t dx) \quad \text{for all } f \in C_c(\mathbb{R}^n)
$$

**Heat kernel measure** $d\mu = p_t dx = \text{Law}(g_t)$

$g_t$ is a (hypoelliptic) Brownian motion

$W$ is either an infinite-dimensional Heisenberg group or $W(M)$, the Wiener space over a sub-Riemannian manifold

$X$ is a vector field on $W$
Area Formulas for Intrinsic Regular Submanifolds in the Heisenberg Group

Valentino Magnani
University of Pisa

Definition. \( \mathbb{H}^n \) is the \( n \)-th Heisenberg group with distance \( d \) and \( \Omega \subset \mathbb{H}^n \) is an open set, \( f : \Omega \to \mathbb{R}^k \) is \( h \)-differentiable with continuous \( h \)-differential \( Df(x) \), satisfying \( (f(xh) - f(x) - Df(x)(h))/d(h) \to 0 \) as \( h \to 0 \). If \( f \) has components \( f_j \), we define the \( h \)-Jacobian

\[ J_Hf(x) = \left| D_Hf_1(x) \wedge \cdots \wedge D_Hf_k(x) \right|. \]

Definition. A Lie subgroup of \( \mathbb{H}^n \) that is closed under intrinsic dilations is a homogeneous subgroup. A homogeneous subgroup \( \mathbb{W} \subset \mathbb{H}^n \) containing the vertical subspace is called vertical subgroup. A homogeneous subgroup \( \mathbb{V} \subset \mathbb{H}^n \) that does not contain the vertical subspace is called horizontal subgroup. If \( \mathbb{W} \cap \mathbb{V} = \{0\} \), then we say that \( \mathbb{W} \times \mathbb{V} \) is a semidirect product of \( \mathbb{H}^n \).

Definition. An intrinsic regular submanifold (of low codimension) \( \Sigma \subset \mathbb{H}^n \) is locally given by \( \Sigma = f^{-1}(0) \), where \( J_Hf(x) > 0 \) for all \( x \in \Omega \) and we may find

\[ \mathbb{H}^n = \mathbb{W} \times \mathbb{V}, \Phi : U \to \mathbb{H}^n, U \subset \mathbb{W}, \text{ and a mapping } \phi : U' \to \mathbb{V} \text{ such that } \Phi(w) = w\phi(w) \in \Sigma. \]

Definition. Let \( U \subset \mathbb{W} \) be an open set. Let \( \phi : U \to \mathbb{V} \) be an intrinsic differentiable function at \( \bar{w} \in U \). We define the intrinsic Jacobian of \( \phi \) at \( \bar{w} \) as

\[ J^\phi \phi(\bar{w}) = \left( 1 + \sum_{\ell=1}^{k} \sum_{I \in I_\ell} (M^\phi_I(\bar{w}))^2 \right)^{1/2}, \]

where \( I_\ell \) are suitable sets of multiindexes and \( M^\phi_I(\bar{w}) \) are minors defined by suitable intrinsic partial derivatives \( \partial^{\phi_i,\phi_j} \).

Theorem 1. If \( \Sigma \subset \mathbb{H}^n \) is an intrinsic regular submanifold and \( \Phi(w) = w\phi(w) \in \Sigma \), \( \mathbb{V} \) orthogonal to \( \mathbb{W} \), then for every Borel set \( B \subset \Sigma \) we have

\[ S^{2n+2-k}_d(B) = \int_{\Phi^{-1}(B)} J^\phi \phi(w) \, d\mathcal{H}^{2n+1-k}_E(w), \]

where the distance \( d \) has suitable rotational symmetries, \( \mathcal{H}^{2n+1-k}_E \) denotes the Euclidean Hausdorff measure and \( S^{2n+2-k} \) is the spherical measure with respect to \( d \).
THE CHENG-YAU GRADIENT ESTIMATE FOR CARNOT GROUPS AND SUB-RIEMANNIAN MANIFOLDS

PHANUEL MARIANO
UNIVERSITY OF NEW HAVEN
(JOINT WORKS WITH SAYAN BANERJEE, FABRICE BAUDOIN AND MASHA GORDINA)

Heisenberg group: \( \mathbb{H}^3 \cong \mathbb{R}^3 \) furnished with the group structure
\[
(x_1, y_1, z_1) \ast (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + (y_2x_1 - x_2y_1)).
\]
The sub-Laplacian on the Heisenberg group is given by \( \Delta_{\mathbb{H}} = \mathcal{X}^2 + \mathcal{Y}^2 \) where \( \mathcal{X} = \partial_x - y\partial_z, \mathcal{Y} = \partial_y + x\partial_z, \mathcal{Z} = \partial_z \) are the left-invariant vector fields.

