The Two-Envelope Problem for General Distributions

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Abstract

The 2-envelope problem is a classical probability conundrum. Two players each receive an envelope: one containing the amount X and the other $2X$. By turns, each player may either keep the amount received or switch envelopes. The conundrum is that if a player assumes the envelopes are equally likely, it is always best to switch, which seems paradoxical. However, having observed one value, the problem becomes essentially one of hypothesis testing based on a single observation, and the conditional probabilities will generally fail to be equally likely. Thus, the player must condition on the observed value, making the problem one of standard statistical inference, and not paradoxical. Here we take a general nonparametric approach and consider finding the envelope containing the larger value of the expectation of any specified function, $v(x)$. The basic result is that if $v(x)$ is bounded, then there is a randomized rule under which the success probability for choosing the larger envelope is greater than 1/2, uniformly over the set of all distribution-pairs. Other criteria are considered and counter-examples are presented to show that more general conditions will not provide success probabilities greater than 1/2.

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1 Introduction

In the classical formulation of the 2-envelope problem, the monetary amounts x and $2x$ are placed in two envelopes and each of two players is offered one of them (at random). The players do not know the contents of the envelope offered, and are given the chance to switch envelopes. The question is to choose the envelope with the larger value. A paradox appears to arise from the fact that if a player (say, player 1) assumes the envelopes are equally likely, then the expected winnings if the player switches is $\frac{1}{2} \cdot \frac{x}{2} + \frac{1}{2} \cdot (2x) = \frac{5}{4}x$, which is strictly greater than the amount, x, player 1 receives by not switching. However, player 2 reasons the same, and so both players appear to do better if they switch!

The paradox is resolved by noting that the amount the player draws is a random observation from a mixture of the (unknown) distribution of a random variable, X, and the distribution of $2X$. Thus, the amount observed is informative, and specifically, the player needs to estimate the conditional probability of the amounts in the envelopes given X. This conditional probability will generally differ from 1/2; and in fact it will depend on the distribution of X.

Zabell, 1988, appears to have introduced the term "2-envelope", but actually referred to the problem as the "exchange paradox". Versions of the problem have arisen numerous times in the past. Bollobás, 1997, attributes one version of the problem to the physicist Erwin Schrödinger. Another version of the problem appears in Kraitchik, M., 1930, who considers two men deciding whether or not to exchange ties. Gardner, 1982, presented a version involving the exchange of wallets whose contents are not known by either player (and thus differs from the 2-envelope problem since nothing is observed).

The 2-envelope problem can be generalized by allowing the amounts in the envelopes to have rather arbitrary distributions. Several problems in operations research seem to have this form, and thus provide potential real-word applications. Egozcue, *et al.*, 2013, applied solutions to the 2-envelope problem to the problem of accepting one of two bids in a real estate market. Two-stage problems in general open markets are clearly closely related. A formal treatment of two-stage pricing strategies in real estate markets is provided by Egozcue, *et al.*, 2017, where sufficient specificity is introduced to provide a useful application. While providing a real-world application, the specificity precludes treating the more general (nonparametric) version here.

There are additional classical problems that provide potential application. The secretary problem consists of a set of payoffs offered sequentially and asks for the optimal stoping time to achieve maximal expected payoff. If there are only two payoffs, the problem is also essentially the 2-envelope problem. The generalized 2-envelope problem is also a 1-observation version of the classical 2-armed bandit problem that was studied extensively in the 1950's and 1960's. In all of these cases, the distributions of the amounts in the two envelopes will be allowed to be random variables.

Thus, we assume that there is a pair of random variables, (X, Y), with X denoting the amount in the first player's envelope. The amounts may be dependent, and their marginal distributions will be allowed to be rather arbitrary differing distributions (thus providing a "nonparametric" setting).

Some recent papers have attacked the problem more formally and have obtained switching rules for maximizing the expected gain under moderately strong restrictions on the distributions, which precludes treating the nonparametric situation here. Brams and Kilgour ,1995, consider special cases for the distribution of X and discuss when switching is optimal. Also, they do not consider randomized switching rules. Agnew, 2004, introduces utility theory to resolve the paradox (in a manner similar to the use of utility in discussing the St Petersburg Paradox), but also avoids randomized procedures. Other more recent papers have also introduced utility considerations, but note that allowing X and Y to have arbitrary distributions obviates the need to consider utilities: by considering the distributions as those of the utilities of X and Y results for the utility of gain follow immediately from results here.

Albers, *et al.*, 2005, allow distributions for both X and Y and introduce randomized procedures. They apply game theory, but restrict attention to special families of distributions. Other results are presented in McDonnell and Abbott, 2009, and McDonnell, *et al.*, 2010; but also only in special cases. The results of Egozcue, *et al.* (2013 and 2015) introduce "threshold" rules to maximize the expected gain in the 2-envelope problem with $Y = 2X$ when the mean and variance are given. See Remark 3 after Theorem 1 for a discussion of their results. Pawitan and Lee, 2017, treated this problem from a likelihood perspective, but avoided formal probabilistic and hypothesis-testing development and, again, did not treat the nonparametric problem.

Here, rather, we treat the case of general distributions and ask for possibly randomized switching rules that do uniformly better than choosing an envelope with fixed probability .5 over nonparametric families of distributions. We focus on choosing the envelope with largest expected value. This would be the appropriate formulation for choosing which of two distributions to use over and over again based on a single observation. We also consider alternatives like maximizing the median or choosing the larger (random) value. The fundamental result here is the following: when the envelopes are offered each with probability .5, there is a strategy for choosing the larger value for the expectation of any bounded function, $v(x)$, with probability greater than 1/2 uniformly over all distributions. As a consequence, when $v(x) = x$, then for any finite interval there is a strategy than chooses the envelope with the larger mean with probability strictly greater than 1/2 uniformly over **all** distributions whose domain is contained in the interval (again when the envelopes are offered to each player with probability .5). Remark 3 after Theorem 1 shows that the same rule also works for maximizing the expected gain over the same nonparametric family.

A converse is also established for the case of choosing the larger mean: if the domain is not bounded, then for any decision rule there are distribution pairs for which the given rule is not as good as always switching at random with probability .5. That is, a pair of distributions can be defined for which the given rule is successful with probability less than or equal to .5, and strictly smaller unless the rule is equivalent to switching with probability .5. Finally, for choosing the larger median or the larger amount, conditions are developed under which the probability of a correct choice is larger than .5. However, these conditions are restrictive to some extent, and it is shown by explicit example that general non-parametric results like those for expectations can not exist.

2 Basic Optimality Result

Consider a joint distribution of the random variable pair $\{X, Y\}$, and let P_1 and P_2 be the two marginal distributions on the real line (that is, for X and Y respectively). Let $p_1(x)$ and $p_2(x)$ be the respective densities for P_1 and P_2 with respect to a dominating measure μ (μ may be taken to be $P_1 + P_2$; for example, see Lehmann, 1986). Let $v(x)$ be any function whose expectation is finite under P_1 and P_2. Suppose a single observation comes from population 1 with probability 1/2 and from population 2 with probability 1/2. The problem is to choose the population, i, for which the expectation $E_i v(X) = \int v(x)\, p_i(x)\, d\mu(x)$ is greater.

Following the theory of hypothesis testing (see, for example, Lehmann, 1986), consider a randomized decision rule (or selection rule) to be given by a function $\phi : \mathcal{X} \to [0, 1]$ such that $\phi(x)$ is the probability of choosing population 1. First consider the case where population 1 has the larger expectation; that is $E_1 v(X) > E_2 v(X)$. Then the probability of selecting the correct population is PC, where

$$\begin{aligned} PC &= \frac{1}{2}\int \phi(x) dP_1(x) + \frac{1}{2}\int (1 - \phi(x)) dP_2(x) \\ &= \frac{1}{2} + \frac{1}{2}\int \phi(x)(p_1(x) - p_2(x))\, d\mu(x)\ . \end{aligned} \tag{1}$$

The analogous equations holds with the roles of P_1 and P_2 interchanged if $E_2 v(X) > E_1 v(X)$; and an entirely analogous argument works.

Intuition: Define $q(x) = p_1(x) - p_2(x)$, and consider trying to find the least favorable densities, that is, to minimize (1) over functions $q(x)$ subject to the conditions: $\int v(x) q(x) d\mu(x) \geq 0$ and $\int q(x) d\mu(x) = 0$ (which are imposed since $q(x)$ is a difference of two densities for which the expectation of $v(X)$ is greater under P_1 than under P_2). This is a constrained linear programming problem on the space of functions, $q(x)$. So, by a Lagrange Multiplier argument, the solution can be obtained by setting to zero the (functional) derivative in direction $h(x)$ of (1) minus a sum of Lagrange multipliers times the constraints. The resulting equations (including the constraints) are often called the Euler conditions or the Kuhn-Tucker equations. Specifically, we seek to solve:

$$\begin{aligned} 0 &= \int \phi(x)\, h(x)\, d\mu(x) \;-\; a \int v(x)\, h(x)\, d\mu(x) \;-\; b \int h(x)\, d\mu(x) \\ &= \int (\phi(x) - av(x) - b)\, h(x)\, d\mu(x) \end{aligned}$$

for all functions $h(x)$ satisfying $\int v(x)h(x)d\mu(x) = 0$ and $\int h(x)d\mu(x) = 0$.

Here, $a > 0$ and b are Lagrange multipliers, and (as such) are determined by the constraints. For the equation to hold for all directions, $h(x)$, the factor in the integral must be zero (almost everywhere). If $v(x)$ is not bounded, this is impossible since $\phi(x) \in [0,\, 1]$. The proofs below avoid formal use of a Lagrange multiplier theorem, and follow from standard hypothesis testing analysis as described in Lehmann, 1986, for example.

Theorem 1 *Suppose $v(x)$ is bounded by a known constant with probability 1; that is, there is a known constant d such that $|v(x)| \le d$ almost everywhere with respect to μ. Then, decision rule,*

$$\phi(x) \equiv (v(x) + d)/(2d) \tag{2}$$

is such that the probability of correctly selecting the population with larger expectation of $v(X)$ is strictly greater than 1/2. Consequently, if $v(x) = x$ then the result holds for all distribution pairs $(P_1,\, P_2)$ whose domains are contained in some compact interval of the form $[-d,\, d]$.

Proof. : Define $\phi(x) \equiv (v(x)+d)/(2d)$. Since $|v(x)| \leq d$, $0 \leq \phi(x) \leq 1$ for all x. From (1) (assuming $E_1 v(X) > E_2 v(X)$),

$$\begin{aligned} PC &= 1/2 + 1/2 \int (v(x)+d)/(2d)\,(p_1(x) - p_2(x)\,d\mu(x) \\ &= 1/2 + (1/(4d)) \left(\int v(x)p_1(x)d\mu(x) - \int v(x)p_2(x)d\mu(x) \right) \\ &\qquad +1/4 \int (p_1(x) - p_2(x))\,d\mu(x) \\ &= 1/2 + (1/(4d))\,(E_1 v(X) - E_2 v(X)) + 0 \ > \ 1/2\,. \end{aligned}$$

An entirely analogous argument holds if $E_1 v(X) < E_2 v(X)$. ■

Remarks:.