Theorem 1. If \( u \) is any positive harmonic function in a ball \( B(x_0, 2r) \subset \mathbb{H}^3 \), then there exists a universal constant \( C > 0 \) not dependent on \( u \) and \( x_0 \) such that
\[
\sup_{B(x_0, r)} \| \nabla_{\mathcal{H}} \log u(x) \|_{\mathcal{H}} \leq \frac{C}{r}.
\]
Moreover, if \( u \) is any positive harmonic function on \( \mathbb{H}^3 \), then \( u \) must be a constant.

The previous theorem is also valid for Carnot groups and sub-Riemannian manifolds satisfying a generalized curvature dimension inequality:

Let \( (M, \mu) \) be a smooth connected manifold with a smooth measure \( \mu \) and a smoothly subelliptic diffusion operator \( L \) satisfying \( L1 = 0 \), and which is symmetric with respect to \( \mu \). Same setting as in Baudoin-Garofalo. We consider the functional \( \Gamma(f) = \Gamma(f, f) \), where
\[
\Gamma(f, g) = \frac{1}{2} (Lf g - f Lg - g Lf), \quad f, g \in C^\infty(M)
\]
known as the carré du champ operator and iterated version of it:
\[
\Gamma_2(f, g) = \frac{1}{2} (\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)), \quad f, g \in C^\infty(M).
\]

We assume \( M \) is equipped with a symmetric, bilinear form \( \Gamma_Z : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) \) satisfying
\[
\Gamma_Z(f g, h) = f \Gamma_Z(g, h) + g \Gamma_Z(f, h) \quad \text{and} \quad \Gamma_Z(f) = \Gamma_Z(f, f) \geq 0.
\]

We say that \( L \) satisfies the generalized curvature-dimension inequality \( CD(\rho_1, \rho_2, \kappa, \nu) \) if there exists \( \rho_1, \rho_2, d \geq 0, \rho_2 > 0 \) such that
\[
\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d} (Lf)^2 + \left( \rho_1 - \frac{\kappa}{\nu} \right) \Gamma(f) + \rho_2 \Gamma_Z(f), \quad f \in C^\infty(M), \nu > 0.
\]
EXTENDING BROWNIAN MOTION TO A FAMILY OF GRUSHIN-TYPE SINGULARITIES

ROBERT NEEL
LEHIGH UNIVERSITY

Here is the basic notation and underlying geometric structure:

- \( M = (\mathbb{R} \setminus \{0\}) \times \mathbb{S}^1 \)
- \( \mathcal{Z} = \{ x = 0 \} \)
- **The metrics:** Let \( \alpha \in \mathbb{R} \). On \( \mathbb{R} \times \mathbb{S}^1 \) with coordinates \((x, \theta)\), consider the following pair of vector fields:
  \[
  X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}.
  \]
Declaring these orthonormal on \( M \) gives the Riemannian metric
  \[
  g = dx^2 + |x|^{-2\alpha} d\theta^2,
  \]
on \( M \), which degenerates on the singularity \( \mathcal{Z} \).
- \( M^- = \{ x < 0 \} \) and \( M^+ = \{ x > 0 \} \).
- **The metric spaces \( M_\alpha \):** Define
  \[
  M_{\text{cylinder}} = \mathbb{R} \times \mathbb{S}^1, \quad M_{\text{cone}} = M_{\text{cylinder}} / \sim,
  \]
where \((x_1, \theta_1) \sim (x_2, \theta_2)\) if and only if \( x_1 = x_2 = 0 \). From Boscain-Prandi (2016), when \( \alpha \geq 0 \) (resp. \( \alpha < 0 \)) the natural extended distance makes \( M_{\text{cylinder}} \) (resp. \( M_{\text{cone}} \)) into a metric space (and a length space) in a way that induces on \( M_{\text{cylinder}} \) (resp. \( M_{\text{cone}} \)) its original topology. Said differently, \( M_{\text{cylinder}} \) (resp \( M_{\text{cone}} \)) gives the metric compactification of \( M \), at \( \mathcal{Z} \), with respect to this distance. We denote these metric spaces by \( M_\alpha \). (Note that \( M^+ \) and \( M^- \) are at distance 0 from each other.)
- \( \omega \) is the Riemannian volume on \( M \).