1. In the 2-envelope problem, the amounts are positive. Hence, if there is a known upper bound, d, on the amounts, then the optimal rule can be taken to be $\phi(x) = x/d$ for $0 \leq x \leq d$. Note that this is the c.d.f. of the uniform distribution on $[0, d]$.

2. In the 2-envelope problem, it is quite natural to assume that the envelopes are equally likely to be given to the player. However, in other versions of the problem it may be reasonable to assume the "larger" envelope is presented to the player with probability r. Egozcue, *et al.*, 2013 and 2015, used this form and derived various results under assumptions on the form of the distributions. Many of these results are relatively immediate consequences of the basic theory of hypothesis testing (see Lehmann, 1986, especially the Neyman-Pearson Lemma and results on most powerful tests using monotone likelihood ratio). The following argument shows that if the ingredients r, P_1, and P_2

are all unknown, then for any $\phi(x)$, there are values of the ingredients for which the probability of choosing the "larger" envelope is no larger than 1/2, and strictly less than 1/2 unless $\phi(x) \equiv 1/2$. Specifically, if $\phi(x) \equiv 1/2$ then the success probability is exactly 1/2. If $\phi(x)$ is constant but not equal to 1/2, then r can be chosen to make the success probability less than 1/2. Otherwise, there there are points $x_1 < x_2$ with $\phi(x_1) \neq \phi(x_2)$. Let P_i be concentrated on $\{x_1, x_2\}$ and define $p_i = P_i(\{x_1\})$ (so that $1 - p_i = P_i(\{x_2\})$). Choose $p_1 < p_2$, so that $p_1 x_1 + (1 - p_1)x_2 > p_2 x_1 + (1 - p_2)x_2$, and P_1 has the larger mean. It follows that the success probability is

$$\begin{aligned} PC &= r\,(p_1\phi(x_1) + (1 - p_1)\phi(x_2)) \\ &\qquad + (1 - r)\,(p_2(1 - \phi(x_1)) + (1 - p_2)(1 - \phi(x_2))) \ . \end{aligned}$$

If either $\phi(x_1) > 1/2$ or $\phi(x_2) > 1/2$, choose $r = 1$ and choose p_1 to put probability greater than .5 on the larger $\phi(x_j)$. Otherwise, $\phi(x_j) < 1/2$ for either $j = 1$ or $j = 2$. Then choose $r = 0$ and choose p_2 to put probability greater than .5 on the larger value of $1 - \phi(x_j)$. In either case, the success probability is less than 1/2.

3. Egozcue *et al.*, 2015, posited the classical 2-envelope problem ($Y = 2X$) and considered choosing the envelope to maximize the expected gain (or, equivalently, the expected return). Their paper focussed on "threshold" rules; that is, rules such that $\phi(x) = 1$ for $x > b$ and $\phi(x) = 0$ otherwise (where b is a constant). Note that by the generalized Neyman-Pearson Lemma (see Lehmann, 1986) such $0 - 1$ rules are optimal in the 2-envelope problem if the distributions are known,

but the set where $\phi(x) = 1$ need not be an interval if the distributions are general. Clearly, such rules may have no favorable properties in the nonparametric version. In fact, in the nonparametric setting, the rule, $\phi(x) = ax + b$, continues to offer uniform improvement in the same sense as in Theorem 1. To show this, let X and Y be the amounts in the envelopes offered with probability .5, and compute the expected return:

$$\begin{aligned}
E &= \frac{1}{2}E\left[\phi(X)X + (1-\phi(X))Y\right] + \frac{1}{2}E\left[\phi(X)X + (1-\phi(X))Y\right] \\
&= \frac{1}{2}E\left[\phi(X)(X-Y) + \phi(Y)(Y-X)\right] + \frac{1}{2}(EX + EY) \qquad (3) \\
&\equiv E_1 + E_0 \, , \qquad (4)
\end{aligned}$$

where E_1 is the expected gain, and E_0 is the expected return for any constant rule (or for any rule that ignores the observed value). Now let $\phi(x) = ax + b$ with $a > 0$ and b such that $P\{aX + b \in [0, 1]\} = 1$. Then,

$$\begin{aligned}
E_1 &= \frac{1}{2}E\left[(aX+b)(X-Y) + (aY+b)(Y-X)\right] \\
&= \frac{1}{2}aE(X^2 + Y^2 - 2XY) + \frac{1}{2}b(EX - EY + EY - EX) \\
&= \frac{1}{2}aE(X-Y)^2 \, .
\end{aligned}$$

Now, $E(X-Y)^2$ is always nonnegative and is strictly positive as long as $P\{X = Y\} < 1$. Thus, the rule $\phi(x) = ax + b$ provides strictly positive gain for all distributions with domain bounded by a known constant and with $P\{X = Y\} < 1$. Note that the gain is identically zero if $P\{X = Y\} = 1$.

4. Since the presence of an observation allows strictly positive gain, one might ask how much a player should pay to make the game fair. If the distributions are known, this involves a straightforward computation (using $\phi(x) = ax + b$). However, the larger the domain, the smaller the expected gain; and so the expected gain can be arbitrarily small. Thus, there is no payment that would always work in the general (non-parametric) case.

The following converse to Theorem 1 shows that no more general condition on the distribution will ensure the existence of a rule with success probability always greater than 1/2.

Theorem 2 *If $v(x)$ is* **not** *bounded by a known constant, then for any decision function, $\phi(x)$, there are densities $p_1(x)$ and $p_2(x)$ (with respect to Lebesgue measure) such that $E_1 v(X) > E_2 v(X)$, but for which the probability of correct selection (given by* (1)*) is strictly less than 1/2 if $\phi(x)$ is one-to-one, and no larger than 1/2 if $\phi(x)$ is not one-to-one.*

Proof. : First note that if $\phi(x)$ were constant, the best constant would be $\phi(x) \equiv 1/2$, for which $PC = 1/2$. Even if $\phi(x_1) = \phi(x_2)$ for two points $x_1 \neq x_2$, P_1 and P_2 could put all probability on $\{x_1, x_2\}$ and again the best value for $\phi(x_1) = \phi(x_2)$ is 1/2 giving $PC = 1/2$. So for the remainder of the proof, assume $\phi(x)$ is one-to-one.

Begin by seeking densities $p_1(x)$ and $p_2(x)$ such that $E_1 v(X) = E_2 v(X)$.

Generalization to the desired case comes later. From (1),

$$\begin{aligned} PC &= 1/2 + 1/2 \int \phi(x)(p_1(x) - p_2(x))\, d\mu(x) \\ &= 1/2 + 1/2 \int (\phi(x) - av(x) - b)(p_1(x) - p_2(x))\, d\mu(x) \\ &= 1/2 + 1/2 \int (\phi(x) - av(x) - b) \left(\frac{p_1(x)}{g(x)} - \frac{p_2(x)}{g(x)} \right) g(x)\, d\mu(x) \,. \quad (5) \end{aligned}$$

The last line above holds for any a and b since $v(X)$ has the same expectation under p_1 and p_2 (and since both integrate to 1), and (trivially) for any density $g(x)$, which will be taken to be strictly positive.

Now, for given $\epsilon > 0$, define $p_1(x)$ and $p_2(x)$ so that

$$\frac{p_1(x)}{g(x)} - \frac{p_2(x)}{g(x)} = -\epsilon\, (\phi(x) - av(x) - b) \ . \quad (6)$$

Since $v(x)$ is unbounded (and $\phi(x) \in [0, 1]$), $(\phi(x) - av(x) - b)$ will differ from zero on a set of positive measure (for any $a \neq 0$ and b). Therefore, from (5), PC will be less than 1/2 (as long a $g(x)$ is chosen so that $g(x)(\phi(x) - av(x) - b)^2)$ is integrable). Clearly, (6) holds if and only if

$$p_1(x) = p_2(x) - \epsilon\, g(x)\, (\phi(x) - av(x) - b) \ . \quad (7)$$

Thus it remains to find a density $p_2(x)$ so that with $p_1(x)$ defined by (7), $p_1(x)$ is a density (that is, $p_1(x)$ is non-negative, and $\int (p_1(x) - p_2(x))\, d\mu(x) = 0$) and $\int v(x)\, (p_1(x) - p_2(x))\, d\mu(x) = 0$. From (7), these conditions become:

$$\begin{aligned} E_g\, \phi(X) \ - \ a\, E_g\, v(X) \ - \ b \quad &= \quad 0 \\ E_g\, \phi(X)\, v(x) \ - \ a\, E_g\, v^2(X) \ - \ b\, E_g\, v(X) \quad &= \quad 0 \ , \quad (8) \end{aligned}$$

where E_g denotes expectation using the density $g(x)$ with respect to the measure, μ.

Since $\phi(x)$ and $v(x)$ are given, we need to show that (8) defines finite values for $a \neq 0$ and b. Clearly, a and b can be defined (uniquely) if the following determinant is non-zero:

$$D \equiv \left| \begin{array}{cc} E_g\, v(X) & 1 \\ E_g\, v^2(X) & E_g\, v(X) \end{array} \right| = (E_g\, v(X))^2 - E_g\, v^2(X)\ . \tag{9}$$

But $(E_g\, v(X))^2 - E_g\, v^2(X) < 0$ as long as $v(x)$ is not constant. Thus, a and b can always be defined so that the side conditions hold. To show that a can be non-zero, the solution to (8) has

$$a = [E_g\, \phi(X)\, E_g\, v(X) \ - \ E_g\, \phi(X)\, v(X)]\, /D\ . \tag{10}$$

The numerator is just minus the covariance of $\phi(X)$ and $v(X)$ under g. Since $v(x)$ is unbounded, there are points x_1 and x_2 such that $v(x_1) \neq v(x_2)$. Since we are considering the case where $\phi(x)$ is one-to-one, $\phi(x_1) \neq \phi(x_2)$. The two planar points $(v(x_1),\, \phi(x_1))$ and $(v(x_2),\, \phi(x_2))$ are distinct. Thus, there is a straight line fitting them; that is, they are linearly related. As a consequence, the correlation between $v(X)$ and $\phi(X)$ on the two point domain is ± 1. Hence, there is a continuous, positive, and bounded density, $g_0(x)$, sufficiently close to a two-point probability distribution on $(x_1,\, x_2)$ so that the covariance is not equal to zero. Similarly, there is δ small enough so that $g(x) \equiv d\, g_0(x)/(1 + \delta v^2(x))$ (where d is chosen to make $g(x)$ a density) still has covariance non-zero, but for which $v^2(x)$ has a finite expectation.

Thus, it remains to show that $p_2(x)$ can be defined so that (7) defines a strictly positive function $p_1(x)$. Take $p_2(x) = g_0(x)$. By (7)

$$p_1(x) = g_0(x)(1 - \epsilon d(\phi(x) - av(x) - b)/(1 + \delta v^2(x)) \ > \ 0$$

for ϵ small enough since the coefficient of ϵ is bounded.

The last step is to consider the case of interest where $E_1 v(X) > E_2 v(X)$. Using the distributions in the equal moment case above, there is ϵ such that $PC < 1/2 - \epsilon$. Let P_{1v} denote the distribution of $v(X)$, and consider the location shift, $P^*_\delta = P_{1v}(x - \delta)$. Thus, P^*_δ will have a larger $Ev(X)$ (for any $\delta > 0$). But P^*_δ tends to P_{1v} as $\delta \to 0$, and the expectations of $v(X)$ also converge. It follows that There is δ such that P_1 has a larger $Ev(X)$, but for which $PC \leq 1/2 - \epsilon/2 < 1/2$. ■

Remark: The argument concerning the existence of $a \neq 0$ and b may seem rather special and artificial. In fact, there is a more general and natural approach based on the method of moment spaces. Let $\mathcal{G}$ be the set of densities, $g(x)$ and consider the set of points

$$M \equiv \{(x,\, y,\, z) \in \mathbb{R}^3 : \; x = E_g v(X),\, y = E_g \phi(X),\, z = E_g v(X)\phi(X)\,;\, g \in \mathcal{G}\}\,.$$

Since the point masses are the extreme points of the set of all distributions, M is the closed convex hull of

$$\{(x,\, y,\, z) \in \mathbb{R}^3 \; : \; x = v(u)\,,\, y = \phi(u)\,,\, z = v(u)\phi(u) \; ; \; u \in \mathbb{R}\,\}\,.$$

The surface, $S \equiv \{z - xy = 0\} \cap M$ represents the set of distributions with $cov(v(X),\, \phi(X)) = 0$. This surface is monotonic and nonlinear; and so if there are at least two points in M, the line segment connecting these two points will also be in M, and at most two points on the segment will lie on S. If there are four different (non-planar) points in M, there will be an open set of distributions with non-zero covariance; and thus there will be a very wide range of densities giving $a \neq 0$ in general. The specific set of such densities will depend on both $v(x)$ and $\phi(x)$, but will tend but a rather general set of densities.

3 Extensions

An immediate question is the following: how special is the focus on expected values? Consider the problem of choosing the envelope with greater median, and consider discrete domains, $\mathcal{X} = \{x_1, x_2, \cdots, x_k\}$. If $k = 2$ (the Bernoulli case), the mean and median are ordered in the same way, and it is possible to achieve $PC > 1/2$. So consider $k \geq 3$. The argument below shows that for any $\phi(x)$, either ϕ is constant and $PC \leq 1/2$ or ϕ is non-constant and there are (discrete) distributions P_1 and P_2 for which $PC < 1/2$.

Specifically, let the decision rule $\phi(x)$ be given. As above, if $\phi(x)$ is constant on two points, the best one can achieve is $PC = 1/2$. So consider the remaining case: there are points $\{x_1, x_2, x_3\}$ (perhaps after relabeling) such that

$$\phi(x_1) > \max\{\phi(x_2), \phi(x_3)\} \quad ; \quad x_2 > x_3 \, . \tag{11}$$

For appropriate $u > 0$, and a small value $\epsilon > 0$, the entries in Table 1 define distributions P_1 and P_2 (depending on $\phi(x)$). It will be shown that there are values for $u > 0$, and $\epsilon > 0$ giving strictly positive distributions with the median being greater under P_1, but for which PC$< 1/2$ (where PC is given by (1)). It is clear that a wide variety of discrete and continuous distributions have this property. Thus, no non-constant rule can assure that the probability of selecting the population with largest median is 1/2 or greater for all populations.

To find appropriate u and ϵ: given $\phi(x)$, define

$$a \equiv \phi(x_1) \; > \; b \equiv \phi(x_2) \quad ; \quad c \equiv \phi(x_3) \, . \tag{12}$$

Table 1: Probabilities for x_i under P_1 and P_2

population	x_1	x_2	x_3	other x (total)
P_1	ϵ	$.5 + 2\epsilon$	$.5 - 4\epsilon$	ϵ
P_2	$u\,\epsilon$	$.5 - \epsilon(u+2)$	$.5 + \epsilon$	ϵ

It is clear that given $u > 0$, ϵ can be chosen so that all the entries are in (0, 1). The row sums are 1, so P_1 and P_2 are probabilities (on $\mathcal{X}$). Also, $P_1(x_2) > .5$ and $P_2(x_3) > .5$, and so x_2 is the median under P_1 and x_3 is the median under P_2; and $x_2 > x_3$ by (11). Thus we can compute the probability of a correct selection as in (1). Using (12),

$$\begin{aligned} PC &= a\,\epsilon\,(1-u) \;+\; b\,(.5 + 2\epsilon - .5 + \epsilon\,(u+2)) \;+\; c\,(.5 - 4\epsilon - .5 - \epsilon) \;+\; 0 \\ &= \epsilon\,(-a(u-1) \;+ b(u+4) - 5c)\;, \end{aligned}$$

which is negative as long as $a(u-1) > b(u+4)$. This holds if (and only if) $a/b > (u-1)/(u+4)$. But by (11), a/b is strictly less than 1, and $(u-1)/(u+4) \to 1$ as $u \to \infty$. Therefore, u can be chosen sufficiently large so that the desired inequality holds; and, hence, $PC < 1/2$ for P_1 and P_2 of the appropriate form of the distributions in Table 1.

Finally, consider the problem of maximizing the probability of choosing the envelope with the larger (random) value. This is an appropriate formulation for the classical Secretary Problem (with 2 applicants), or when "winning" is the main object (for example, in the classical 2-envelope problem with specified amounts). Again, consider a joint distribution with marginals P_1 and P_2, and let $X \sim P_1$ and $Y \sim P_2$. Let R be a single observation that comes from population 1 with probability 1/2 and from population 2 with probability 1/2, and let S denote the unobserved random variable (that is, if $R = X$ then $S = Y$ and if $R = Y$ then $S = X$). The aim is to decide whether $R > S$ or $R < S$; that is, to claim either R is the larger of the two draws, or R is the smaller.

To allow for the possibility that $Pr\{X = Y\} > 0$, we need to define the right-continuous version of a distribution function. So, letting $X \sim P$, where $P(x) = Pr\{X \leq x\}$ is the usual (left-continuous) distribution, we denote the right-continuous version as $\tilde{P}$ with $\tilde{P}(x) \equiv Pr\{X < x\}$. Specifically, we will need the function $\tilde{P}_1$ and $\tilde{P}_2$ being the right-continuous versions of P_1 and P_2.

In this problem, the (randomized) decision rule is given by a function ϕ such that $\phi(x)$ is the probability of claiming $R > S$ if $R = x$ is observed and $(1 - \phi(x))$ is the probability of claiming $R < S$. Let PC denote the probability of a correct inference.

Lemma 1 *Using the above definitions and specifications, and* **not** *assuming X and Y are independent*

$$\begin{aligned} PC = \;& \tfrac{1}{2}Pr\{X \neq Y\} + \tfrac{1}{2}E(\phi(\max\{X, Y\}) - \phi(\min\{X, Y\})) && (13) \\ = \;& \tfrac{1}{2} + \tfrac{1}{2}(E(\phi(\max\{X, Y\}) - \phi(\min\{X, Y\})) - Pr\{X = Y\}) \,. && (14) \end{aligned}$$

Proof. Let $PC_{X<Y}$ denote the conditional probability of a correct inference given (X, Y) on the set $\{X < Y\}$; and let $PC_{X>Y}$ denote the conditional probability on the set $\{X > Y\}$. On the set $\{X < Y\}$, the inference is correct if $R = X$ (probability 1/2) and the player claims $R < S$ (probability $(1 - \phi(R)) = (1 - \phi(X))$) or $R = Y$ (probability 1/2) and the player claims $R > S$ (probability $\phi(R) = \phi(Y)$). Thus,

$$PC_{X<Y} = \frac{1}{2}(1 - \phi(X)) + \frac{1}{2}\phi(Y) \;=\; \frac{1}{2} + \frac{1}{2}(\phi(Y) - \phi(X)) \;. \qquad (15)$$

Similarly,

$$PC_{X>Y} = \frac{1}{2}\phi(X) + \frac{1}{2}(1-\phi(Y)) \;=\; \frac{1}{2} + \frac{1}{2}(\phi(X)-\phi(Y)) \;. \qquad (16)$$

Note that if $X = Y$, any claim of strict inequality is false. Therefore, with $I\{\cdot\}$ denoting the usual indicator function, it follows that

$$PC = EI\{X<Y\}\big(\frac{1}{2}+\frac{1}{2}(\phi(Y)-\phi(X))\big)+EI\{X>Y\}\big(\frac{1}{2}+\frac{1}{2}(\phi(X)-\phi(Y))\big)$$

$$= \frac{1}{2}Pr\{X\neq Y\}+\frac{1}{2}E\big(I\{X<Y\}(\phi(Y)-\phi(X))+I\{X>Y\}(\phi(X)-\phi(Y))\big)\,, \qquad (17)$$

from which (13) and (14) follow immediately. ■

Lemma 2 *Continuing Lemma 1, if the domains of* P_1 *and* P_2 *are contained in a finite interval* $[a, b]$ *(that is, the domain of the joint distribution is contained in the rectangle with corners* (a, a) *and* (b, b)*), and if* $\phi(x) = (x-a)/(b-a)$*, then* $PC > \frac{1}{2}$ *if (and only if)*

$$E(\max\{X, Y\} - \min\{X, Y\}) > (b-a)\,Pr\{X = Y\} \;. \qquad (18)$$

Finally, if X *and* Y *are independent,* PC *can also be computed as*

$$PC = \frac{1}{2} \;+\; \frac{1}{2}(A_1 + A_2) - \frac{1}{2}\,Pr\{X = Y\} \;, \qquad (19)$$

where

$$A_1 \;=\; \int \Big[\phi(x)p_1(x)\tilde{P}_2(x) \;-\; p_2(x)(1-P_1(x)\Big]\; d\mu(x) \qquad (20)$$

$$A_2 \;=\; \int \phi(x)\,\Big[p_2(x)\tilde{P}_1(x) \;-\; p_1(x)(1-P_2(x)\Big]\; d\mu(x) \;. \qquad (21)$$

Here $\tilde{P}_1$ *and* $\tilde{P}_2$ *are the right-continuous versions defined above. Note that if* P_j *is continuous at* x *(equivalently, there is no point mass at* x *under* P_j*), then* $\tilde{P}_j(x) = P_j(x)$*. Also note that* $A_1 + A_2$ *can be written as*

$$\int \phi(x)\big[p_1(x)(\tilde{P}_2(x) - 1 + P_2(x)) \;+\; p_2(x)(\tilde{P}_1(x) - 1 + P_1(x))\big]\, d\mu(x)\,. \qquad (22)$$

Proof. To obtain (18), note that $Pr\{X \neq Y\} = 1 - Pr\{X = Y\}$, and simply insert the given $\phi(x)$ into (14).

To obtain (19), (20), and (21), let A_1 and A_2 denote the expectations of the last two terms of (17). Then, recalling that $\tilde{P}_j$ is the right-continuous version,

$$\begin{aligned} A_1 &= \int \int_{(-\infty,\, y)} \phi(y) p_1(x) p_2(y)\, d\mu(x) d\mu(y) \\ &\qquad - \int \int_{(x,\, \infty)} \phi(x) p_1(x) p_2(y)\, d\mu(y) d\mu(x) \\ &= \int_{-\infty}^{\infty} \phi(y)\, \tilde{P}_1(y)\, p_2(y)\, d\mu(y) - \int_{-\infty}^{\infty} \phi(x)\, (1 - P_2(x))\, p_1(x))\, d\mu(x) \ , \end{aligned}$$

and similarly,

$$A_2 = \int_{-\infty}^{\infty} \phi(x)\, \tilde{P}_2(x)\, p_1(x)\, d\mu(x) - \int_{-\infty}^{\infty} \phi(y)\, (1 - P_1(y))\, p_2(y))\, d\mu(y) \ .$$

Since x and y are dummy variables (variables of integration), we can replace y by x (and simplify) to get (20) and (21). Equation (22) follows immediately. ■

If $Pr\{X = Y\} = 0$ (for example, if the distributions are continuous), then PC will exceed 1/2 as long as the second term in (14) is positive. This will be true quite often, as shown in the following theorem. However, Theorem 4 below shows that if $Pr\{X = Y\} > 0$ then there will be examples with $PC < 1/2$.

Theorem 3 *Assume the hypotheses for Lemmas 1 and 2. Let* $Pr\{X = Y\} = 0$. *Then* $PC > 1/2$ *if either* ϕ *is strictly monotonic or* P_1 *and* P_2 *are known and independent and the factor in brackets in* (22) *is strictly positive on a set of positive measure.*

Proof. The first claim follows from (14): the first term contributes 1/2 (since $Pr\{X = Y\} = 0$) and the second is strictly positive if ϕ is strictly monotonic. That the second term is positive under the alternative assumption above follows directly from (19) and (22) by taking $\phi(x)$ to be 1 if the factor in brackets is positive, and 0 otherwise. ■

Theorem 4 *As a partial converse of Theroem 3, if* $Pr\{X = Y\} > 0$ *then there are (discrete) distributions for which* $PC < 1/2$.

Proof. Suppose $Pr\{X = Y\} = \epsilon$. The result depends on the following fact: if ϕ is measurable, then there is a sequence $\{x_1, , x_2, \cdots\}$ converging to x_0 such that $\phi(x_n) \to \phi(x_0)$. This follows (for example) from Lusin's Theorem: for any $\epsilon > 0$ and any interval $[a, b]$ there is a compact subset, D, on which ϕ is continuous almost everywhere and such that D has measure greater than $(b-a)-\epsilon$. Let $[a, b]$ have strictly positive probability under the distributions of $\max\{X, Y\}$ and $\min\{X, Y\}$. Then, by Lusin's Theorem, there is a compact set, D, containing an uncountable number of points on which ϕ is continuous. Thus, there is x_0 and a sequence tending to x_0 along which ϕ is continuous. Hence, for any positive ϵ, there is a (finite) subsequence $\{y_1, y_2, \cdots, y_n\}$ with $\frac{1}{n} < \epsilon$ such that $|\phi(y_i) - \phi(y_j)| < \epsilon/2$ (for all y_i and y_j in the subsequence). Define P_1 and P_2 to be independent and equal, putting probability $\frac{1}{n}$ on each of y_i . Then

$$Pr\{X = Y\} = n \cdot \left(\frac{1}{n}\right)^2 = \frac{1}{n} < \epsilon ,$$

and, with $I(\cdot)$ denoting the indicator function,

$$
\begin{aligned}
&E(\phi(\max\{X, Y\}) - \phi(\min\{X, Y\})) \\
&\quad = E(I\{X = Y\}) + I\{X \neq Y\})\,(\phi(\max\{X, Y\}) - \phi(\min\{X, Y\})) \\
&\leq\ 0 + (1 - \frac{1}{n})\,\max\{(\phi(y_i) - \phi(y_j))\}\ <\ \epsilon/2\ .
\end{aligned}
$$

Hence, from (14), $PC < \frac{1}{2}\,(1-\epsilon) + \frac{1}{2}\,\epsilon/2 < \frac{1}{2}$. ■

Remarks.

1. The condition for PC to exceed 1/2 is quite general for smooth distributions. Specifically, if P_1 and P_2 have strictly positive densities, then P_1 and P_2 are continuous and strictly increasing to 1; and so the left- and right-continuous versions are the same, and for all x sufficiently large, both $P_1(x) > \frac{1}{2}$ and $P_2(x) > \frac{1}{2}$. Thus, the term in brackets in (22) is strictly positive for x large enough. As a consequence, for any pair of distributions with strictly positive densities, there is a constant, d, such that the rule, $\phi(x) = 1$ for $x \geq d$ and $\phi(x) = 0$ otherwise will obtain the larger value with probability strictly greater than 1/2.

2. While the counterexample above seems rather special (using discrete distributions), if ϕ is assumed to be smooth except on a set of probability ϵ where $X = Y$, then somewhat more general examples are easily constructed. Basically, any smooth ϕ will be nearly constant on some (small) interval, and choosing the smooth part of P_1 and P_2 to concentrate probability on this interval will allow the computation in the proof to go through.

3. Suppose we only want to claim $R \geq S$ or $R \leq S$. Let $\phi(x)$ be the probability of claiming $R \geq S$ if $R = x$ is observed. Similar to the proof above, let $PC_{X \leq Y}$ be the conditional probability of a correct claim given (X, Y) on the set $\{X \leq Y\}$. Then (as above) $PC_{X \leq Y}$ is exactly the same as in equation (15). With a bit more thought, $PC_{X>Y}$ can be seen to be exactly the same as in equation (16). However, if $X = Y$, the claim is always correct, and so we can write (using notation introduced in (15) and (16)):

$$PC_{X=Y} = Pr\{X = Y\} = EI(X = Y) \ . \tag{23}$$

Adding (15), (16) and (23),

$$PC = \frac{1}{2} + \frac{1}{2}E(\phi(\max\{X, Y\}) - \phi(\min\{X, Y\})) + \frac{1}{2}Pr\{X = Y\} \ .$$

It follows that PC is greater than 1/2 for claiming $R \geq S$ vs. $R \leq S$ whenever ϕ is strictly increasing.

4 Acknowledgements and Some History

I would like to thank James Stein for bringing the 2-envelope problem to my attention, describing the form of the solution, and pointing me to Wapner, 2012. Our discussions have been extremely helpful.

Wapner referred to a more general version of the 2-envelope problem as “Blackwell’s bet”, and obtained what is essentially the solution here for means. Wapner’s introduced this eponym with the citation of Blackwell, 1951. This citation may be misleading, as Blackwell, 1951 has no reference to anything like a 2-envelope problem, or even to a bet. The paper does mention randomized decision rules, but perhaps better references for the use of randomized procedures are Blackwell, 1950, or Blackwell and Girshick, 1954. The solution described by Wapner and Stein and developed as an optimal rule here amounts to considering ϕ as the cumulative distribution function of a uniform random variable (see Remark 1 after Theorem 1) and taking a random draw from this distribution as a "pointer" to the larger envelope (when compared to the offered amount, X). Following Wapner's reference to "Blackwell's bet", Wapner and Stein referred to this with the rather appealing tag: "Blackwell pointer". While randomized decision rules considerably predated Blackwell's work, it is not unreasonable to honor Blackwell's extensive work on randomized procedures; and this does provide a good example of Stigler's law of eponymy (see Stigler, 1980).

In fact, Blackwell, 1951, deals with the admissibility of location estimators and is fascinating to those of us who worked in statistical decision theory. I first learned of this paper in a course on decision theory taught by Charles Stein in the mid 1960's. Blackwell's paper presented an example where the best invariant estimator of a single location parameter is inadmissible, but it depended on treating the real line (the domain of the location parameter) as a vector space of dimension 4. Stein considered this somewhat "pathological", but conjectured that the best invariant estimate of a single co-ordinate of a d-dimensional location parameter would be admissible when $d \leq 3$ and inadmissible when $d \geq 4$ in some generality. Portnoy, 1975, provided a version of this result in somewhat special cases and Berger, 1976, gave a more general version.

5 Conflict of Interest Statement

The author hereby states that there is no conflict of interest.

References

[1] Albers, C. J., Kooi, B. P., and Schaafsma, W. (2005). Trying to resolve the two-envelope problem. *Synthese, 145(1)*, 89-109.

[2] Agnew, R. A. (2004). On the two-box paradox. *Mathematics Magazine, 77(4)*, 302-308.

[3] Berger, J.O. (1976). Inadmissibility results for generalized Bayes estimators of coordinates of a location vector, *Ann. Statist., 4*, 302-333.

[4] Blackwell, D. (1950). On randomization in statistical games with k terminal actions, *Contributions to the theory of games, 2*, 183-187.

[5] Blackwell, D. (1951). On the translation parameter problem for discrete variables, *Ann. Math. Statist., 22*, 393-399.

[6] Blackwell, D, and Girshick (1954). *Theory of Games and Statistical Decisions*, John Wiley and Sons, London.

[7] Bollobás, B. (1997). *Littlewood's miscellany*. Cambridge University Press., Cambridge, UK.

[8] Brams, S. J. and Kilgour, D. M. (1995). The box problem: to switch or not to switch. *Mathematics Magazine, 68(1)*, 27-34.

[9] Egozcue, M., García, L.F., and Zitikis, R. (2013). An optimal strategy for maximizing the expected real-estate selling price: accept or reject an offer, *Journal of Statistical Theory and Practice, 7*, 596-609.

[10] Egozcue, M. and García, L.F. (2015). An optimal threshold strategy in the two-envelope problem with partial information, *Journal of Applied Probability, 52*, 298-304.

[11] Egozcue, M., Wu, J., and Zitikis, R. (2017). Optimal two-stage pricing strategies from the seller's perspective under the uncertainty of buyer's decisions. *Journal of Statistical Distributions and Applications, 4(1)*, 13.

[12] Gardner, M. (1982). *Aha! Gotcha: Paradoxes to Puzzle and Delight*, W. H. Freeman and Co., San Francisco, CA.

[13] Kraitchik, M. (1930). *Le paradoxe des cravates.* Imprimerie Stevens Frères: La Mathematique des Jeux, Bruxelles.

[14] Lehmann, E. L. (1986). *Testing Statistical Hypotheses*, Second Edition, John Wiley and Sons, New York.

[15] McDonnell, M. and Abbott, D. (2009). Randomized switching in the two-envelope problem, *Proc. Royal Soc. A, 465*, 3309-3322.

[16] McDonnell, M., Grant, A., Land, I., Vellambi, B.N., Abbott, D., and Lever, K. (2011). Gain from the two-envelope problem via information asymmetry: on the suboptimality of randomized switching, *Proc. Royal Soc. A, 467*, 2825-2851.

[17] Pawitan, Y. and Lee, Y. (2017). Wallet game: probability, likelihood, and extended likelihood, *The Amer. Statist., 71*, 120-122.

[18] Portnoy, S. (1975). Admissibility of the best invariant estimator of one co-ordinate of a location vector, *Ann. Statist., 3*, 448-450

[19] Stigler, S. (1980). Stigler's law of eponymy, *Transactions of the New York Academy of Sciences, 39*, 147-157.

[20] Wapner, Leonard M. (2012). *Unexpected Expectations: The Curiosities of a Mathematical Crystal Ball*, CRC Press: Taylor & Francis Group, Boca Raton, FL.

[21] Zabell, S. (1988). Loss and gain: the exchange paradox. In *Bayesian statistics, Proc. of the Third Valencia Int. Meeting* (ed: J. M. Bernardo, *et al.*, Clarendon Press, Oxford, UK, 233–236.